Applications of Group Cohomology

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March 10, 2017

Abstract

In this talk, I will explore group cohomology from various perspectives (categorical, algebraic, and geometric). I will then demonstrate some neat applications of group cohomology to geometric, algebraic, and topological problems. At the end I will discuss a powerful tool for computing group cohomology, the Lyndon-Hochschild-Serre Spectral Sequence.

1 What is Group Cohomology?

If \( A, B \) are abelian categories, and \( F : A \to B \) be a left exact covariant functor \((0 \to A \to B \to C \to 0 \text{ goes to } 0 \to F(A) \to F(B) \to F(C))\). Then we want to know how to extend this to a LES. This can be done in many ways, but can be done “canonically” if \( A \) has enough injectives, and we get a LES

\[
0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots
\]

The \( R^nF(-) \) are the right derived functors of \( F \), and they can be thought of as how far \( F \) is from being exact.

What this is really doing is extending a functor \( F : A \to B \) to a functor of the derived categories \( RF : D(A) \to D(B) \).

We can embed

\[
A \to D(A) \cong K_\ast(P_A) \hookrightarrow K_\ast(A) \to K_\ast(B)
\]

where the last arrow is the functor on chain complexes induced by \( F \). But this does not factor through \( D(B) \cong K_\ast(P_B) \hookrightarrow K_\ast(B) \). Taking the derived functor forces this to factor.

Example 1.1 \( R \) a ring, then take \( A = R\text{-Mod} \), (this has enough projectives = injectives), \( B = \text{Ab} \), and \( F = \text{Hom}(X, -) \) for a fixed \( R \)-module \( X \). Then the right derived functors are \( \text{Ext}^n(X, -) \).

The derived categories are chain complexes modulo quasi-isomorphisms, which is the same as chain complexes of projective modules.

In particular, we will take \( R = kG \), the group ring of a finite group \( G \). \( k \) is a commutative ring, usually a Dedekind domain. We will think of \( k \) as a field or as \( \mathbb{Z} \). Let \( M \) be a \( kG \)-module, and we then define group cohomology to be the right derived functors of \( \text{Hom}(k, -) \), where \( k \) is the trivial \( kG \)-module.

\[
H^n(G, M) := \text{Ext}^n_{kG}(k, M)
\]

Remark 1.2 We can also define group homology in the following way: \( H_n(G, M) := \text{Tor}^n_{kG}(k, M) \). These are the left derived functors of the right exact functor \( - \otimes M \).

1.1 Algebraic Perspective

That’s nice, but how do we compute it? Recall how we compute \( \text{Ext}^n(N, M) \):

We take a projective resolution of the module \( N \) (can do when we have a finite \( k \)-algebra). \( P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0 \)
Then apply \( \text{Hom}_{kG}(-, M) \) to get a cochain complex, and then take the homology to get \( \text{Ext} \). This is isomorphic taking an injective resolution of \( M \), applying \( \text{Hom}_{kG}(N, -) \), and taking the cohomology, as one can see by looking at the double complex. We like working with projective resolutions, so we’ll do this instead. (This also works since in \( kG\text{-Mod} \) projective modules are precisely the injective modules).

This may seem to depend on the base ring \( k \), but it doesn’t. This is basically because you can take a free \( \mathbb{Z} \)-resolution, and then tensor with \( k \), which preserves exactness, and \( k \otimes \mathbb{Z}G \cong kG \).

We have the bar resolution to work with, which is nice, sometimes we get lucky and have a nice minimal resolution. (can always do with bar resolution).

For simplicity, we will take \( M = k \). This is because we can recover \( H^n(G, M) \) from \( H^n(G, k) \) and the computation of a (simpler) \( \text{Ext} \) group via the Universal Coefficient Theorem.

**Theorem 1.3 (Universal Coefficient Theorem)** Let \( k \) be a field/Dedekind domain, let \( X \) be a \( k \)-free complex, and \( M \) a \( k \)-module. Then we have exact sequences

\[
0 \to \text{Ext}^1_k(H_{*+1}(X), M) \to H^*(\text{Hom}(X, M)) \to \text{Hom}(H_*(X), M) \to 0
\]

In particular,

**Proposition 1.4** If \( k \) is a field, then we have that

\[
H^*(G, k) \otimes M \cong H^*(G, M)
\]

\[\text{Sending } \alpha \otimes m \mapsto [f] \otimes m \mapsto f(-)m \mapsto [f(-)m] \]

Remark: why is it called \( \text{Ext} \)? The extension problem of \( R \)-modules!

**Definition 1.5** If \( M \) and \( N \) are \( R \)-modules, then an extension of \( M \) by \( N \) is a SES of \( R \)-modules:

\[
0 \to N \to E \to M \to 0
\]

**Definition 1.6** Two extensions \( 0 \to N \to E \to M \to 0 \) and \( 0 \to N \to E' \to M \to 0 \) are equivalent if there is a commutative diagram, middle is isomorphism

**Proposition 1.7** The set of equivalence classes of extensions has a group structure coming from Baer sum (i.e. pullback over \( M \))

**Theorem 1.8** There is a bijective correspondence between equivalence classes of extensions and \( \text{Ext}^1(N, M) \)

### 1.2 Geometric Perspective

We also have a geometric way of looking at group cohomology: it is the cohomology of the classifying space \( BG \).

Why is it called a classifying space? Well, that’s because isomorphism classes of principal \( G \)-bundles \( \gamma : Y \to Z \) is the same as homotopy classes of maps \( Z \to BG \).

Since we are working with discrete groups \( G \), then \( BG \) is the Eilenberg-Maclane space \( K(G, 1) \).

**Definition 1.9** The Eilenberg Maclane space \( K(G, n) \) is a connected topological space such that \( \pi_n(K(G, n)) \cong G \) and all other homotopy groups are trivial.

Such a space always exists, is a CW complex, (kill higher homotopy groups via Postnikov towers) and is unique up to weak homotopy equivalence.

**Example 1.10**

1. \( K(\mathbb{Z}, 1) = S^1 \)
2. $K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty$
3. $K(\mathbb{Z}/2, 2) = \mathbb{C}P^\infty$
4. $K(G \times H, n) = K(G, n) \times K(H, n)$

**Theorem 1.11** there is an (NATURAL?) isomorphism

$$H^n(G; M) \cong H^n(BG; M)$$

1.3 **Examples**

If $k$ is $\mathbb{F}_p$, and $G$ is $\mathbb{Z}/p$, then we can calculate $H^*(G, k)$ either algebraically with Ext or geometrically with $BG$ and Universal Coefficient Theorem.

**Remark 1.12** We have a nice minimal resolution in the algebraic setting.

**Example 1.13** If $k$ is a field of characteristic $p$, and $G$ is $\mathbb{Z}/p$, then

$$H^*(G, k) = \begin{cases} \mathbb{F}_p[x] & |x| = 1, p = 2 \\ \Lambda[x] \otimes \mathbb{F}_p[y] & |x| = 1, |y| = 2, p \neq 2 \end{cases}$$

Since $k$ is a field, we can extend this very easily using the Kunneth formula. (Exterior algebra is polynomial algebra that is graded commutative and if $a$ is of odd degree, then $a \cdot a = 0$)

2 **Applications**

Why is this useful? Well, oftentimes low dimensional cohomology tells us interesting things:

2.1 $H^0$ as fixed points

This one is kind of silly.

$$\text{Ext}^0_{kG}(k, M) = \text{Hom}_{kG}(k, M) = \text{Hom}(k, M)^G = M^G.$$

The second equality comes from the fact that since $k$ is the trivial module, we must have by module homomorphism that $g \cdot f(1) = f(g \cdot 1) = f(1)$. Furthermore, we have a $G$ action on $\text{Hom}(k, M)$ by conjugation. The $G$ fixed points under conjugation are precisely the $G$-equivariant maps. The last comes from Yoneda.

Therefore, we have that $H^0(G; M)$ is precisely the $G$ fixed points of $M$.

2.2 Parametrizing the rational points on the circle

**Theorem 2.1** (Hilbert’s Theorem 90, homological version) Let $G$ be the Galois group of a finite extension $K$ of a field $k$. Then

$$H^1(G; k^*) = 0$$

Now, there is a very cool application:

**Proposition 2.2** If $G$ is furthermore cyclic with generator $s$, then an element $a \in K$ has norm one if and only if it can be written as $a = 1/(s(t))$ for some $t \in K$.

Now let us consider $k = \mathbb{Q}$, and $K = \mathbb{Q}(i)$. Then $G = \mathbb{Z}/2$, and is generated by complex conjugation.

Then by our proposition, we have that $x + iy \in K$ satisfies the equation $x^2 + y^2 = 1$ iff

$$x + iy = \frac{u + iv}{u - iv} = \frac{u^2 - v^2}{u^2 + v^2} + i \frac{2uv}{u^2 + v^2}$$

What does this tell us? Well, that the rational points on the circle must be of the form $(u^2 - v^2, 2uv)$

This can be computed elementarily in other ways, such as via trigonometry and intersection of circle with lines of rational slope, but man is this awesome!
2.3 \( H^2(G, A) \) as group extensions

**Definition 2.3** A central extension of a group \( G \) is a SES of groups

\[
1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1
\]

such that \( A \in Z(E) \)

**Theorem 2.4** There is a bijective correspondence between equivalence classes of central extensions and \( H^2(G, A) \).

2.4 Classifying Central Simple Algebras over \( \mathbb{R} \)

Okay, so what might \( H^2(G; M) \) tell us?

**Definition 2.5** A central division algebra over a field \( k \) is a division ring \( D \) such that \( Z(D) \cong k \) and \( \dim_k(D) < \infty \).

**Definition 2.6** The Brauer group associated to the field \( k \), denoted \( Br(k) \), is the set of isomorphism classes of central division algebras.

**Theorem 2.7** Let \( G \) be the Galois group of a finite separable extension \( K \) of a field \( k \). Then

\[
Br(k) \cong H^2(G; K^*)
\]

Then we can classify all the central division algebras up to isomorphism!

If we take \( k = \mathbb{R} \), and suppose we can calculate \( H^2(G; K^*) \) to be \( \mathbb{Z}/2 \). (By classical results) we know that \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) are division algebras over \( k \). Furthermore we can show that only \( \mathbb{R} \) and \( \mathbb{H} \) are central division algebras. Hence we have classified all of the central division algebras over \( \mathbb{R} \)!

**Remark 2.8** We can give a categorical definition of the Brauer Group over an arbitrary category. Given a symmetric monoidal category \( C \), we can define Azumaya algebras over \( C \), and ask for an equivalence relation (Morita equivalence, \( A \otimes B^{op} \cong \text{End}(V) \) for a dualizable \( V \)). The way this works is we view the classical Brauer group as the Brauer group of \( k \) vector spaces. Then the result is that

\[
Br(C) \cong H^2_{et}(\text{Spec} R, \mathbb{G}_m)
\]

3 The LHS Spectral Sequence

A powerful tool to compute group cohomology.

We would like to understand the group cohomology by understanding the group cohomology of simpler groups. (Such as cyclic groups or maybe Sylow-p-subgroups).

**Theorem 3.1** Let \( M \) be a \( kG \)-module. And suppose we have a short exact sequence of groups

\[
1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1
\]

Then we have a first quadrant spectral sequence with

\[
E_2^{p,q} \cong H^p(Q, H^q(N, M))
\]

that converges to \( H^*(G, M) \).

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3.1 What is a first quadrant spectral sequence?

It is a sequence of bigraded differential graded modules $E^{p,q}_n$ such that $d_n : E^{p,q}_n \to E^{p+n,q+1-n}_n$, $E_{n+1} = H^*(E_n)$ (taken with respect to $d_n$), and $E^{p,q}_n = 0$ if $p < 0$ or $q < 0$.

The first condition is that the differential has total degree 1.

The first quadrant tells us that there are only finitely many non-zero differentials.

In nice situations, this stabilizes to what is called the $E_\infty$ page, which we can use to read off what the spectral sequence converges to.

Let us run through a quick computational example first:

**Example 3.2**

Claim: We have a fibration $U(n-1) \to U(n) \to S^{2n-1}$

This follows since $U(n)$ (the matrices whose conjugate transpose is its inverse; this is a Lie group) has a transitive action on $S^{2n-1} \subset \mathbb{C}^n$.

Let us take $n = 2$. We can apply the Serre spectral sequence since we have a simple fibration ($\pi_1(B)$ action on $F$ is trivial)

Then since we know that $U(1) \cong S^1$ and we know the cohomology of $S^3$, we can recover the cohomology of $U(2)$.

Okay, now that we know how to calculate with a spectral sequence, where does it come from?

It comes from a double complex, such that $d' + d'' = 0$. We take the cohomology of the total complex in two ways: First take cohomology with respect to $d'$, so $d''$ induces a differential on $H'_{\ast}(A)$, and then take cohomology again.

We need to extend this process to approximate the cohomology of the total space, and you do this via a filtration coming from the double complex. Then the $E_n$ page is the cocycles of the total boundary map mod coboundaries.

3.2 What is the double complex?

Let $X \to k$ be a $kG$-projective resolution. Then by restriction, $X \to k$ is a $kN$-projective resolution as well. Let $Y \to k$ be a $kQ$-projective resolution.

Then note that $G$ acts on $\text{Hom}_{kH}(X, M)$ by conjugation. Note also that $H$ then acts trivially, so we can view $\text{Hom}_{kH}(X, M)$ as a $kQ$ complex.

Then we have a double complex $\text{Hom}_{kQ}(Y, \text{Hom}_{kH}(X, M))$. Taking one composition gives us the $E_2$ page, and the other shows that it abuts to $H^\ast(G, M)$.

The second perspective is via Grothendiecks’ spectral sequence of fixed point functors or something.

(we have an isomorphism of functors $\text{Hom}_{kQ}(k, \text{Hom}_{kN}(k, -)) \cong \text{Hom}_{kG}(k, -)$ )

**Remark 3.3** This can also be seen as the Serre SS for the fibration of classifying spaces arising from our SES.

There are some nice settings in which this SS collapses:

**Theorem 3.4** If $\text{char } k \nmid [G : N]$, then Maschke’s theorem tells us that $H_p(Q, -) = 0$ for $p \geq 1$. Hence the SS collapses to give us the isomorphism

$$H^\ast(G, M) \cong H^0(Q, H^\ast(N, M)) \cong H^\ast(N, M)^Q$$

**Theorem 3.5** If $\text{char } k \nmid [N]$, then again Maschke’s theorem tells us that $H^q(N, M) = 0$ for $q \geq 1$. Hence the SS collapses to give us the isomorphism

$$H^\ast(G, M) \cong H^\ast(Q, H^0(N, M)) \cong H^\ast(Q, M^N)$$

These two examples allow us to compute the Group cohomology of $S_3$, for example.
**Theorem 3.6** If \( p \neq 2, 3 \), then \( H^n(S_3, k) \cong \begin{cases} k & n = 1 \\ 0 & \text{else} \end{cases} \)

If \( p = 2 \), then \( H^n(S_3, k) \cong k[e_1^*], \) with \( |e_1^*| = 1 \)

If \( p = 3 \), then \( H^n(S_3, k) \cong \wedge (ke_1^*e_2^*) \otimes_k \text{Sym}(k(e_2^*)^2), \) with \( |e_1^*| = 1, |e_2^*| = 2 \)

**Proposition 3.7** If our SES is a central extension, (i.e. \( N \leq Z(G) \)), then the Universal Coefficient Theorem tells us that \( E_2^{p,q} \cong H^p(Q, k) \otimes H^q(N, k) \).

Use LHS to calculate cohomology of dihedral group, which is isomorphic to the relative bordism group of Fox n-coloured knots. Also important in number theory.