Bordism and the Pontryagin-Thom Theorem

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1 Introduction

Given the classification of low dimensional manifolds up to equivalence relations such as diffeomorphism or homeomorphism, one would hope to be able to continue to classify higher dimensional manifolds. Unfortunately, this turns out to be difficult or impossible, and so one solution would be to turn to some weaker equivalence relation. One such equivalence relation would be to consider manifolds up to bordism, which we will define below. Though it is a weaker notion of equivalence, bordism still captures some important invariants of manifolds, which we will not discuss here. Furthermore, it turns out that bordism has a very rich structure and deep connections to algebraic topology, which one can exploit and makes it possible to classify manifolds up to bordism.

2 Unoriented Bordism

We begin with unoriented closed manifolds, where closed means compact and without boundary. All manifolds and maps will be smooth, and by convention, we consider the empty set \( \emptyset \) to be a manifold in every dimension. We then have the notion of a bordism between manifolds.

Definition 2.1. An (unoriented) bordism between two closed \( n \)-manifolds \( Y_0, Y_1 \), is the data \((X,p,\theta_0,\theta_1)\), where

- \( X \) is a \((n+1)\)-manifold with boundary.
- \( p : \partial X \rightarrow \{0,1\} \) is a partition of the boundary of \( X \).
- \( \theta_0 : [0,1) \times Y_0 \rightarrow X \) is an embedding.
- \( \theta_1 : (-1,0] \times Y_1 \rightarrow X \) is an embedding.

Furthermore, this data is subject to the condition that \( \theta_i(0,y) = p^{-1}(i) \) for \( i = 0,1 \). We think of the \( \theta_i \) as inclusion of a collar neighborhood of the boundary of \( X \). We say \( Y_0 \) and \( Y_1 \) are bordant if there is a bordism between them. If the context or choices are clear, we may refer to the bordism simply by \( X \).

In other words, \( Y_0 \) and \( Y_1 \) are bordant if there is a manifold \( X \) with boundary \( Y_0 \sqcup Y_1 \). However, we give the above definition so that we can generalize to oriented and framed bordism, where we need to keep track of orientations. Almost immediately, we observe that unoriented bordism is an equivalence relation, which justifies the use of the phrase “are bordant”:

Proposition 2.2. If \( Y_0 \) is bordant to \( Y_1 \), then \( Y_1 \) is bordant to \( Y_0 \) via the dual bordism, \((X^\vee,p^\vee,\theta_0^\vee,\theta_1^\vee)\).

Proof. We take \( X^\vee = X \), \( p^\vee = 1 - p \), \( \theta_0^\vee(t,y) = \theta_1(-t,y) \), and \( \theta_1^\vee(t,y) = \theta_0(-t,y) \). It then follows that we have a bordism between \( Y_1 \) and \( Y_0 \).

\[ \square \]
Dual bordism is not too interesting for unoriented manifolds, since there is not much structure to work with. However, we will note later on that there is some subtleties in the case of oriented and framed bordism.

**Theorem 2.3.** Unoriented bordism defines an equivalence relation on closed $n$-manifolds.

**Proof.** It is clear that bordism is reflexive, since for an $n$-manifold $Y$, we can take $X = [0, 1] \times Y$, and $p, \theta_0, \theta_1$ to be the obvious choices.

Proposition 2.2 shows that bordism is symmetric.

Unoriented bordism is also transitive. Suppose we are given a bordism $(X', p', \theta'_0, \theta'_1)$ between $Y_0$ and $Y_1$, and a bordism $(X'', p'', \theta''_0, \theta''_1)$ between $Y_1$ and $Y_2$. We construct a bordism between $Y_0$ and $Y_2$ in the following way. We glue $X'$ and $X''$ via the identity map along the boundary components $p'^{-1}(1)$ and $p''^{-1}(0)$. This gives us a smooth manifold $X$, along with a natural partition of the boundary $p$, and we take $\theta_0 = \theta'_0$ and $\theta_1 = \theta''_1$.

Now that we have shown that bordism is an equivalence relation, we can talk about the set of equivalence classes of closed $n$-manifolds under bordism, $\Omega_n$. It turns out that $\Omega_n$ has a very rich structure.

**Proposition 2.4.** $(\Omega_n, \sqcup)$ is an abelian group, where $\sqcup$ is the operation of disjoint unions of manifolds. Furthermore, $Y \sqcup Y$ is null-bordant.

**Proof.** Let $Y_0, Y_1$ be $n$-manifolds. We define $[Y_0] + [Y_1] = [Y_0 \sqcup Y_1]$.

Let us see that this is well-defined. Suppose $Y'_0$ is another representative of $[Y_0]$. Then let $X$ be a bordism between $Y_0$ and $Y'_0$. Then it is clear that $X \sqcup [0, 1] \times Y_1$ with the obvious choices of $p, \theta_0, \theta_1$ is a bordism between $[Y_0 \sqcup Y_1]$ and $[Y'_0 \sqcup Y_1]$.

This is clearly associative and commutative, as disjoint union is associative and commutative. We take the identity to be $[\emptyset]$, as we recall that we defined it to be a manifold in every dimension. Finally, we note that $X = [0, 1] \times Y$ is a bordism of $Y \sqcup Y$ with the empty set if we let $p^{-1}(0) = \partial X$ and $p^{-1}(1) = \emptyset$. Therefore, $[Y] + [Y] = [\emptyset]$, and so we have inverses.

The second part of our proposition implies that $\Omega_n$ is a product of copies of $\mathbb{Z}/2\mathbb{Z}$. As an example, we can calculate $\Omega_0 = \mathbb{Z}/2\mathbb{Z}$, since a closed 0-manifold is a finite disjoint union of points. The lemma above shows us that the disjoint union of two points is null-bordant. Therefore, we simply need to see that a single point is not the boundary of a compact 1-manifold. However, by the classification of compact 1-manifolds, any such manifold is a finite disjoint union of circles and line segments, hence the boundary of such a manifold must be even number of points. Again by the classification theorem, $\Omega_1 = 0$, as every closed 1-manifold is a disjoint union of circles, which are the boundary of a disjoint union of disks. Finally, we note but do not prove that $\Omega_2 = \mathbb{Z}/2\mathbb{Z}$, with generator $[\mathbb{RP}^2]$.

In addition to an abelian group structure on $\Omega_n$, we have further structure if we consider the following set: $\Omega = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Omega_n$.

**Proposition 2.5.** $(\Omega, \sqcup, \times)$ has a graded commutative ring structure, where $\times$ is the Cartesian product.

**Proof.** Let $Y$ be an $n$-manifold and $Z$ be a $k$-manifold. We define $[Y] \ast [Z] = [Y \times Z]$.

Let us see that this is well-defined. Suppose $Y'$ is another representative of $[Y]$. Then let $X$ be a bordism between $Y$ and $Y'$. Then $X \times Z$ is a bordism between $[Y \times Z]$ and $[Y' \times Z]$. This follows since $Z$ is closed and $X$ is a manifold with boundary, then $\partial (X \times Z) = (\partial X) \times Z$, and take $p, \theta_0, \theta_1$ to be the obvious choices.

This is clearly associative and commutative, as cartesian product is associative and commutative. Furthermore, we check that it is distributive: $([Y_0] + [Y_1]) \ast [Z] = [Y_0 \sqcup Y_1] \ast [Z] = [Y_0 \sqcup Y_1 \times Z] = [Y_0 \times Z] + [Y_1 \times Z]$. Finally, we take the identity to be the bordism class of a point $[pt]$.

Finally, for completeness, we state but do not prove the following theorem of Thom, which allows us to compute bordism classes of unoriented manifolds.

**Theorem 2.6.** (Thom) As graded rings,

$$\Omega \cong \mathbb{Z}/2\mathbb{Z}[x_2, x_4, x_5, x_6, \ldots]$$

where there is a polynomial generator of degree $k$ for each positive integer $k$ not of the form $2^i - 1$. 

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3 Oriented Bordism

We now briefly consider closed oriented $n$-manifolds. The same ideas for unoriented bordism will carry through to oriented bordism, with some minor adjustments. In particular, we define oriented bordism in the following way:

**Definition 3.1.** An oriented bordism between two closed $n$-manifolds $Y_0$, $Y_1$, is the data $(X,p,\theta_0,\theta_1)$, where

- $X$ is an oriented $(n+1)$-manifold with boundary.
- $p : \partial X \rightarrow \{0,1\}$ is a partition of the boundary of $X$.
- $\theta_0 : [0,1) \times Y_0 \rightarrow X$ is an orientation-preserving embedding.
- $\theta_1 : (-1,0] \times Y_1 \rightarrow X$ is an orientation-preserving embedding.

Furthermore, this data is subject to the condition that $\theta_i(0,y) = p^{-1}(i)$ for $i = 0, 1$.

Oriented bordism also defines an equivalence relation on closed oriented $n$-manifolds. However, we note that the dual bordism must be reconsidered. Rather than being a bordism from $Y_1$ to $Y_0$, we should think of it as a bordism from $Y_0^\vee$ to $Y_1^\vee$. For unoriented manifolds, it turned out that $Y_i^\vee = Y_i$, but in general, this is not true. In fact, for oriented manifolds, $Y_i^\vee = -Y_i$, where $-Y_i$ is the manifold $Y_i$ with the opposite orientation. Then it follows that $-X^\vee$ is the desired oriented bordism from $Y_1$ to $Y_0$.

We then have the set of oriented bordism classes, which we denote at $\Omega_{SO}^n$. Again, each $\Omega_{SO}^n$ is an abelian group under disjoint union, and letting $\Omega_{SO}^0 = \oplus_{n \in \mathbb{Z}_{\geq 0}} \Omega_{SO}^n$, we again have a graded ring structure with cartesian product. As above, we have a few simple calculations from the classification of 0, 1, and 2-manifolds. We have that $\Omega_{SO}^0 = \mathbb{Z}$, generated by $[pt.]$, the point with a positive orientation. We note that we no longer have that $[Y \sqcup Y'] = [\emptyset]$.

We again have that $\Omega_{SO}^1 = 0$, as every oriented closed 1-manifold is a disjoint union of circles, which are the boundary of a disjoint union of disks. Furthermore, $\Omega_{SO}^2 = 0$, as every oriented closed 2-manifold is a connect sum of tori, which is the boundary of a handlebody.

We also have the following theorem of Thom which we will state, but not prove.

**Theorem 3.2.** (Thom) As graded rings,
\[ \Omega_{SO}^n \otimes \mathbb{Q} \cong \mathbb{Q}[y_4, y_8, y_{12}, \ldots] \]
where there is a polynomial generator of degree $4k$ for each positive integer $k$.

4 Framed Bordism and the Pontryagin-Thom Theorem

Now we turn our attention to bordism classes of framed $n$-dimensional submanifolds $Y$ of some ambient manifold $M$. Recall that we have the following short exact sequence of vector bundles
\[ 0 \rightarrow TY \rightarrow TM|_Y \rightarrow \nu \rightarrow 0 \]
where $TY$ is the tangent bundle of $Y$ and $\nu$ is the normal bundle of $Y$ in $M$. We can now define the notion of a framing of $Y$.

**Definition 4.1.** An **framing** of an $n$-dimensional submanifold $Y$ of an ambient manifold $M$ of dimension $m$ is a trivialization of the normal bundle $\nu$. That is, an isomorphism of vector bundles $\mathbb{R}^{m-n} \rightarrow \nu$. 

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Framed submanifolds of \( M \) of codimension \( n \) occur naturally in the following way. Let \( N \) be a connected, oriented manifold of dimension \( n \), and let \( f : M \to N \) be a smooth map. Let \( p \) be a regular value of \( f \). Then \( f^{-1}(p) = Y \) is a submanifold of \( M \) of codimension \( n \). Furthermore, if we fix a basis \( \{e_1, \ldots, e_n\} \) of \( T_p(N) \), we can pull this back via \( df_p : T_yM \to T_pN \) to a basis for the normal bundle \( \nu_Y \) at each point \( y \in Y \). This follows since we have that \( T_yM = T_yN \oplus \nu_y \) and that \( df_y|_{T_yY} = 0 \) (since \( f \) is constant on \( Y \)). Then since \( p \) is a regular value, \( df_p \) factors to an isomorphism \( \nu_y \to T_pN \), which is exactly a framing for \( Y \).

Let us now define the notion of framed bordism:

**Definition 4.2.** A framed bordism between two closed \( k \)-submanifolds \( Y_0, Y_1 \), of an ambient manifold \( M \), is the data \((X, p, \theta_0, \theta_1)\), where

- \( X \) is a framed \((k+1)\)-submanifold with boundary of \([0,1] \times M\) such that \( X \cap ([i] \times M) = Y_i \)
- \( p : \partial X \to \{0,1\} \) is a partition of the boundary of \( X \).
- \( \theta_0 : [0,1] \times Y_0 \to X \) is an embedding that preserves framing.
- \( \theta_1 : (-1,0] \times Y_1 \to X \) is an embedding that preserves framing.

Furthermore, this data is subject to the condition that \( \theta_i(0,y) = p^{-1}(i) \) for \( i = 0, 1 \).

We once again get an equivalence relation generated by framed bordism, and we denote the set of classes as \( \Omega^{fr}_{k;M} \).

We note, but will not prove that if \( N \) is connected and oriented, and if \( p_0 \) and \( p_1 \) are regular values of \( f \), then \( f^{-1}(p_0) = Y_0 \) is framed bordant to \( f^{-1}(p_1) = Y_1 \). The fact that \( N \) is connected allows us to transport the framing along a path in \([0,1] \times M\), and the fact that \( N \) is oriented allows us to consistently choose a framing at all points of \( N \). Therefore, \( f \) determines a framed bordism class. In fact, homotopic maps yield the same framed bordism class.

**Proposition 4.3.** We have a well defined map \( \Phi : [M,N] \to \Omega^{fr}_{m-n;M} \).

We will not prove this in general, rather we will prove this specifically for the case \( N = S^n \).

Framing is an instance of what is known as a tangential structure, and we can define various other tangential structures to obtain other variations on bordism. For example, orientation is another example of a tangential structure, and from it we obtain oriented bordism \( \Omega^SO_n \). Furthermore, for other bordisms arising from tangential structures, we can obtain results similar to the proposition above and the theorem below. In particular, we are interested in framed cobordism because of the following theorem.

**Theorem 4.4 (Pontryagin-Thom).** If we take \( N = S^n \), then we have an isomorphism \( \Phi : [M,S^n] \to \Omega^{fr}_{m-n;M} \).

We identify \( S^n = \mathbb{R}^n \cup \{\infty\} \), and fix \( p \in \mathbb{R}^n \). Suppose we have a representative map \( f : M \to S^n \). Then we claim that we can homotope \( f \) to a map \( f_0 : M \to S^n \) such that \( p \) is a regular value of \( f_0 \).

By Sard’s theorem, the set of regular values of \( f \) is dense, so \( f \) has a regular value \( q \in \{ x \in S^n : ||x-p|| < \varepsilon \} \). In fact, we can take \( \varepsilon \) small enough so that this neighborhood of \( p \) only contains regular values of \( f \). Then we can choose a smooth family of rotations \( r_t : S^n \to S^n \) such that \( r_1(q) = p \), and \( r_t \) is the identity for \( 0 \leq t < \varepsilon' \), \( r_t = r_1 \) for \( 1 - \varepsilon' < t \leq 1 \), and each \( r_t^{-1}(z) \) lies on the great circle from \( y \) to \( z \), and is hence a regular value of \( f \). Then we have the homotopy \( F : [0,1] \times M \to \mathbb{R}^n \) given by \( F(t,x) = r_t(f(x)) \).

Then we define \( \Phi([f]) = \{ f_0^{-1}(p) \} \). To see that \( \Phi \) is well defined, suppose \( F : [0,1] \times M \to S^n \) is a smooth homotopy from \( f_0 \) to \( f_1 \), where \( p \) is a regular value of both maps. Then by the extension theorem, there is a perturbation \( F' : [0,1] \times M \to S^n \) of \( F \) that is transverse to \( \{p\} \) and is equal to \( F \) in a neighborhood of \( \partial([0,1] \times M) \subset [0,1] \times M \). Then \( (F')^{-1}(p) \) is a framed bordism from \( f_0^{-1}(p) \) to \( f_1^{-1}(p) \).

Let us now construct an inverse map \( \Psi : \Omega^{fr}_{m-n;M} \to [M,S^n] \), which we call Pontryagin-Thom collapse. Let \( Y \subset M \) be a framed submanifold of codimension \( n \). By the tubular neighborhood theorem, there exists a tubular neighborhood, which has the following data: a projection from \( \nu \to Y \) and the inclusion of its zero section from \( Y \to \nu \), an open neighborhood \( U \subset M \) of \( Y \) and a diffeomorphism \( \phi : \nu \to U \) such that \( \phi|_Y = \text{id}_Y \), and the following diagram commutes.
Lemma 4.5. Let \( M \) be a closed manifold, \( Y \subset M \) a framed submanifold of codimension \( n \), and \( f_0, f_1 : M \to S^n \) such that \( f_0^{-1}(p) = f_1^{-1}(p) = Y \) and \( df_0|_Y = df_1|_Y \). Then \( f_0 \) is homotopic to \( f_1 \).

Proof. We construct this homotopy in two steps. First we choose a tubular neighborhood \((U, \varphi)\) of \( Y \) such that neither \( f_0 \) nor \( f_1 \) take on the value \( \infty \) in \( U \). Then the framing gives us an identification \( U \cong Y \times \mathbb{R}^n \), and under this identification, we have maps \( g_0, g_1 : Y \times \mathbb{R}^n \to \mathbb{R}^n \). For a cutoff function \( \rho \) we can define the smooth homotopy \( G(t, y, \xi) = g_0(y, \xi) + t\rho(|\xi|)(g_1(y, \xi) - g_0(y, \xi)) \) between \( g_0 \) and \( g_1 \). Let \( g = G(1, y, \xi) \).

Then we obtain a map \( f : M \to S^n = \left\{ g : x \in U \right\} \). Then by construction \( f = f_1 \) in a neighborhood \( U\subset Y \) of \( Y \), and \( f = f_0 \) on the complement of \( U \), and \( f_0 \) \( f \). Furthermore, we now choose a specific \( \rho \) such that \( f \) does not take the value \( p \) in \( U \setminus V \).

We now wish to construct a homotopy from \( f \) to \( f_1 \). We note that we only need a homotopy on the complement of \( Y \). Then restricting \( f \) and \( f_1 \) to \( M \setminus V \), and writing \( S^n = \mathbb{R}^n \cup \{p\} \), we obtain functions \( h, h_1 : M \setminus V \to \mathbb{R}^n \), since both \( f \) and \( f_1 \) miss \( p \). Then we have a smooth homotopy \( H(t, x) = h(x) + t(h_1(x) - h(x)) \) between \( h \) and \( h_1 \), which we can then upgrade to a smooth homotopy from \( f \) to \( f_1 \) in the same way as above. Therefore, we have obtained our desired result. \( \square \)

5 Connection with stable homotopy theory

The astute reader may have noticed that we did not state a theorem about \( \Omega^r_M \), that allows us to compute the framed bordism classes of submanifolds of \( M \). There is a good reason for this, and it reveals a deep relationship between differential topology and algebraic topology, in particular homotopy theory.

We note that by the Whitney embedding theorem, we can embed any manifold into \( \mathbb{R}^{n+k} \), and hence into \( S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \) via the natural inclusion map. Therefore, if we want to understand manifolds up to bordism, it is really enough to consider framed submanifolds of \( S^{n+k} \). Then by the Pontryagin-Thom theorem, we have an isomorphism \( \pi_{n+k}(S^n) = [S^{n+k}, S^n] = \Omega^r_{n+k} \). In other words, we have a way of going between algebraic topology and homotopy groups of spheres to differential topology and bordism classes of framed manifolds, which has been historically very fruitful.

For instance, we have in Algebraic Topology the Freudenthal suspension theorem, which says that the homotopy groups of spheres stabilize for large enough \( k \); \( \pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^n) \). In other words, the stable
homotopy groups of spheres $\pi_S^n := \lim_{k \to \infty} \pi_{n+k}(S^n)$ exists. Via Pontryagin-Thom, this also tells us that the framed cobordism groups also stabilize, and it turns out one can prove stability of the framed cobordism groups and obtain the Freudenthal suspension theorem as a corollary!

We can also generalize these ideas and techniques from framed cobordism to various other notions of cobordism, such as oriented and unoriented cobordism. In particular, the Pontryagin-Thom construction generalizes, and we obtain, for example, a way to calculate the $n$-th unoriented bordism group as the stable homotopy group of some topological space, namely the Thom space of the universal $n$-plane bundle. The fact that we can calculate bordism as the stable homotopy groups of some space reflects the fact that bordism is a generalized cohomology theory, which can be represented as the homotopy groups of a spectrum, which is in spirit how Thom proved his results about the graded ring structure of unoriented and oriented bordism.
References


