THE STRUCTURE OF STABLE HOMOTOPY THEORIES

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Abstract. In this talk, I will first motivate the study of stable homotopy theory and introduce axiomatic frameworks that capture the salient features of the stable homotopy category. This allows one to use homotopy theoretic techniques to study areas such as algebraic geometry or modular representation theory. I will then talk about results that describe the structure of various stable homotopy categories: namely the classifications of thick/localizing tensor ideal subcategories.

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1. Stable Homotopy Theory

We begin by considering the category of (pointed) topological spaces. This is not well-behaved, so we instead consider the category of “convenient” topological spaces, i.e. compactly gen. weak Hausdorff spaces. This category contains CW complexes, has function objects Hom(X, Y), and is closed under limits and colimits.

With this in mind, given a space X, we can form its suspension ΣX. In fact, Σ is a functor. So if we have a map f : X → Y, we get a map Σf : ΣX → ΣY. This gives us a system of homotopy classes of maps:

[X, Y] → [ΣX, ΣY] → [Σ²X, Σ²Y] → [Σ³X, Σ³Y] → ···

If we take X = S^k, we have the following system:

\[ \pi_k(Y) \to \pi_{k+1}(\Sigma Y) \to \pi_{k+2}(\Sigma^2 Y) \to \pi_{k+3}(\Sigma^3 Y) \to \cdots \]

The Freudenthal Suspension theorem tells us that these groups stabilize: eventually these maps become isomorphisms. If we take Y = S^n, this pattern manifests as the stabilization of homotopy groups of spheres:
The goal of stable homotopy theory is to understand this phenomenon. To this end, we would like to look for a category whose objects and morphisms encode this stable information.

One first guess would be to simply choose the morphisms between our objects to be the colimit of the suspensions of morphisms. If we also note that we now only care about arbitrarily high suspensions, it now makes sense to consider formal desuspensions of spaces as well. So one could consider a category whose objects are topological spaces with suspension data, and the morphisms to be the colimits under suspension. This leads to the definition of the following category:

**Definition 1.1.** The Spanier-Whitehead Category has objects pairs \((X, n)\) consisting of pointed [finite] CW complexes \(X\) and integers \(n\). We set \(\text{Hom}((X, n), (Y, m)) = \text{lim}_{n \to q} [\Sigma^{n+q}X, \Sigma^{m+q}Y]\).

This is actually a pretty poorly-behaved category. For example, it does not have all coproducts. Furthermore, we would like a good point-set model of maps, rather than working with homotopy classes of maps. Hence we’d like a category that is more well-behaved. This leads to the development of spectra, which captures information about stable homotopy theory.

**Definition 1.2** (Boardman). A sequential pre-spectrum is a sequence \((\mathbb{N} \text{ graded})\) of spaces \(X_n\), along with structure maps \(\Sigma X_n \to X_{n+1}\). A morphism of sequential pre-spectra \(f : X \to Y\) is a collection of morphisms \(f_n : X_n \to Y_n\) that are compatible with the structure maps.

\[
\begin{array}{c}
\Sigma X_n \xrightarrow{\Sigma f_n} \Sigma Y_n \\
\sigma_n \downarrow \quad \downarrow \sigma_n'
\end{array}
\]

\(X_{n+1} \xrightarrow{\Sigma f_{n+1}} Y_{n+1}\)

This category is Boardman’s category of spectra, and was the first category of spectra to have good properties (it is triangulated, cocomplete, and has Brown representability). We can ask for even more properties (a good symmetric monoidal smash product, cofibrant sphere spectrum), and we get other flavors of spectra (symmetric, orthogonal, etc.). However, we can’t have all the nice things that we might want (thanks, Gaunce Lewis).

Among pre-spectra, we have a distinguished collection objects that are homotopically well-behaved, called \(\Omega\)-spectra. (the \(\Omega\)-spectra are the fibrant model of maps). To define these objects, first note that by the suspension-loop adjunction, the structure maps \(\Sigma X_n \to X_{n+1}\) can be transformed into maps \(X_n \to \Omega(X_{n+1})\).

**Definition 1.3.** A sequential pre-spectrum is called an \(\Omega\)-spectrum if furthermore the structure maps \(X_n \to \Omega(X_{n+1})\) are weak homotopy equivalences.

Given a pre-spectrum, we can replace it with an \(\Omega\)-spectrum.

Furthermore, we can consider homotopy classes of maps of pre-spectra, where a homotopy of maps of prespectra is given by \(X \wedge [0, 1], \to Y\) satisfying the following diagram:

\[
\begin{array}{c}
\Sigma X_n \xrightarrow{\Sigma f_n} \Sigma Y_n \\
\sigma_n \downarrow \quad \downarrow \sigma_n'
\end{array}
\]

\(X_{n+1} \xrightarrow{\Sigma f_{n+1}} Y_{n+1}\)

We can then consider the associated homotopy category, which has \(\Omega\)-spectra as its objects. However, it has as morphisms the homotopy classes of maps (of pre-spectra). This category is called the Stable Homotopy Category \(\text{SHC}\).
**Definition 1.4.** The homotopy groups of a pre-spectrum $E$ (with associated $\Omega$-spectrum $F$) are defined as follows:

$$\pi_n(E) = \lim_{k \to \infty} \pi_{n+k}(F_k)$$

Note in particular, that a spectrum $E$ may have negative homotopy groups. This means that now the loop space functor, denoted $\Omega$, is the inverse to suspension in the stable homotopy category.

How does the Spanier Whitehead category fit into this? It is the full subcategory of compact objects in the Stable Homotopy Category. (We restricted to finite to make the map $\lim_{q \to q}[\Sigma^{n+q}X, \Sigma^{m+q}Y] \to \text{Hom}(\Sigma^n X, \Sigma^n Y)$ a bijection)

**Definition 1.5.** An object $X$ in a category $C$ is compact if the functor $\text{Hom}(X, -) : C \to \text{Set}$ preserves filtered colimits.

Think of this as a kind of finiteness condition.

### 1.1. Axiomatic Framework

#### 1.1.1. Model Categories

How can we capture the structure of the stable homotopy category axiomatically? We begin by considering a categorical structure that captures the homotopy-theoretic information.

**Definition 1.6.** A model category is a complete and cocomplete category along with a choice of 3 classes of morphisms: fibrations, cofibrations, and weak equivalences such that

1. The class of weak equivalences contain all isomorphisms, and is closed under the 2 out of 3 property
2. If $f$ is a retract of $g$ and $g$ is a fibration, cofibration, or weak equivalence, then so is $f$. DIAGRAM.
3. Cofibrations have the LLP w/rt acyclic fibrations and acyclic cofibrations have the LLP w/rt fibrations.
4. Any morphism $f$ can be factored as a cofibration followed by an acyclic fibration, or as an acyclic cofibration followed by a fibration.

This parametrizes the homotopy theory of our category. The weak equivalences are what we want to be homotopy equivalences. The cofibrations/fibrations help make the homotopy category tractable. Think of these as your injections/surjections. The terms come from algebraic topology (homotopy extension property, homotopy lifting property). Let us assume our category is pointed.

**Definition 1.7.** An object $X$ is cofibrant if $\ast \to X$ is a cofibration. Dually, an object $X$ is fibrant if $X \to \ast$ is a fibration.

In a model category, we can define a cylinder object for $X$, written $Cyl(x)$, as an object given by factorizing the codiagonal map $X \sqcup X \to X$. (I.E. $X \times [0,1]$). This allows us to define a notion of (left) homotopy as we are used to: a morphism $Cyl(X) \to Y$ that makes the appropriate diagram commute. There are some subtleties here, but we can define the homotopy category of a model category $C$.

**Definition 1.8.** The homotopy category of our model category $C$ is denoted $\text{Ho}(C)$ and has as objects the cofibrant-fibrant objects of $C$, and has morphisms homotopy classes of morphisms.

Recall that in homotopy theory, Whitehead’s theorem says that a weak homotopy equivalence between CW complexes is a homotopy equivalence. We have the following abstract theorem:

**Theorem 1.9** (Whitehead’s Theorem). *Given a model category on $C$, a weak equivalence between cofibrant-fibrant objects is a homotopy equivalence.*

We remember that we can define suspension and loop functors of a topological space $X$ as the homotopy pushout and homotopy pullback of the diagrams $\ast \leftarrow X \to \ast$, $\ast \to X \leftarrow \ast$, respectively. Unfortunately, a homotopy pushout/pullback is not a pushout/pullback in the homotopy category. Instead, one applies co/fibrant replacement to the diagrams $\ast \leftarrow X \to \ast$, $\ast \to X \leftarrow \ast$, and then takes the pushout/pullback of the resulting diagram.

**Definition 1.10.** A stable model category is a pointed model category such that the suspension and loop space functors are inverse equivalences on the homotopy category.
1.1.2. \(\infty\)-Categories

Another (perhaps more modern) way to parameterize homotopy theory is through the use of infinity categories.

**Definition 1.11.** A quasicategory (which is our model for an \(\infty\)-category) is a simplicial set such that all inner horns have fillers. Furthermore, composition is defined up to a contractible space of choices aka coherence.

**Definition 1.12.** The homotopy category of a quasicategory \(\mathcal{C}\) has objects the vertices, and has morphisms homotopy classes of edges of \(\mathcal{C}\). (Where homotopy is given by a 2-cell).

Thus one should think of an \(\infty\)-category as a category that keeps track of maps, homotopy classes of maps, homotopies between homotopies, etc. The inner horn restriction guarantees composition of morphisms, and the simplicial set is a way of encoding all higher coherence data.

**Definition 1.13.** A stable \(\infty\)-category is an \(\infty\)-category \(\mathcal{C}\) that satisfies

1. \(\mathcal{C}\) has a 0 object
2. Every morphism in \(\mathcal{C}\) admits a kernel and cokernel.
3. A triangle is a pushout iff it is a pullback.

A stable model category is a 1-category structure used to present a stable \((\infty,1)\)-category.

1.2. Examples

**Example 1.14 \(K(R)\)**

Let \(R\) be a commutative ring. We can consider the category of chain complexes of modules over \(R\), denoted \(\text{Ch}(R)\). This has a stable model category structure with weak equivalences being chain homotopies, and the obvious suspension functor (shifting the indexes by 1). The associated homotopy category is denoted \(K(R)\). The compact objects are the bounded chain complexes.

However, it turns out that this is not quite the “right” stable model category to consider. In particular, we usually only care about the homology of our chain complexes. Therefore, we should localize at the quasi-isomorphisms (maps of complexes that induce isomorphisms on all homology groups):

**Example 1.15 \(D(R)\)**

\(\text{Ch}(R)\) has a stable model category structure with weak equivalences being quasi-isomorphisms, and the obvious suspension functor (shifting the indexes by 1).

The associated homotopy category is denoted \(D(R)\). The subcategory of compact objects is denoted \(D^{perf}(R)\) and has objects the chain complexes that are quasi-isomorphic to a bounded chain complex of finite projective modules.

Here is an example with a slightly different flavor:

**Example 1.16 \(\text{StMod}(kG)\)**

Let \(k\) be a field of positive characteristic, and \(G\) be a finite group such that \(k \nmid |G|\). We would like to look at the representations \(\hat{\rho} : G \to \text{End}_k(V)\). We can then extend this to a group algebra representation \(\rho : kG \to \text{End}_k(V)\), and this is the same as looking at modules over the group algebra \(kG\).

Mashke’s theorem does not apply in this scenario, since we cannot divide by \(G\). This means that not every module is projective. This lets us define the following category:

**Definition 1.17.** The stable module category \(\text{StMod}(kG)\) has objects \(kG\)-modules, and has a vector space of morphisms \(\text{Hom}_{kG}(M,N) = \text{Hom}_{kG}(M,N)/\text{PHom}_{kG}(M,N)\), where \(\text{PHom}_{kG}(M,N)\) is the linear subspace of projective maps (those that factor through a projective module).

This category comes from a stable model category structure on \(kG\) modules, where the weak equivalences are the stable equivalences, the fibrations are surjections, and the cofibrations are injections.

**Definition 1.18.** We say a map \(f : M \to N\) is a stable equivalence if there exists a map \(g : N \to M\) such that \(fg - Id, gf - Id\) factor through a projective module.
The suspension is denoted \( \Omega^{-1} \), and is defined to be the cokernel of an injective map into an injective module. This is well defined up to a direct sum of a projective module. The subcategory of compact objects is denoted \( \text{stmod}(kG) \) and has objects the finitely generated modules.

**Proposition 1.19.** As stable \( \infty \)-categories, there is an equivalence \( \text{StMod}(kG) \cong \text{Mod}((Hk)^{tG}) \), where \( G \) acts trivially on \( Hk \), the Eilenberg-MacLane spectrum of the field \( k \).

For those that are more topologically minded:

**Example 1.20** \( \text{Mod}(R) \) for \( R \) a commutative ring spectrum.

**Definition 1.21.** \( R \) is a (commutative, \( E_\infty \)) ring spectrum if it is a (commutative) ring object in (symmetric, orthogonal) spectra.

This is a point-set level construction that corresponds to a (commutative) monoid in the homotopy category, and can be defined in various ways (operadically, etc). Examples include \( S, KU, HR \). As such, it encapsulates the previous example of the derived category of a ring \( R \).

We can define modules over a commutative ring spectrum, and we can consider the category \( \text{Mod}(R) \). This is also a stable model category with weak equivalences being the weak equivalences of underlying (orthogonal) ring spectra (induces iso on all homotopy groups). Furthermore, this category has a tensor product: Given \( R \)-modules \( M, N \), the tensor product is denoted \( M \otimes_R N \) and is the coequalizer of \( M \otimes R \otimes N \to M \otimes N \).

The suspension is smashing with the ring spectrum \( R \), denoted \( \wedge^R \).

The subcategory of compact objects is denoted \( \text{Mod}^{perf}(R) \). It is the smallest full subcategory of \( \text{Mod}(R) \) which contains \( R \) and is closed under finite colimits, desuspensions, and passage to direct summands. If we consider modules over \( HR \), we recover the perfect complexes.

2. Thick and Localizing Subcategories

We would like to understand the structure of the homotopy category of a stable homotopy theory. We note that the homotopy category of a stable model category/stable infinity category is triangulated. This structure is interesting in its own right.

**Definition 2.1.** A category \( \mathcal{T} \) is triangulated if

1. It is an additive category (that is, enriched over abelian groups, admits all finitary prod/coprod).
2. It has a suspension functor \( \Sigma \) that is an automorphism of \( \mathcal{T} \).
3. There is a collection of diagrams (distinguished triangles) of the form \( X \to Y \to Z \to \Sigma X \) satisfying certain axioms.
   a. \( X \xrightarrow{id} X \to 0 \to \Sigma X \) is distinguished.
   b. Any map \( f : X \to Y \) can be completed to a distinguished triangle. The object \( Z \) is called the mapping cone of \( f \).
   c. Any triangle isomorphic to a distinguished triangle is distinguished. That is, given a distinguished triangle \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \), and isomorphisms \( X \xrightarrow{u} X', Y \xrightarrow{v} Y', Z \xrightarrow{w} Z' \), the bottom row of this commutative diagram is also distinguished.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{u} & & \downarrow{v} \\
X' & \xrightarrow{f} & Y'
\end{array}
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{w} & & \downarrow{w} \\
Y' & \xrightarrow{g} & Z'
\end{array}
\begin{array}{ccc}
Z & \xrightarrow{h} & \Sigma X \\
\downarrow{w} & & \downarrow{w} \\
Z' & \xrightarrow{h} & \Sigma X'
\end{array}
\]

(d) (Tilting Axiom) Given a distinguished triangle \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \), the following two distinguished triangles are also distinguished:

\[
Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \quad \Sigma^{-1} Z \xrightarrow{\Sigma^{-1} h} X \xrightarrow{f} Y \xrightarrow{g} Z
\]

(e) Given two distinguished triangles with maps \( X \xrightarrow{u} X', Y \xrightarrow{v} Y' \), then there exists a (not necessarily unique) map \( Z \xrightarrow{w} Z' \) making the diagram commute.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{u} & & \downarrow{v} \\
X' & \xrightarrow{f} & Y'
\end{array}
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{w} & & \downarrow{w} \\
Y' & \xrightarrow{g} & Z'
\end{array}
\begin{array}{ccc}
Z & \xrightarrow{h} & \Sigma X \\
\downarrow{w} & & \downarrow{w} \\
Z' & \xrightarrow{h} & \Sigma X'
\end{array}
\]
(f) (Octahedral Axiom) Given three distinguished triangles $X \to Y \to Z', X \to Z \to Y'$, and $Y \to Z \to W$ fitting into the below diagram (we omit the suspension maps for a cleaner diagram), then there exists maps $f : Z' \to Y'$ and $g : Y' \to W$ such that $Z' \xrightarrow{f} Y' \xrightarrow{g} W$ is a distinguished triangle and the below diagram commutes.

This axiom is like the third isomorphism theorem, if we interpret distinguished triangles as cofiber sequences. That is, setting $Z' = Y/X$, $Y' = Z/X$ and $W = Z/Y$. Then we have that $(Z/X)/(Y/X) = Z/Y$.

Furthermore, we consider nice triangulated categories (essentially small, rigid, idempotent complete).

**Definition 2.2.** A triangulated category $\mathcal{T}$ is compactly generated if there is a set of compact objects $\mathcal{G}$ that generate $\mathcal{C}$. That is, there is no proper triangulated subcategory of $\mathcal{C}$ that is closed under coproducts containing $\mathcal{G}$.

This implies the following: If $\text{Hom}(G, X) = 0$ for all $G \in \mathcal{G}$, then $X = 0$. It is a theorem of Neeman that this condition on a triangulated category implies Brown representability.

**Example 2.3** The stable homotopy category $\text{SHC}$ has suspension $\Sigma$, and it has distinguished triangles the homotopy fiber/cofiber sequences $\ker f \to A \xrightarrow{f} B \to \Sigma \ker f$. These fit into a homotopy pullback/pushout diagram:

$$
\begin{array}{c}
\ker f \\
\downarrow \\
A \\
\downarrow f \\
B
\end{array}
\quad
\begin{array}{c}
* \\
\downarrow r \\
\Sigma \ker f
\end{array}
$$

**Example 2.4** $\text{D}(R)$ has the obvious suspensions and it has distinguished triangles the ones that are isomorphic to $X \xrightarrow{f} Y \to C(f) \to X[1]$.

**Example 2.5** $\text{StMod}(kG)$ is a triangulated category, with suspension given by $\Omega^{-1}$, which is defined to be the cokernel of an injective map into an injective module. The sequence $A \to B \to C \to \Omega^{-1}A$ is triangulated iff we have short exact sequences $0 \to A \to B \oplus \text{(proj)} \to C \to 0$ and $0 \to B \to C \oplus \text{(proj)} \to \Omega^{-1}A \to 0$ with the appropriate maps equal in the stable category.

One thing that we can do is classify certain subcategories of the (triangulated) homotopy category:

**Definition 2.6.** A triangulated subcategory $\mathcal{X}$ of a triangulated category $\mathcal{T}$ is a full subcategory containing $0$ that is closed under triangles. That is, for any distinguished triangle in $\mathcal{T}$, $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, if two of $X, Y, Z$ belong to $\mathcal{X}$, then so does the third.

**Definition 2.7.** A triangulated subcategory $\mathcal{X}$ of $\mathcal{T}$ is called thick if it is closed under finite direct summands.

In terms of $\infty$-categories: A full subcategory $\mathcal{X}$ of a stable $\infty$-category $\mathcal{C}$ is called thick if $\mathcal{X}$ is a stable subcategory (closed under finite limits and colimits), and $\mathcal{X}$ is closed under retracts.

One reason that we care about these subcategories is because they are the kernels of exact functors between triangulated categories.
Example 2.8 Suppose $F : T \to A$ is a cohomological functor into an abelian category $A$. That is, it is contravariant and takes distinguished triangles in $T$ to LES in $A$. Then $\cap_{n \in \mathbb{Z}} \text{Ker} \left( F \circ \Sigma^n \right)$ is a thick subcategory of $T$.

Remark 2.9. Since thick subcategories are only closed under finite limits and colimits, we restrict to talking about the subcategory of compact objects. This is because the subcategory of compact objects is the biggest proper thick subcategory.

Definition 2.10. A localizing subcategory is a thick subcategory that is closed under arbitrary direct sums.

In terms of $\infty$-categories, a localizing subcategory is a thick subcategory that is closed under arbitrary colimits.

Generally, these subcategories are harder to classify (since we are now looking at unbounded chain complexes, infinite dimensional modules, non-finite spectra). For example, in $\text{StMod}(kG)$, the classification of localizing categories even for something simple like $\mathbb{Z}/2 \times A_4$ is not known. There’s some kind of obstruction to this (the nucleus).

However, in the cases that we care about, we also have a symmetric monoidal product, such as a smash product or a tensor product, that plays nicely with the homotopy theory in the sense of Quillen’s SM07. In terms of axiomatic homotopy theory, this is the structure of a monoidal model category. This, among other things, allows us to define (commutative) ring objects in spectra that correspond to (commutative) ring objects in the stable homotopy category. We will leverage this structure, as it is actually kind of difficult to classify just the thick/localizing subcategories.

Let us now assume that $T$ is a triangulated category with a symmetric monoidal product denoted $\otimes$.

Definition 2.11. A triangulated subcategory $S$ is tensor-ideal if for all $X \in T$ and $Y \in S$, $X \otimes Y \in S$.

So our goal is to try to classify the thick tensor-ideal subcategories, or maybe the localizing ones. This is the goal of Paul Balmer’s program in tensor-triangulated $(\otimes, \Delta)$ geometry. Balmer introduced a general framework for classifying the thick subcategories of tensor-triangulated categories.

Definition 2.12. Let $P$ be a thick tensor-ideal subcategory of a triangulated $T$. We say that $P$ is prime if $a \otimes b \in P$, then either $a \in P$ or $b \in P$.

Definition 2.13. Let $T$ be a tensor triangulated category. Then we define the Balmer spectrum $\text{Spec}(T)$ to be the set of all primes of $T$. We can give this set a Zariski topology in the following way. Given a collection of objects $S \in T$, we let $Z(S) = \{ P \in \text{Spec}(T) \mid P \cap S = \emptyset \}$ be our closed sets.

Definition 2.14. A subset $U$ of $\text{Spec}(T)$ is specialization closed if it is a union of closed subsets.

Theorem 2.15. Under reasonable conditions on $T$, the thick tensor ideal subcategories in $T$ are in correspondence with specialization closed subsets of $\text{Spec}(T)$. See section 3 for the details.

Therefore, we can even restrict to classifying the prime thick tensor-ideal subcategories!

Later, Benson-Iyengar-Krause developed machinery of local cohomology and supports to classify the localizing tensor-ideal subcategories of $\text{StMod}(kG)$. Dell’Ambrogio-Stanley have used this machinery to classify the localizing categories of $\text{Mod}(R)$ for $R$ an evenly concentrated, regular, $E_\infty$ ring. But more about this later.

2.1. Why is this interesting?

This was all motivated by Devinatz-Hopkins-Smith’s work on Nilpotence & Periodicity in the stable homotopy category. They built spectra/cohomology theories that detect periodicity in order to understand families of elements in the stable homotopy groups of spheres. Their proof led them to classify the thick subcategories of the finite $p$-local stable homotopy category.

Their work also led to the classification of Bousfield classes of finite spectra. That is, they classified equivalence classes of spectra that induce the same cohomology theory. In general, Iyengar-Krause talks about the Bousfield lattice of a (well-generated) tensor triangulated category and show that it’s a set.

- If one wishes to prove a property of all finite ($p$-local) spectra (i.e. the sphere), then it suffices to show that the property is thick and that a single finite spectrum with nontrivial rational homology satisfies it. In general, if an object $X$ satisfies a thick property, then the thick subcategory generated by $X$ satisfies that property.
Definition 2.16. A property is thick if it is closed under triangles and retracts.

Example 2.17 A spectrum $X$ is said to be of finite type if $\pi_k(X)$ is finitely generated for each $k$. That it is thick follows by examining the LES of homotopy groups induced by cofibers of spectra.

- In modular representation theory, a classification of thick subcategories allows us to know when we can build $M$ from $N$. The development of machinery of local cohomology and supports allows us to classify the localizing subcategories, but also allows one to determine when $\text{Hom}^*_k(M,N) = 0$ and $M \otimes N$ are projective.
- Given a commutative ring $R$, one can recover $\text{Spec}(R)$ from $D(R)$ - this provides a link between homotopy theory and algebraic geometry which is neat!
- A similar flavor is the following theorem of Balmer: for $X$ a topologically noetherian scheme, the derived category of perfect complexes $D^{perf}(X)$ fully characterizes $X$. Furthermore, Balmer’s proof is constructive!

3. Thick Subcategory Theorems

3.1. $p$-local Stable Homotopy Category

As a consequence of the Nilpotence theorem, Devinatz-Hopkins-Smith classified the thick subcategories of the finite $p$-local stable homotopy category, $\text{SHC}_{fin}$. This consists of the finite spectra that have $\pi_*(X) \otimes \mathbb{Z}(p) \cong \pi_*(X)$. This is Bousfield localization at the Moore spectrum of $\mathbb{Z}(p)$ (the acyclics). Recall that we look at finite spectra because we only care about the compact objects when classifying thick subcategories.

To say what the thick subcategories are, we must first discuss the Morava K-theories $K(n,p)$. It is standard to omit the prime $p$.

Definition 3.1. For a fixed prime $p$ and for each $n \geq 1$, there is a spectrum $K(n,p)$ whose coefficient ring $K(n,p)_*$ satisfies

$$\pi_*(K(n,p)) \cong \mathbb{F}_p[v_n^\pm] \quad \text{with} \quad |v_n| = 2(p^n - 1)$$

Remark 3.2. $K(n,p)_*$ is a graded field, that is, every graded module over it is free. This leads to some of the following consequences.

Proposition 3.3. The spectra $K(n,p)$ satisfy these nice properties:

1. For every spectrum $X$, $K(n,p) \wedge X$ has the homotopy type of a wedge of suspensions of $K(n,p)$.
2. (Künneth isomorphism) For spectra $X,Y$, we have that

$$K(n,p)_*(X \wedge Y) \cong K(n,p)_*(X) \otimes_{K(n,p)_*} K(n,p)_*(Y)$$

3. Let $X$ be a $p$-local finite spectrum. If (reduced) $K(n+1)_*(X) = 0$, then (reduced) $K(n,p)_*(X) = 0$.
4. Let $X$ be a non-contractible $p$-local finite spectrum. Then

$$\text{(reduced)} \ K(n,p)_*(X) \cong \text{(reduced)} \ H_*(X;\mathbb{F}_p)$$

for $n$ sufficiently large. In particular, this means $K(n,p)_*(X) \neq 0$ for $n$ sufficiently large.
5. (Nilpotence theorem) If $R$ is a non-contractible ring spectrum, then there exists an $n$ ($0 \leq n \leq \infty$) such that $K(n,p)_*R \neq 0$.

Example 3.4 $K(1,p) = KU/p$, that is, mod $p$ complex $K$-theory.

Example 3.5 $K(\infty,p) = HF_p$, that is, the Eilenberg Maclane spectrum associated to the field $\mathbb{F}_p$.

Example 3.6 It is reasonable to define $K(0,p) := H\mathbb{Q}$, that is, the Eilenberg Maclane spectrum associated to the field $\mathbb{Q}$. Note that this is independent of the prime $p$.

To define these spectra rigorously, we need to fix a prime $p$ and invoke number theory to define the height $n$ formal group laws. In particular we need to describe the $n$th Honda formal group law, which has $p$-series $xp^n$.

Once we do so, there is a way to go from spectra to formal group laws (via complex orientations and Chern characters). Applying this functor to $K(n,p)$ gives us the $n$th Honda formal group law.

Under certain hypotheses, there is a theorem (the Landweber Exact Functor Theorem) that allows us to go from formal group laws to spectra. Unfortunately, the $n$th Honda formal group law does not satisfy these properties, so the Morava K-theories $K(n,p)$ must be constructed explicitly.
But this is the story of chromatic homotopy theory, which takes us too far afield.

**Definition 3.7.** We define the following subcategories of $\mathbf{SHC}_p^{\text{fin}}$:

$$C_{n,p} = \{ X \in \mathbf{SHC}_p^{\text{fin}} \mid K(n, p)_* X = 0 \}$$

**Theorem 3.8** (Hopkins-Smith). The proper thick tensor-ideal subcategories of $\mathbf{SHC}_p^{\text{fin}}$ are precisely the subcategories $C_{n,p}$. Furthermore, these subcategories form a nested decreasing filtration of $\mathbf{SHC}_p^{\text{fin}}$:

$$C_{\infty,p} \subset \ldots \subset C_{n+1,p} \subset C_{n,p} \subset \ldots \subset C_0 \subset \mathbf{SHC}_p^{\text{fin}}$$

**Corollary 3.9.** Every proper thick tensor-ideal subcategory of $\mathbf{SHC}_p^{\text{fin}}$ is prime.

**Proof.** This follows from the Künneth isomorphism. □

Thus the (prime) thick tensor-ideal subcategories of $p$-local stable homotopy theory are the $K(n, p)$-acyclic spectra.

We can then assemble this data to understand the prime thick tensor-ideal subcategories of the finite stable homotopy category $\mathbf{SHC}_p^{\text{fin}}$.

**Theorem 3.10** (Balmer). The Balmer spectrum of $\mathbf{SHC}_p^{\text{fin}}$ is the following topological space:

Furthermore, we have a surjective continuous map

$$\rho : \text{Spec}(\mathbf{SHC}_p^{\text{fin}}) \to \text{Spec}(\mathbb{Z})$$

(since we have that $\text{End}_{\mathbf{SHC}_p^{\text{fin}}}(1) \cong \mathbb{Z}$).

This looks a lot like the picture of the moduli space of formal groups over $\text{Spec}(\mathbb{Z})$!

### 3.2. $D(R)$ for various rings $R$

**Example 3.11** Thick and Localizing subcategories of $D(R)$ for $R$ a commutative Noetherian ring.

We first classify the thick subcategories of $D^\text{perf}(R)$.

**Definition 3.12.** Given $M \in D(R)$, we write $H^*(M)$ for the total cohomology of the complex $M$. That is,

$$H^*(M) = \oplus_{n \in \mathbb{Z}} H^n(M)$$

Note that $H^*(M)$ is a graded $R$-module.

**Definition 3.13.** Given a graded $R$-module $M$, and $p \in \text{Spec}(R)$, we write $M_p$ for the homogeneous localization of $M$ at $p$. That is,

$$M_p = (R/p)^{-1} M$$

**Definition 3.14.** Let $M \in D(R)$. We define the (big) support of $M$, $\text{Supp}_R(M)$:

$$\text{Supp}_R(M) = \{ p \in \text{Spec}(R) \mid H^*(M_p) \neq 0 \}$$

Note that passing to cohomology commutes with localization.
Lemma 3.15. For any subset $U$ of $\text{Spec}(R)$, the full subcategory 

$$C_U = \{ M \in D^{\text{perf}}(R) \mid \text{Supp}_R(M) \subseteq U \}$$

is a thick subcategory of $D^{\text{perf}}(R)$.

Theorem 3.16 (Hopkins). Let $R$ be a commutative Noetherian ring. Thick subcategories of $D^{\text{perf}}(R)$ are in correspondence with specialization closed subsets of $\text{Spec} R$. The right direction comes from $\text{Supp}_R(\_)$.

Now we discuss localizing subcategories of $D(R)$. In the 60s, Gabriel proved a bijection between localizing subcategories of $\text{Mod}(R)$ and specialization-closed subsets of $\text{Spec}(R)$.

Definition 3.17. Let $M \in D(R)$. We define the (small) support of $M$, $\text{supp}_R(M)$:

$$\text{supp}_R(M) = \{ p \in \text{Spec}(R) \mid \text{k}(p) \otimes_R M \neq 0 \text{ in } D(R) \}$$

Where $\text{k}(p)$ is the residue field $R_p/(pR_p)$. Equivalently,

$$\text{supp}_R(M) = \{ p \in \text{Spec}(R) \mid H^*_p R_p(M_p) \neq 0 \}$$

Theorem 3.18 (Hopkins-Neeman). Let $R$ be a commutative Noetherian ring. Localizing subcategories of $D(R)$ are in correspondence with arbitrary subsets of $\text{Spec}(R)$. The right direction comes from $\text{supp}_R(\_)$.

Example 3.19 Thick subcategories for $R$ a commutative ring.

Definition 3.20. A Thomason subset $Y$ of $X$ is a subset that can be expressed as $Y = \cup_\alpha Y_\alpha$, where $X \setminus Y_\alpha$ is quasi-compact and open.

Theorem 3.21 (Thomason). Let $R$ be a commutative ring. Then the thick subcategories of $D^{\text{perf}}(R)$ are in correspondence with Thomason subsets of $\text{Spec}(R)$.

This can be reformulated to be about the thick subcategories of $\text{Perf}(X)$ for $X$ a quasi-compact, quasi-separated scheme.

A classification of localizing subcategories of $D(R)$ for $R$ commutative non-noetherian ring is wide open. However, we know that a similar type of theorem using support varieties (so a correspondence with subsets of $\text{Spec}(R)$) cannot be true, as there are rings with many localizing subcategories but $\text{Spec}(R)$ is a point. For example, consider the ring

$$R = k[x_2, x_3, x_4 \ldots]/(x_2^2, x_3^3, x_4^4, \ldots)$$

This is in Neeman’s Oddball Bousfield Classes.

4. Tensor-Triangulated Geometry

We would now like to prove theorems like this in greater generality. This leads to Balmer’s tensor-triangulated geometry, as well as Benson-Iyengar-Krause’s theory of local cohomology and supports for a triangulated category.

Definition 4.1. Let $\mathcal{T}$ be a tensor triangulated category. We define the Balmer spectrum $\text{Spc}(\mathcal{T})$ to be the set of all prime thick tensor ideals of $\mathcal{T}$.

We can give this set a Zariski topology by defining the closed subsets in the following way. Given a collection of objects $S \in \mathcal{T}$, we let $Z(S) = \{ P \in \text{Spc}(\mathcal{T}) \mid P \cap S = \emptyset \}$. We take these to be our closed sets.

Definition 4.2. In an arbitrary tensor triangulated category $\mathcal{T}$, the support of an object $X \in \mathcal{T}$ is the set

$$\text{supp}(X) = \{ P \in \text{Spc}(\mathcal{T}) \mid a \notin P \}$$

Proposition 4.3. $\text{Spc}(\mathcal{T})$ is Noetherian iff any closed subset is the support of some object in $\mathcal{T}$.

Definition 4.4. A triangulated category is rigid if the natural evaluation map

$$\text{Hom}(X, 1) \otimes Y \to \text{Hom}(X, Y)$$

is an isomorphism.

Definition 4.5. A category $\mathcal{C}$ is idempotent complete if any idempotent morphism $e : B \to B$ splits as $B \xrightarrow{r} A \xrightarrow{s} B$, where $r \circ s = \text{id}_A$. 

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4.1. Local Cohomology and Supports

We would like to mimic the classification of localizing subcategories of $D(R)$ for $R$ a graded commutative Noetherian ring. This leads to Benson-Iyengar-Krause’s theory of triangulated categories $\mathcal{T}$ with a central ring action by a graded commutative Noetherian ring $R$.

**Definition 4.8.** A central ring action of $R$ on $\mathcal{T}$ is given by maps $\Phi_X : R \to \text{End}_{\mathcal{T}}(X)$ such that for any $\alpha : \text{Hom}_{\mathcal{T}}(X,Y)$ and $r \in R$, then

$$\Phi_Y(r) \circ \alpha = (-1)^{|r||\alpha|} \alpha \circ \Phi_X(r)$$

This is the appropriate notion of an $R$-linear triangulated category. We always have a central ring action coming from $\text{End}(\mathcal{I})$, given by composition with tensoring. The condition of a central ring action comes from the universal property of the graded center of a category, which is a $k$-algebra.

**Definition 4.9.** If $R$ is a graded commutative Noetherian ring, then a map $R \to \text{End}_{\mathcal{T}}(\mathcal{I})$ induces a central ring action. Such an action is said to be canonical.

We want to classify localizing subcategories of $\mathcal{T}$ via the action of $R$ (via $\text{Spec}(R)$).

**Definition 4.10.** Given a subset $\nu \in \text{Spec}(R)$, we can define a subcategory

$$\mathcal{T}_\nu = \{ x \in \mathcal{T} \mid \text{Hom}^*(c,x) \text{ is } \nu\text{-torsion for all compact } c \}$$

**Definition 4.11.** Given a prime ideal $p \in \text{Spec}(R)$, we can define a subcategory

$$\mathcal{T}_p = \{ x \in \mathcal{T} \mid \text{Hom}^*(c,x) \text{ is } p\text{-local for all compact } c \}$$

We can drop the compact hypotheses when we are considering a stable homotopy category.

**Proposition 4.12.** These two subcategories are localizing, and correspond to localization functors denoted $\Gamma_{\nu}(-)$ and $(-)_p$.

**Definition 4.13.** The local cohomology functor $\Gamma_{\nu}$ is given by

$$\Gamma_{\nu}(X) := (\Gamma_{\nu}(p)X_p) \cong \Gamma_{\nu}(p)(X_p)$$

**Remark 4.14.** If we have a canonical ring action then we can compute this on the unit object $\mathcal{I}$ and tensor up. This is the case in $\text{StMod}(kG)$.

**Definition 4.15.** Given a canonical central ring action $R$ on $\mathcal{T}$, we can define the support of an object $X \in \mathcal{T}$:

$$\text{supp}_R(X) := \{ p \in \text{Spec}(R) \mid \Gamma_{\nu}(X) \neq 0 \}$$

**Definition 4.16.** Given a canonical central ring action $R$ on $\mathcal{T}$, we define the support of $\mathcal{T}$ to be

$$\text{supp}_R(\mathcal{T}) := \{ p \in \text{Spec}(R) \mid \Gamma_{\nu}(\mathcal{T}) \neq 0 \}$$

This allows us to say what tensor ideal localizing subcategories are when we have a nice canonical central ring action (which we call stratification):

**Theorem 4.17** (local-to-global). *Given a canonical central ring action $R$ on $\mathcal{T}$, the tensor ideal localizing subcategory generated by an object $X$, denoted $\text{Loc}^\otimes(X)$, is the smallest localizing subcategory containing $\{ \Gamma_{\nu}(X) \mid p \in \text{Spec}(R) \}$.***

**Definition 4.18.** $\mathcal{T}$ is stratified by $R$ if for each $p \in \text{Spec}(R)$ either $\Gamma_{\nu}(p) = \{0\}$ or $\Gamma_{\nu}(p)$ has no proper tensor ideal localizing subcategory.

**Remark 4.19.** The condition above is really called minimality. It turns out that $\mathcal{T}$ is stratified by $R$ if and only if we have minimality and a local-to-global theorem. Luckily, the latter is guaranteed by the canonical central ring action.

**Theorem 4.20.** *Suppose $\mathcal{T}$ is stratified by $R$. Then tensor-ideal localizing subcategories of $\mathcal{T}$ are in correspondence with subsets of $\text{supp}_R(\mathcal{T})$.***
5. More Examples of Thick/Localizing Subcategory Theorems

Example 5.1 StMod(kG) and Modular Representation Theory (for certain groups G)

This is an example of a triangulated category StMod(kG) being stratified by the ring $H^*(G,k)$. We restrict to G a p-group (or a slightly weaker condition) so that StMod(kG) has a single compact generator.

In the case of StMod(kG), the ring End(1) can be identified with the Tate cohomology ring $H^*(G,k)$. Unfortunately, this is not Noetherian. However, it is a theorem of Evens-Venkov that the group cohomology ring $H^*(G,k)$ is Noetherian! So we can apply the theory of local cohomology and supports.

It turns out that the only prime ideal of $H^*(G,k)$ that is not in $\text{supp}_R(T)$ is the maximal ideal. This gives us the following theorem:

**Theorem 5.2.** Tensor ideal localizing subcategories of StMod(kG) are in correspondence with subsets of Proj($H^*(G,k)$).

**Corollary 5.3.** Thick tensor ideal subcategories of Stmod(kG) are in correspondence with specialization closed subsets of Proj($H^*(G,k)$) The right directions comes via support.

Example 5.4 Mod$^{perf}_R$(R)

Let R be an even periodic $E_\infty$ ring spectrum with $\pi_0(R)$ regular Noetherian. The idea of Mathew’s proof is the following: We reduce to considering support on $\pi_0(R)$. This gives us a collection of thick subcategories.

We then build a spectrum/cohomology theory that detects support and nilpotence. This allows us to show that these are all of the thick subcategories.

Because of the conditions on R and because we are considering perfect modules, we can reduce to considering $\pi_0(M) \oplus \pi_1(M)$ and Spec($\pi_0(R)$). Since a module M is perfect iff $\pi_0(M) \oplus \pi_1(M)$ is finitely generated over $\pi_0(R)$.

**Definition 5.5.** The support of an $R$-module $M$ is the support of $\pi_0(M) \oplus \pi_1(M)$, which defines a closed subset of Spec($\pi_0(R)$).

**Definition 5.6.** Let $Z \subseteq \text{Spec}(\pi_0 R)$ be a specialization closed subset. We define

$$\text{Mod}^{perf}_Z(R) = \{M | \text{Supp}(M) \subseteq Z\}

**Theorem 5.7** (Mathew). The thick subcategories of $\text{Mod}^{perf}_Z(R)$ are precisely the set $\{\text{Mod}^{perf}_Z(R)\}$ for $Z \subseteq \text{Spec}(\pi_0(R))$ closed under specialization, and these are all distinct.

Example 5.8 The prime ideals for SHC(G) for G a finite group.

Let G be a finite group and let SHC(G) denote the G-equivariant stable homotopy category with respect to a complete G-universe. Balmer-Sanders have computed the prime ideals of SHC(G), which allows them to draw conclusions about the classification of thick tensor ideals of SHC(G) for G a finite group.

**Definition 5.9.** For every subgroup $H \leq G$, there is a geometric fixed point functor $\Phi^H : \text{SHC}(G) \rightarrow \text{SHC}$. (it is induced by restriction the H-fixed points).

**Proposition 5.10.** $\Phi^H : \text{SHC}(G) \rightarrow \text{SHC}$ is a tensor-triangulated functor.

**Theorem 5.11** (Balmer-Sanders). All prime thick tensor ideals in SHC(G)$^c$ are obtained by pulling back prime thick tensor ideals in SHC$^c$ with respect to the subgroups $H \leq G$.

$$P(H, n, p) := (\Phi^H)^{-1}(C_{n,p})$$

Furthermore, two prime ideals in SHC(G)$^c$, $P(H, n, p)$, $P(H', n', p')$ coincide only if $H$ and $H'$ are conjugate in $G$, and the chromatic primes $C_{n,p}$ and $C_{n', p'}$ coincide in SHC$^c$.

Example 5.12 John Greenlees has proved a thick subcategory theorem for rational $G$-spectra for $G$ compact Lie. Something about cotoral inclusion.

**Example 5.13** Classification for localizing subcategories of $E_\infty$ rings concentrated in even degrees, with $\pi_*$ regular.

In fact, they prove a more general result:
Theorem 5.14 (Dell’Ambrogio-Stanley). Let $\mathcal{T}$ be an essentially small tensor triangulated category, and let $R = \text{End}_\mathcal{T}(1)$ be the graded endomorphism ring of the unit.

If $\mathcal{T}$ furthermore satisfies

1. $\mathcal{T} = \text{Thick}(1)$
2. $R$ is graded Noetherian concentrated in even degrees and weakly regular. That is, if for every homogeneous prime ideal $p$ of $R$, the maximal ideal of the local ring $R_p$ is generated by a finite regular sequence of homogeneous non-zero-divisors.

Then the localizing subcategories of $\mathcal{T}$ are in bijection with the specialization closed subsets of $\text{Spec}(R)$.

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