THE CALABI CONJECTURE

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Abstract. In this paper we aim to explore the Geometric aspects of the Calabi Conjecture and highlight the techniques of nonlinear Elliptic PDE theory used by S.T. Yau [SY] in obtaining a solution to the problem. Yau proves the existence of a Geometric structure using differential equations, giving importance to the idea that deep insights into geometry can be obtained by studying solutions of such equations. Yau’s proof of the existence of a specific class of metrics have found a natural interpretation in recent developments in Theoretical Physics most notably in the formulation of String Theory. We will also attempt to explore the importance of a special case of Yau’s solution known as Calabi-Yau Manifolds in the context of holonomy.

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1. Kähler Geometry

There are many mathematical reasons to study Kähler geometry. However, we wish to give a physical motivation to study Kähler manifolds. In string theory, the universe is conjectured to be 10 dimensional (9 space dimensions, 1 time dimension) with 6 of the space dimensions compactified into a Planck scale manifold, let it be denoted as $\mathcal{M}$. During the early days of string theory, it was believed that $\mathcal{M}$ was a flat hypertorus, i.e. $\mathcal{M} = S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1$. But the hypertorus based string theory could not incorporate chiral aspects of the Standard Model. Thus, in order for String theory to be a unified field theory, there must be a more sophisticated choice for $\mathcal{M}$. It turns out that the correct choice for $\mathcal{M}$ are called Calabi-Yau manifolds [POL] which are examples of Kähler manifolds. Kähler geometry also naturally occurs in the context of supersymmetry and $\sigma$-models. So in addition to its inherent mathematical interest, Kähler geometry is also of fundamental importance to high energy physics research. For the remainder of this section, we will give an introduction to some of the Kähler geometry concepts necessary for the Calabi Conjecture.
Definition 1. A complex manifold of complex dimension $n$ (hence real dimension $2n$) is a smooth manifold equipped with an atlas of charts in which all the transition functions are holomorphic functions.

Complex manifolds come equipped with a complex structure, denoted by $J$. In particular, we have the following definition:

Definition 2. An almost complex manifold $M$ is a (real) smooth manifold with a globally defined $(1, 1)$ tensor $J$ which is an endomorphism of the tangent bundle $TM$ such that:

\[(1) \quad J^2 = -I,\]

$J$ is called an almost complex structure.

Now let us consider the situation locally. For any given point $p \in M$, we have an endomorphism $J_p : T_p M \to T_p M$ which satisfies $J_p^2 = -I_p$ depending smoothly on $p$, where $I_p$ is the identity operator on the tangent space $T_p M$. Given a local coordinate system $x^\mu, \mu = 1, \ldots, \dim(M)$, then we locally have the representation:

\[(2) \quad J_p = J_\mu^\nu(p) \frac{\partial}{\partial x^\nu} \otimes dx^\mu.\]

It should be noted that $J_\mu^\nu(p)$ is a real function in the real basis. Hence, for any vector field $X = X^\mu \frac{\partial}{\partial x^\mu}$, our endomorphism $J$ acts on $X$ according to:

\[(3) \quad J(X) = (X^\mu J_\mu^\nu) \frac{\partial}{\partial x^\nu},\]

\[(4) \quad \Rightarrow J^2(X) = X^\sigma J_\sigma^\nu J_\nu^\mu \frac{\partial}{\partial x^\mu}.\]

Thus, in local coordinates the conditions for an almost complex structure is equivalent to the following matrix equation:

\[(5) \quad J_\sigma^\nu(p)J_\nu^\mu(p) = -\delta_\sigma^\mu.\]

Globally, the existence of an almost complex structure $J$ on a smooth manifold means that we can define $J_p$ in any patch and glue them together without any obstructions or singularities.

Observe that given a complex manifold $M$, we may view it as a complex manifold of complex dimension $n$ or we may view it as the underlying real manifold of dimension $2n$. If we have complex coordinates $z^j$, for $j = 1, \ldots, n$, we have the corresponding real coordinates $x^j, y^j$ for $j = 1, \ldots, n$, where we use the identification $z^j = x^j + iy^j$. Thus we get a set of $2n$ coordinates $z^j, z^j$, where $z^j = x^j - iy^j$ for $j = 1, \ldots, n$. We make the following definitions:

\[(6) \quad \frac{\partial}{\partial z^j} \equiv \frac{1}{2} \left( \frac{\partial}{\partial x^j} - i \frac{\partial}{\partial y^j} \right),\]

\[(7) \quad \frac{\partial}{\partial z^j} \equiv \frac{1}{2} \left( \frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j} \right),\]

\[(8) \quad dz_j \equiv dx_j + idy_j,\]

\[(9) \quad dz^j \equiv dx_j - idy_j.\]
We then have the following differential operators:

\[ \partial = \frac{1}{2} \left( \partial_{\partial x^j} - i \partial_{\partial y^j} \right) (dx^j + idy^j), \]

\[ \overline{\partial} = \frac{1}{2} \left( \partial_{\partial x^j} + i \partial_{\partial y^j} \right) (dx^j - idy^j). \]

With these definitions in hand, we have the following proposition on our \( \partial, \overline{\partial} \) differential operators:

**Proposition 1.** The exterior derivative \( d \) satisfies:

\[ d = \partial + \overline{\partial}. \]

Furthermore,

\[ \partial \partial = 0, \overline{\partial} \overline{\partial} = 0, \]

\[ \partial \overline{\partial} = -\overline{\partial} \partial. \]

**Proof.** We have the following:

\[ \partial + \overline{\partial} = \frac{1}{2} \left( \partial_{\partial x^j} - i \partial_{\partial y^j} \right) (dx^j + idy^j) + \frac{1}{2} \left( \partial_{\partial x^j} + i \partial_{\partial y^j} \right) (dx^j - idy^j) \]

\[ = \frac{1}{2} \partial_{\partial x^j} dx^j + \frac{1}{2} \partial_{\partial y^j} dy^j = d. \]

Therefore,

\[ 0 = d^2 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \overline{\partial} \overline{\partial} + \overline{\partial} \partial + \overline{\partial} \partial \]

Decomposing the above equation into types yields that each piece \( \partial^2, \overline{\partial}^2, \overline{\partial} \partial + \partial \overline{\partial} \) vanishes. This proves the theorem. \( \square \)

With these \( \partial, \overline{\partial} \) differential operators, we may decompose \( \Omega^k \) the space of \( k \)-forms into subspaces \( \mathcal{A}^{p,q} \) where \( p + q = k \). Namely, \( \mathcal{A}^{p,q} \) is locally spanned by forms of the type

\[ \omega(z) = \eta(z) dz^j_1 \wedge \cdots \wedge dz^j_p \wedge \overline{dz}^{j_1} \wedge \cdots \wedge \overline{dz}^{j_q}. \]

The almost complex structure \( J \) allows us to complexify our real \( 2n \) dimensional tangent space, let \( (T_p M)^C \) denote the complexified tangent space. We reinterpret \( J \) as a complex linear map still squaring to \(-I\) at each point \( p \in M \). Due to the fact that it squares to \(-I\), the eigenvalues of \( J_p \) can only be \( \pm i \) and this allows us to decompose our tangent space \( (T_p M)^C \):

\[ (T_p M)^C = T_p M^+ \oplus T_p M^-, \]

where \( T_p M^+ \) is the eigenspace of \( i \) and \( T_p M^- \) is the eigenspace of \(-i\). We note that the complex structure \( J \) is an endomorphism of the tangent space given in local coordinates by the following two conditions:

\[ J(\partial/\partial x^j) = \partial/\partial y^j \]

\[ J(\partial/\partial y^j) = -\partial/\partial x^j, \]

so clearly \( J^2 = -I \).

A natural question to now ask is what is the relationship between complex and almost complex manifold?
Theorem 1. Any complex manifold \( M \) is also an almost complex manifold.

Proof. Recall that a complex manifold admits a holomorphic atlas, giving us a complex coordinate system \( z^a \) in a neighborhood \( U \) of an arbitrary point \( p \in M \). Thus, we may define the tensor:

\[
J = i \frac{\partial}{\partial z^j} \otimes dz^j - i \frac{\partial}{\partial \bar{z}^j} \otimes d\bar{z}^j.
\]

It is important to note that the above equation is well defined in a patch \( U \) as opposed to only a point (which would be the case for an almost complex manifold) since we may define complex coordinates that vary holomorphically in the patch \( U \) for complex manifolds, but we cannot do this for almost complex manifolds. Thus, we have an almost complex structure defined in any given patch. Next, we need to check if it is well defined in the overlaps of patches \( (U, z) \cap (V, w) \).

As the coordinate transformation functions \( z^j(w) \) are holomorphic, it follows from basic transformations of vectors and one forms that we have the following:

\[
\frac{\partial}{\partial z^j} \otimes dz^j = \frac{\partial z^j}{\partial w^k} \frac{\partial}{\partial w^k} \otimes dw^j = \frac{\partial}{\partial w^j} \otimes dw^j.
\]

Thus we may conclude that in the overlap, \( J \) takes the form:

\[
J = i \frac{\partial}{\partial w^j} \otimes dw^j - i \frac{\partial}{\partial \bar{w}^j} \otimes d\bar{w}^j,
\]

which proves the theorem. \( \square \)

Definition 3. Let \( M \) be a complex manifold with Riemannian metric \( g \) and complex structure \( J \). If \( g \) satisfies:

\[
g(JX, JY) = g(X, Y)\]

i.e. a compatibility condition,

for any two sections \( X, Y \) of the tangent bundle, then \( g \) is said to be a Hermitian metric and the pair \( (M, g) \) is called a Hermitian manifold.

Proposition 2. A complex manifold \( (M, J) \) always admits a Hermitian metric.

Proof. From rudimentary Riemannian geometry, we know that any manifold admits a Riemannian metric \( g \) (we always locally have one, and we patch these together via a partition of unity). To obtain a Hermitian metric, simply define:

\[
h(X, Y) = \frac{1}{2}(g(X, Y) + g(JX, JY)).
\]

Observe that

\[
h(JX, JY) = \frac{1}{2}(g(JX, JY) + g(J^2X, J^2Y))
\]

\[
= \frac{1}{2}(g(JX, JY) + g(-X, -Y))
\]

\[
= \frac{1}{2}(g(JX, JY) + g(X, Y)) = h(X, Y).
\]

Thus \( h \) is indeed a Hermitian metric. \( \square \)

Definition 4. Given the data \( (M, g, J) \), where \( M \) is a complex manifold, \( g \) is a Riemannian metric and \( J \) is a compatible complex structure, we may associate a 2 form (in particular a real \((1,1)\) form) denoted \( \omega \), defined by:

\[
\omega(v, w) = g(Jv, w).
\]
We are well within our rights to wonder why is $\omega$ a 2 form? Observe that:

$$\omega(X, Y) = g(JX, Y) = g(J^2X, JY) = -g(X, JY) = -g(JY, X) = -\omega(Y, X).$$

So we see that $\omega$ is indeed a antisymmetric 2 tensor, i.e. it is a 2-form.

In local coordinates, we have the following:

$$g = \frac{1}{2} \sum_{i,j} h_{ij}(z, \bar{z})(dz_i \otimes d\bar{z}_j + d\bar{z}_j \otimes dz_i),$$

$$\omega = i \sum_{i,j} h_{ij}(z, \bar{z})dz_i \wedge d\bar{z}_j,$$

where $h_{ij}$ is positive definite and Hermitian matrix.

**Definition 5.** Let $(M, g, J)$ be a Hermitian manifold with associated 2 form $\omega$. If $\omega$ is closed, i.e. $d\omega = 0$, then $M$ is called a Kähler manifold, $g$ is the Kähler metric and $\omega$ the Kähler form.

It goes without saying that there are some marvelous properties of Kähler manifolds. We will now quote one of the nicer properties of Kähler manifolds:

**Proposition 3.** (The Global $\partial \bar{\partial}$ Lemma): If $M$ is a compact, Kähler manifold and $\alpha \in A^{p,q}(M)$ and $d\alpha = 0$, then the following are equivalent:

- $\alpha$ is $d$-exact
- $\alpha$ is $\partial$-exact
- $\alpha$ is $\bar{\partial}$-exact
- $\alpha$ is $\partial \bar{\partial}$-exact.

The Global $\partial \bar{\partial}$ Lemma allows us to characterize “cohomologous” Kähler metrics $g$ and $\tilde{g}$:

**Proposition 4.** For $M$ a compact, Kähler manifold and $g, \tilde{g}$ two Kähler metrics and $\omega, \tilde{\omega}$ their respective Kähler forms. Suppose that $\omega, \tilde{\omega}$ are cohomologous, i.e. $[\omega] = [\tilde{\omega}] \in H^2(M, \mathbb{R})$. Then there exists a smooth, real function $\varphi$ on $M$ such that $\tilde{\omega} = \omega + i\partial \bar{\partial} \varphi$. Moreover, $\varphi$ is unique up to a constant.

**Proof.** Because we have that $\omega, \tilde{\omega}$ are cohomologous, we have that $\omega - \tilde{\omega}$ is $d$-exact, real $(1,1)$ form. Thus $\omega - \tilde{\omega}$ is actually $i\partial \bar{\partial}$-exact by the Global $\partial \bar{\partial}$ Lemma. Hence $\omega - \tilde{\omega} = i\partial \bar{\partial} \varphi$.

Suppose we have two solutions $\varphi_1, \varphi_2$. Then $\partial \bar{\partial}(\varphi_1 - \varphi_2) = 0$. Thus again by Global $\partial \bar{\partial}$ Lemma, we have that $d(\varphi_1 - \varphi_2) = 0$, i.e. $\varphi_1 - \varphi_2$ is constant, so any two solutions differ by a constant. 

Thus we see that if two Kähler metrics $g, \tilde{g}$ have cohomologous Kähler forms, then the metrics $g$ and $\tilde{g}$ are related by:

$$\tilde{g}_{jk} = g_{jk} + \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \varphi.$$

Another consequence of Kähler manifolds comes from the form $\omega$ being closed, $d\omega = 0$: the existence of a Kähler potential. The Kähler potential condition says that for any point $p \in M$ and any local patch $U \ni p$, there exists a smooth, real valued function $\varphi$ such that locally, we have:

$$g_{ij} = \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}.$$
However, this $\varphi$ is only really defined locally. In general, since $d\omega = 0$, it defines a cohomology class $[\omega] \in H^2(M, \mathbb{R})$, let it be denoted as the Kähler class. Since the real dimension of the underlying real manifold is $2n$, we have that $\omega^n$, the $n$-times wedge product is equal to $n!$ times $\text{Vol}_g$:

$$\int_M \omega^n = n! \text{Vol}(M) > 0.$$  

As $\int_M \omega^n$ is an invariant of the cohomology class, we conclude that $\omega^n \neq 0$. On the other hand as $i\partial \bar{\partial} \varphi$ is an exact form, we have that $[i\partial \bar{\partial} \varphi] = [0]$. Thus, on a compact Kähler manifold, it is impossible to find a global Kähler potential.

We also have a necessary and sufficient condition for a Hermitian manifold $(M, g, J)$ to be a Kähler manifold in terms of the Riemannian geometry data, namely the Levi-Civita connection $\nabla$:

**Theorem 2.** A Hermitian manifold $(M, g, J)$ is Kähler if and only if the complex structure $J$ satisfies the following parallel transport equation:

$$\nabla J = 0,$$

where $\nabla$ is the Levi-Civita connection of $g$.

**Proof.** ($\Leftarrow$) Assume that $\nabla J = 0$. Recall that $J$ is a globally defined $(1,1)$-tensor. Then $\nabla J = 0$ means that the components of $J$ are covariantly constant, $\nabla_{\mu} J_{\nu}^{\rho} = 0$. Furthermore, since $\nabla$ is the Levi-Civita connection, the components of the metric $g$ are also covariantly constant. Thus the Kähler form $\omega$ satisfies $\nabla \omega = 0$. This clearly implies that $d\omega = 0$. Hence $M$ is Kähler.

($\Rightarrow$) For a proof of the converse, we reference Andy Neitzke’s Complex Geometry lecture notes [ANCG].

Let’s now discuss the Riemannian Geometry of Kähler manifolds. Recall that for any arbitrary Riemannian manifold $(M, g)$, let $\nabla$ denote the Levi-Civita connection for $g$. Given any set of local coordinates $x^\mu$, we have that the Christoffel symbols are given by the following equation:

$$\Gamma^{\nu}_{\mu \rho} = \frac{1}{2} g^{\nu \sigma} \left( \partial_{\mu} g_{\sigma \nu} + \partial_{\nu} g_{\mu \sigma} - \partial_{\sigma} g_{\mu \nu} \right).$$

More specifically, in local coordinates, the connection $\nabla$ defines a covariant derivative on tensors. So for $T$ a $(1,1)$ tensor, we have the following:

$$\nabla_{\mu} T_{\nu}^{\rho} = \partial_{\mu} T_{\nu}^{\rho} + \Gamma^{\sigma}_{\mu \rho} T_{\nu}^{\sigma} - \Gamma^{\sigma}_{\mu \nu} T_{\sigma}^{\rho}.$$

Not surprisingly, for a Kähler manifold, we have an elegant prescription for our Christoffel symbols:

**Lemma 1.** For a Kähler manifold $(M, g, J)$, we have that in complex coordinates $z^j, \bar{z}^\ell$ in a neighborhood of a point $z_0 \in M$, the only non-vanishing components of the Christoffel symbols are:

$$\Gamma^{\ell}_{jk} = (\partial_{j} g_{k \bar{\ell}}) g^{\bar{m} \ell}, \Gamma^{\bar{m}}_{jk} = (\Gamma^{\ell}_{jk})^*.$$

Moreover:

$$\Gamma^{\ell}_{jk} = \partial_{j} (\log \sqrt{\text{det} g}).$$
Proof. The first condition follows two facts. The first fact is that $g$ is locally given by the Hermitian positive definite matrix $h_{ij}$, therefore $h_{ij} = h_{ji} = 0$, i.e. the only nonzero components of our metric are the components corresponding to mixed holomorphic and antiholomorphic indices. The second fact is that since $g$ is a Kähler metric, $d\omega = 0 \Rightarrow \partial_j h_{k\bar{l}} = \partial_k h_{j\bar{l}}$ and the complex conjugate of these statements. For a proof of this statement, we refer the reader to [JST]. Thus we have that:

\begin{align}
\Gamma_{jk}^k &= (\partial_j g_{k\bar{l}}) g^{\bar{l}k} \\
&= \frac{1}{2} \text{Tr}[g^{-1} \partial_j g] \\
&= \frac{1}{2} \partial_j [\log \det g] \\
&= \partial_j \log \sqrt{\det g}.
\end{align}

\[\square\]

A byproduct of all these nice Kähler identities is that the Ricci tensor takes a particularly nice form. Again, the pure holomorphic and pure antiholomorphic components of the Ricci tensor will vanish. The nonvanishing components of the Ricci tensor will given by:

\[R_{j\bar{k}} = \partial_j \partial_{\bar{k}} [\log \sqrt{\det g}].\]

Given this fact, we may define the Ricci Form given by $R = iR_{j\bar{k}} dz^j \wedge d\bar{z}^k$. This Ricci Form $R$ defines a cohomology class, which we call the first Chern class $c_1 = [\frac{i}{2\pi} R]$.

We now have introduced all the relevant concepts that will allow us to begin discussing the Calabi Conjecture. However, before we move on to the statement and philosophy of the Calabi Conjecture, we will actually introduce and prove what will turn out to be a crucial little lemma:

**Lemma 2.** Let $(M, g)$ be a Kähler manifold with Kähler form $\omega$. For a $f \in C^0(M)$, we may define $A$ via:

\[A = \int_M e^f dV_g = Vol_g(M).\]

Suppose that $\varphi \in C^2(M)$ satisfies $(\omega + i\partial \bar{\partial} \varphi)^n = Ae^f \omega^n$ on $M$. Then we have that $\omega + i\partial \bar{\partial} \varphi$ is a positive $(1,1)$ form.

**Proof.** Consider any set of holomorphic coordinates $z^j, j = 1, \ldots, n$ on a connected patch $U \subset M$. Then since $\omega$ and $\omega + i\partial \bar{\partial} \varphi$ are cohomologous, we have a new metric $\tilde{g}$ in the patch $U$ given by:

\[\tilde{g}_{j\bar{k}} = g_{j\bar{k}} + \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k}.
\]

Up to the reader’s conventions, we have that $\omega^n = n! d\text{Vol}_g$, thus $\tilde{g}_{j\bar{k}}$ is a Hermitian metric if and only if $\omega + i\partial \bar{\partial} \varphi$ is a positive $(1,1)$ form, i.e. if and only if the eigenvalues of $\tilde{g}_{j\bar{k}}$ are all positive. Hence, it suffices to show that $\tilde{g}_{j\bar{k}}$ has positive eigenvalues.
To this end, by hypothesis, we have that:

\[(\omega + i\partial\bar{\partial}\varphi)^n = Ae^f \omega^n \text{ on } M.\]

This immediately implies that:

\[\det(\tilde{g}) = Ae^f \det(g).\]

Thus \(\det(\tilde{g}) > 0\) and hence must have no zero eigenvalue. But by continuity, if the eigenvalues of \(\tilde{g}\) are positive at some point \(z_0\), they must be positive in a neighborhood of \(z_0\). By connectedness, we conclude that the eigenvalues must be positive on all of \(M\).

By compactness of \(M\) and continuity of \(\varphi\), \(\varphi\) obtains a minimum at some point \(p \in M\). Let \(U\) be a patch about a minimum point \(p\). Since \(p\) is a point where \(\varphi\) achieves its minimum, we conclude that the matrix

\[\begin{bmatrix}
\partial^2 \varphi \\
\partial z_i \partial \bar{z}_j
\end{bmatrix}
\]

has positive eigenvalues. We may conclude this since in higher dimensions, the Hessian of \(\varphi\) must be positive definite at a minimum point \(p\). As \(g\) is a Hermitian metric, it has only positive eigenvalues as well. Thus we conclude that \(\tilde{g}\) must have positive eigenvalues and thus defines a Hermitian metric and hence we may conclude that it’s Kähler form \(\omega + i\partial\bar{\partial}\varphi\) is a positive \((1,1)\) form. \(\square\)

2. Introduction to Calabi Conjecture

We start by assuming that \(M\) is a compact Kähler Manifold with Kähler metric \(g = \sum_{i,j} g_{ij} dz^i \otimes \bar{dz}^j\) and Ricci Tensor \(R = \sum_{i,j} R_{ij} dz^i \otimes \bar{dz}^j\). We showed above that

\[R_{ij} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g_{kl}).\]

Note: Up to sign choices and the constant \(\frac{1}{2}\), our Ricci tensor agrees with Yau’s. Note in Equation 32, we have a leading \(\frac{1}{2}\) factor, while Yau does not and the minus sign comes from choice of \(\sqrt{-1}\) convention.

This implies that the closed real \((1,1)\) form \(\frac{1}{2\pi} \sum_{i,j} R_{ij} dz^i \wedge \bar{dz}^j\) can equivalently be written as \(-\frac{1}{2\pi} \partial\bar{\partial} \log \det(g_{kl})\). According to a theorem proved by S.S. Chern [SS] the cohomology class of this particular \((1,1)\) form depends only on the Complex Structure of \(M\). Furthermore this closed, real \((1,1)\) form is exactly equal to the first Chern Class of \(M\). The Calabi Conjecture is to look at the converse of this statement.

**Theorem 3.** (Calabi Conjecture 1954) Let \(M\) be a compact, complex manifold and \(g\) a Kähler metric on \(M\) with Kähler form \(\omega\). Suppose \(\tilde{R}\) is a closed, real \((1,1)\) form on \(M\) with \([\tilde{R}] = c_1(M)\). Then there exists unique Kähler metric \(\tilde{g}\) on \(M\) with Kähler form \(\tilde{\omega}\) such that \([\tilde{\omega}] = [\omega] \in H^2(M, \mathbb{R})\) and the Ricci form of \(\tilde{\omega}\) is \(\tilde{R}\).

Calabi was able to show that if such a \(\tilde{g}\) exists then it must be unique. The key insight to solving the problem was to reduce the problem to a nonlinear Elliptic PDE of Monge-Ampère type.

Assume that the Calabi Conjecture is in fact true. Since the metrics have been shown to be cohomologous, we know by the Global \(\partial\bar{\partial}\)-Lemma, that \(\exists\) a smooth real function \(\varphi\) unique up to the addition of a constant, such that \(\tilde{\omega} = \omega + i\partial\bar{\partial}\varphi\). Furthermore locally we have the following representation for our metrics: \(\tilde{g}_{ij} = g_{ij} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \).
Note that this allows us to parameterize a class of Kähler metrics by a single smooth function $\varphi$.

Turning our attention to their respective Ricci forms let $\tilde{R} = \frac{i}{2\pi} \sum_{i,j} \tilde{R}_{ij} dz^i \wedge \overline{dz}^j$ represent the first Chern Class of our manifold $M$. Recall also that $\tilde{R}_{ij} = -\frac{\partial^2}{\partial z^i \partial \overline{z}^j} [\log \det(g_{k\bar{l}})]$. Since $[R - \tilde{R}] = 0$, another application of the Global $\partial \overline{\partial}$-Lemma demonstrates that there exists a smooth function $F$ defined on $M$ such that: $\tilde{R}_{ij} - R_{ij} = \frac{\partial^2 F}{\partial z^i \partial \overline{z}^j}$. Hence:

\[(50)\quad \partial \overline{\partial} \log \frac{\det(\tilde{g}_{ij})}{\det(g_{ij})} = \partial \overline{\partial} F.\]

Since $M$ is a compact manifold, and $\frac{\det(\tilde{g}_{ij})}{\det(g_{ij})}$ is a globally defined function (this can be demonstrated by looking at the metric locally and applying a change of coordinates), a further application of the Global $\partial \overline{\partial}$-Lemma shows that $\frac{\det(\tilde{g}_{ij})}{\det(g_{ij})} - F$ is a constant function. Hence there exists a constant $C > 0$ such that:

\[(51)\quad \det(\tilde{g}_{ij}) = C \exp (F) \det(g_{ij}).\]

Recalling the fact that our Kähler forms are cohomologous we find that the above equation is equivalent to:

\[(52)\quad \det \left( g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j} \right) = C \exp (F) \det(g_{ij}).\]

This equation is a nonlinear, elliptic, second-order partial differential equation in $\varphi$. It is of Monge-Ampère type (Note: Ellipticity follows from the fact that the metric is assumed to be Hermitian). We now run the argument backwards and argue that if we can find a constant $C > 0$ such that there exists a smooth solution $\varphi$ satisfying the integrability condition $\int_M \varphi = 0$ and $\tilde{g} = \sum_{i,j} g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j} dz^i \otimes \overline{dz}^j$ defines a Kähler Metric then we will have a solution to the Calabi Conjecture. Note the integrability condition is imposed so that $\varphi$ is chosen to be unique. Not imposing this integrability condition gives us a solution to the Calabi Conjecture up to a constant.

Moreover one readily sees that the constant $C$ must satisfy the following compatibility condition:

\[(53)\quad C \int_M \exp \{F\} = \text{Vol}(M).\]

We now have an equivalent formulation of the Calabi Conjecture:

**Theorem 4.** (Calabi Conjecture Reformulation) Let $M$ be a compact, complex manifold and $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $F$ be a smooth function on $M$ and let $C > 0$ be defined via the compatibility condition $C \int_M \exp \{F\} = \text{Vol}(M)$. Then $\exists$ unique smooth function $\varphi$ such that:
\( \hat{\varphi} = \sum_{i,j} g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j \) defines a Kähler Metric

(ii) \( \int_M \varphi = 0 \)

(iii) \( \det \left( g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) = C \exp(F) \det(g_{ij}) \)

Note: Assuming such a \( \varphi \) exists, (i) of the theorem follows from (iii) and the lemma proved in the last section. Before closing the present section we prove that our solution to the Complex Monge Ampère equation is unique.

**Theorem 5. (Uniqueness of Kähler Metric)** Under the assumptions of the previous theorem, the smooth function \( \varphi \) is unique.

**Proof.** Suppose we have \( \varphi_1 \) and \( \varphi_2 \) solving the Complex Monge Ampère Equation. Further assume that \( \varphi_1, \varphi_2 \in C^3(M) \). Let \( \omega_1 = \omega + i\partial \bar{\partial} \varphi_1 \) and \( \omega_2 = \omega + i\partial \bar{\partial} \varphi_2 \). By the lemma in the previous section we know that both \( \omega_1 \) and \( \omega_2 \) are positive (1,1)-forms. Let \( g_1 \) and \( g_2 \) be the respective Kähler Metrics.

We introduce the operator \( d^c = i(\bar{\partial} - \partial) \). We also introduce the Hodge Star operator \( \star \). Let \( V \) be an oriented vector space of dimension \( m \). Furthermore let \( \{ e_i \} \) be a positive oriented orthonormal basis for \( V \). Then Hodge Star operator \( \star \) is defined by \( \alpha \wedge \star \beta = g(\alpha, \beta) \text{vol.} \)

We have:

\[
0 = \omega_1^m - \omega_2^m = d\omega_1^c \wedge (\varphi_1 - \varphi_2) - d\omega_2^c \wedge (\varphi_1 - \varphi_2) \wedge \sum_{k=0}^{m-1} \omega_1^k \wedge \omega_2^{m-k-1}.
\]

Multiply both sides by \( (\varphi_1 - \varphi_2) \) We have:

\[
0 = (\varphi_1 - \varphi_2) d\omega_1^c \wedge (\varphi_1 - \varphi_2) - (\varphi_1 - \varphi_2) d\omega_2^c \wedge (\varphi_1 - \varphi_2) \wedge \sum_{k=0}^{m-1} \omega_1^k \wedge \omega_2^{m-k-1},
\]

which in turns implies the following:

\[
0 = d \left[ (\varphi_1 - \varphi_2) d^c (\varphi_1 - \varphi_2) \wedge \sum_{k=0}^{m-1} \omega_1^k \wedge \omega_2^{m-k-1} \right] - d (\varphi_1 - \varphi_2) d^c (\varphi_1 - \varphi_2) \wedge \sum_{k=0}^{m-1} \omega_1^k \wedge \omega_2^{m-k-1}.
\]

We integrate over \( M \) and use Stokes Theorem to get:

\[
\sum_{k=0}^{m-1} \int_M d (\varphi_1 - \varphi_2) \wedge d^c (\varphi_1 - \varphi_2) \wedge \omega_1^k \wedge \omega_2^{m-k-1} = 0.
\]

Since \( \omega_1 \) and \( \omega_2 \) define Kähler Metrics \( g_1 \) and \( g_2 \) we can choose a local holomorphic coordinate system around each \( x \in M \) where \( \{ e_1 \ldots e_m, J e_1 \ldots J e_m \} \) is an orthonormal basis with \( e_\alpha = \frac{\partial}{\partial z_\alpha}(x) \) and \( J e_\alpha = \frac{\partial}{\partial y_\alpha}(x) \). Also:

\[
\omega_1 = \sum_{j=1}^{m} e_j \wedge J e_j
\]
\[ \omega_2 = \sum_{j=1}^{m} a_j e_j \wedge Je_j \]

Where \( a_j \) are strictly positive local functions. This shows that:

\[ \omega^k \wedge \omega^{m-k-1} = \left( \sum_{j=1}^{m} b_j^k e_j \wedge Je_j \right) \]

where \( b_j^k = k!(m-k-1)! \sum_{\begin{subarray}{c} j_1 \neq \cdots \neq j_k \\ j_1 < \cdots < j_k \end{subarray}} a_j_1 \cdots a_j_k \).

This shows that the integrand is strictly positive unless \( 0 = d (\varphi_1 - \varphi_2) \). Thus \( \varphi_1 - \varphi_2 \) is a constant. Since we imposed the integrability condition the constant is equal to 0. Hence \( \varphi_1 = \varphi_2 \). \( \square \)

In the next section we look at some of the important results from Elliptic PDE theory that helped Yau in proving the Calabi Conjecture.

3. Elliptic PDE Theory

The theory of Partial Differential Equations is enormously varied, yet a consistent strategy employed in solving both quasilinear and nonlinear PDE’s has been to prove that solutions to the PDE must satisfy certain \textit{a priori} estimates. When one assumes that a solution to a PDE lies in a specific function class one is in fact imposing a regularity condition on solutions to the PDE. Proving an \textit{a priori} estimate amounts to showing that under the basic assumption that a solution to a PDE belongs to a given function class (i.e. has sufficient regularity) that we can in fact find a uniform bound for all solutions of that function class. This of course says nothing about the actual existence of a solution to a particular PDE.

To prove the existence of a solution to a quasilinear or nonlinear PDE, one employs fixed point methods or continuity arguments that links the nonlinear PDE under consideration to a linear PDE for which one can show the existence of sufficiently regular solutions. In the previous section it was demonstrated that a solution to the Calabi Conjecture would follow from proving the existence of a smooth solution to a nonlinear elliptic PDE of Monge-Ampère type. Much of the theory we discuss in this section will help us understand the structure of Yau’s argument and help to prove existence and uniqueness of a smooth solution.

Let us recall the general Second Order Linear Elliptic PDE:

\[ \sum_{i,j} a_{ij} \partial_{ij} u + \sum_i b_i \partial_i u + cu = f. \]

We rewrite this as \( Lu = f \) where \( L \) is a Second Order Linear Elliptic Operator. Ellipticity follows from the fact that the coefficient matrix \([a_{ij}]\) is positive definite in the domain of the respective arguments. For what follows we assume that we are working in an open domain \( \Omega \subseteq \mathbb{R}^n \). We also assume that \( c \leq 0 \). Furthermore we assume that our operator is \textit{Uniformly Elliptic}: \( \forall x \in \Omega, \xi \in \mathbb{R}^n, \lambda > 0. \)
\[
\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 .
\]

We start by defining some function classes that will be of importance in our analysis:

**Definition 6.** A function \( f \) is uniformly Hölder Continuous with exponent \( 0 < \alpha < 1 \) if

\[
[f]_{\alpha, \Omega} = \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.
\]

We define the Hölder Spaces \( C^{k,\alpha}(\bar{\Omega}) \) \((C^{k,\alpha}(\Omega))\) to be the function space consisting of functions whose \( k \)-th order partial derivatives are uniformly Hölder Continuous. Note: \( C^{k,0}(\bar{\Omega}) = C^{k}(\bar{\Omega}) \) \((C^{k,0}(\Omega) = C^{k}(\Omega))\) where \( C^{k}(\Omega) \) \((C^{k}(\bar{\Omega}))\) is the space of continuous functions whose \( k \)-th order partial derivatives are continuous (up to the boundary). We define the following norms:

\[
\|u\|_{C^{k}(\bar{\Omega})} = \sum_{j=0}^{k} [D^j u]_{0, \Omega},
\]

\[
\|u\|_{C^{k,\alpha}(\bar{\Omega})} = \|u\|_{C^{k}(\bar{\Omega})} + [D^k f]_{\alpha, \bar{\Omega}}.
\]

To understand the estimates below it is important to make a comment in regards to how one goes about obtaining them. One begins by first finding a priori estimates on suitable subdomains \( \Omega' \subseteq \Omega \). Since one can go about establishing a Maximum Principle for Linear Elliptic PDEs one will find that solutions to such PDEs will have their maximum on the boundary \( \partial \Omega \). Hence if the domain \( \Omega \) has sufficiently smooth boundary and the solution to our PDE is sufficiently regular, one can extend these interior estimates to the boundary and obtain global estimates.

**Definition 7.** A bounded domain \( \Omega \subset \mathbb{R}^n \) and its boundary are of class \( C^{k,\alpha} \) if at each point \( x_0 \in \partial \Omega \) there is a ball \( B = B(x_0) \) and a one-to-one mapping \( \psi \) of \( B \) onto \( D \subset \mathbb{R}^n \) such that:

(i) \( \psi(B \cap \Omega) \),

(ii) \( \psi(B \cap \partial \Omega) \subset \partial \mathbb{R}^n_+ \),

(iii) \( \psi \in C^{k,\alpha}(B), \psi^{-1} \in C^{k,\alpha}(D) \).

One must refer to Potential Theory to establish existence and uniqueness of solutions for the Laplace and Poisson Equation. But we mention that one can prove the unique existence of a \( C^{2,\alpha}(\bar{\Omega}) \) solution for the Poisson Equation \( \Delta u = f \) when \( f \in C^{0,\alpha}(\Omega) \). In the course of this problem one in fact obtains an a priori estimate for this unique solution:

\[
\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \left( \sup_{\Omega} |u| + ||f||_{C^{0,\alpha}(\bar{\Omega})} \right) \quad \text{where} \quad C = C(n, \alpha).
\]

The Poisson Equation is a Linear Elliptic PDE with constant coefficients, yet one can show that the estimate above can be extended to the case of the general Linear Second Order Elliptic PDE with variable coefficients. In fact one can show even
more: that if the coefficients of the general Second Order Linear PDE are Hölder Continuous then there exists an \emph{a priori} estimate for $C^{2,\alpha}(\Omega)$ solutions for this class of PDE’s. The idea is to treat the equation locally as a perturbation of constant coefficient equations. Similar in spirit to the estimate obtained above for the Poisson Equation, one first considers interior estimates and then assuming sufficient regularity of our solutions and sufficient smoothness up to the boundary, one then extends these estimates to the boundary to obtain global estimates.

**Theorem 6.** \textit{(Schauder Estimate)} Let $\Omega$ be a $C^{2,\alpha}$ domain in $\mathbb{R}^n$ and let $u \in C^{2,\alpha}(\Omega)$ be a solution of $Lu = f$ in $\Omega$ where $f \in C^0(\Omega)$ and coefficients of $L$ satisfy for positive constants $\lambda, \Lambda$:

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \forall x \in \Omega, \xi \in \mathbb{R}^n, \lambda > 0 \text{ (Ellipticity Condition)}$$

$$[a_{ij}]_{\alpha,\Omega}, [b_i]_{\alpha,\Omega}, [c]_{\alpha,\Omega} \leq \Lambda \text{ (Coefficients are Hölder Continuous)}$$

Assume further that $\varphi \in C^{2,\alpha}(\Omega)$ and $u = \varphi$ on $\partial\Omega$. Then,

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C (\|u\|_{0,\Omega} + \|\varphi\|_{2,\alpha,\Omega} + \|f\|_{0,\alpha,\Omega})$$

$$C = C(n, \alpha, \lambda, \Lambda, \Omega)$$

We now have an \emph{a priori} estimate for solutions in the class $C^{2,\alpha}$. What remains to be considered is to show that there in fact does exist a unique $C^{2,\alpha}$ solution to our second order Linear Elliptic PDE. To prove existence of a solution we will use functional analytic methods. The following is known as the \textit{Linear Continuity Method}.

**Theorem 7.** \textit{(Linear Continuity Method)} Let $(B_1, \|\cdot\|_{B_1})$ and $(B_2, \|\cdot\|_{B_2})$ be Banach Spaces. Let $L_0, L_1 : B_1 \to B_2$ linear and bounded. Set $L_t = (1-t)L_0 + tL_1$. Furthermore $\|x\|_{B_1} \leq C\|L_t x\|_{B_2}$ $\forall x \in X, \forall t \in [0,1]$. Then the following are equivalent:

- $L_0 : B_1 \to B_2$ isomorphism.
- $L_1 : B_1 \to B_2$ isomorphism.
- $\exists t_0 \in [0,1]$ such that $L_{t_0} : B_1 \to B_2$ isomorphism.

**Proof.** Assume $L_s$ is onto for some $s \in [0,1]$. By the bound in the assumption we have that $L_s$ is also injective hence $L_s$ is an isomorphism.

$\Rightarrow L_s^{-1} : B_2 \to B_1$ exists.

$\Rightarrow \forall y \in B_2 \exists x \in B_1 \text{ s.t. } y = L_s x.$

$\Rightarrow \|L_s^{-1} y\|_{B_1} = \|x\|_{B_1} \leq C\|L_s x\|_{B_2} = C\|y\|_{B_2}$

Let $t \in [0,1]$. Now we know that:

$$(66) \quad L_t x = y \iff L_s x = (L_s - L_t)x + y = y + (t-s)L_0 x - (t-s)L_1 x.$$  

$$(67) \quad \iff x = L_s^{-1} y + (t-s)L_s^{-1}(L_0 - L_1)x.$$  

Define map $T : X \to X$ such that $T(x) = L_s^{-1} y + (t-s)L_s^{-1}(L_0 - L_1)x.$

We have a string of inequalities:

$$(68) \quad \|Tx - T\|_{B_1} \leq |t-s|\|L_s^{-1}(L_0 - L_1)(x - x')\|_{B_1}$$  

$$(69) \quad \leq |t-s|C(L_0 + \|L_1\|)|x - x'|_{B_1}$$
If we choose $t$ close enough to $s$ we see that from the last inequality that the map $T$ is contractive. In particular if $|t - s| \leq \frac{1}{2\|L_0\| + \|L_1\|}$ we see that:

$$\|Tx - Tx'\|_{B_1} \leq \frac{1}{2}\|x - x'\|_{B_1}.$$ 

Hence by the Banach Fixed Point Theorem:

If $|t - s| \leq \delta$ then $\exists!$ fixed point for $T \Rightarrow L_t$ is onto $\forall t$, $|t - s| \leq \delta$. Since our map is surjective in a uniform neighborhood we can iterate our argument to show surjectivity for $L_t$ $\forall t \in [0, 1]$. □

We now establish a Maximum Principle for solutions of 2nd Order Linear Elliptic PDE. This will allow us to successfully apply the Linear Continuity Method and prove existence of a unique $C^{2,\alpha}$ solution.

**Lemma 3.** *(Maximum Principle)* Assume $Lu = f$ in bounded domain $\Omega \subseteq \mathbb{R}^n$. Suppose $u \in C^0(\Omega) \cap C^\alpha(\Omega)$, $c \leq 0$, $a_{ij} \geq \lambda I$, $b_i \in L^\infty$. Then,

$$\sup_{\Omega} u \leq \sup_{\partial \Omega} u^+ + \frac{C_0 \sup_{\Omega} f^-}{\lambda}$$

Where $u^+$ is positive part of function $u$, $f^-$ is negative part of function $f$, and $C_0 = C_0(\Omega, \lambda, \|b\|_{L^\infty})$.

**Proof.** Let $L_0 = a_{ij} \partial_{ij} + b_i \partial_i$

We translate and rotate $\Omega$ such that in the $x_1$ direction it is bounded between $[0, \delta]$. We know that $L_0 e^{\alpha x_1} = (a_{11} \alpha^2 + b_1 \alpha) e^{\alpha x_1} \geq (\lambda \alpha^2 - \|b\|_{L^\infty} \alpha) e^{\alpha x_1} \geq \lambda$ if $\alpha >> 1$.

Set $v := \sup_{\partial \Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \sup_{\partial \Omega} f^- + \varepsilon \geq 0$

Now $L_0 v = -\lambda \sup_{\partial \Omega} f^- + \varepsilon \leq -\sup_{\partial \Omega} f^- - \varepsilon$.

Since $v \geq 0$ and $c \leq 0$, $L v = L_0 v + cv \leq L_0 v$.

$$\Rightarrow L(v - u) \leq -\lambda \left(\sup_{\partial \Omega} f^- + \varepsilon\right) \leq 0 \text{ and } v - u \geq 0 \text{ on } \partial \Omega.$$ 

We want to show that our functions $u$ and $v$ never cross. Let $x_0$ be a maximum point of $v - u$. Assume by contradiction that $(v - u)(x_0) < 0$.

$$\Rightarrow L(v - u)(x_0) = a_{ij} \partial_{ij}(v - u)(x_0) + b_i \partial_i(v - u)(x_0) + c(v - u)(x_0) \geq 0$$

But $L(v - u)(x_0) \leq 0$. Hence we have derived a contradiction.

$$\Rightarrow \text{In } \Omega: u \leq v.$$ Let $\varepsilon \to 0$ to obtain bound. □

With the help of the Maximum Principle and the Linear Continuity Method we prove the existence of a unique $C^{2,\alpha}$ solution for 2nd Order Linear Elliptic PDE.

**Theorem 8.** *(Schauder)* Assume $\Omega \subseteq \mathbb{R}^n$ and its boundary $\partial \Omega$ is of class $C^{2,\alpha}$. Suppose also that $a_{ij}, b_i, c \in C^{0,\alpha}(\Omega); f \in C^{0,\alpha}(\Omega); a_{ij} \geq \lambda I, b_i \in L^\infty, \text{ and } c \leq 0$. Then, $\forall \varphi \in C^{2,\alpha}(\Omega)$ $\exists! u \in C^{2,\alpha}(\Omega)$ solving the Boundary Value Problem:

$$Lu = f \text{ in } \Omega$$

$$u = \varphi \text{ on } \partial \Omega$$

**Proof.** Without loss of generality we assume that $\varphi = 0$. Otherwise we can replace $u$ with $v = u - \varphi$. Define $L_t = (1 - t)\Delta + tL$. 

$$\|T x - T x'\|_{B_1} \leq \frac{1}{2}\|x - x'\|_{B_1}.$$
Note that $L_t : X \to Y$, where $X = \{u \in C^{2,\alpha}(\Omega) \mid u = 0 \text{ on } \partial \Omega\}$ and $Y = \{C^{0,\alpha}(\Omega)\}$

By the maximum principle $\forall w \in X \|w\|_{L^\infty(\Omega)} \leq C \|L_tw\|_{L^\infty(\Omega)}$.

Our $C^{2,\alpha}$ a priori estimate implies that:

$$
\|w\|_{C^{2,\alpha}(\Omega)} \leq C \left( \sup_{\Omega} |w| + \|f\|_{C^{0,\alpha}(\Omega)} \right) \leq C \|L_tw\|_{C^{0,\alpha}(\Omega)}.
$$

$\Rightarrow \|w\|_X \leq C \|L_tw\|_Y \text{ \forall w \in X, } \forall t \in [0,1].$

We know that $L_0$ is simply the Laplacian operator and we stated that there exists a unique $C^{2,\alpha}$ solution for this operator. Hence by the continuity method $L_1$ is an isomorphism.

In order to successfully study the Complex Monge Ampère Equation we have to also consider Nonlinear Elliptic PDE theory. Let us recall the General Second-Order Nonlinear Elliptic PDE on domain $\Omega \subseteq \mathbb{R}^n$. Our nonlinear operator will be a real function defined on $\Gamma = \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$:

(70) \quad $F[u] = F(x, u, Du, D^2u) = 0$.

A typical point $\gamma \in \Gamma$ will be indexed by $\gamma = (x, z, p, r)$.

**Definition 8.** $F$ is elliptic in $\mathfrak{A} \subseteq \Gamma$ if this matrix $[F_{ij}(\gamma)]$ given by $F_{ij}(\gamma) = \frac{\partial^2 F}{\partial r_i \partial r_j}(\gamma) > 0 \forall \gamma \in \mathfrak{A}$. Furthermore let $\lambda(\gamma)$ and $\Lambda(\gamma)$ be the minimum and maximum eigenvalues of matrix $[F_{ij}(\gamma)]$. $F$ is uniformly elliptic if $\frac{\lambda}{\Lambda} < \infty$.

To show the existence of a solution to a Nonlinear Elliptic PDE one can use a nonlinear version of the Continuity Method. We will now develop this idea further. We assume the reader is familiar with Fréchet Differentiability for operators on a Banach Space.

**Theorem 9.** (Implicit Function Theorem for Banach Spaces) Let $B_1, X, B_2$ be Banach Spaces and suppose that $G : B_1 \times X \to B_2$ is a $C^1$ mapping defined at least in a neighborhood of a point $(u_0, \sigma_0)$. Denote by $y_0$ the image of $G(u_0, \sigma_0)$. Suppose $D_2G(u_0, \sigma_0)$ is an isomorphism. Then $\exists$ open sets $W \subseteq B_1$, $U \subseteq X$, $V \subseteq B_2$ with $u_0 \in W$, $\sigma_0 \in U$ and $y_0 \in V$ and a unique $C^1$ mapping $g : W \times V \to U$ such that

$$
G(u, g(u, y)) = y \quad \forall (u, y) \in W \times V.
$$

In order to apply the Implicit Function Theorem to nonlinear PDE Theory we assume that $F$ is a mapping from an open subset $\mathfrak{A} \subseteq B_1$ into $B_2$. Let $\psi$ be a fixed element in $\mathfrak{A}$ and define for $u \in \mathfrak{A}$, $t \in \mathbb{R}$ the mapping $G : \mathfrak{A} \times \mathbb{R} \to B_2$ where

$$
G[u, t] = F[u] - tF[\psi].
$$

Define $S \subseteq [0,1]$ such that:

$$
S = \{t \in [0,1] \mid G[u, t] = 0 \text{ for some } u \in \mathfrak{A}\}.
$$

**Note:** $S \neq \emptyset$ since $t = 1 \in S$. If we further assume that the map $F$ is $C^1$ it follows from the Implicit Function Theorem that the set $S$ is open. If we can show that the set $S$ is also closed then by connectivity of the set $[0,1]$, $S = [0,1]$. Hence in particular there exists a $u \in \mathfrak{A}$ such that $F[u] = 0$. This is the solution to our Nonlinear PDE. As we will see when we apply these ideas to the Complex
Monge Ampère Equation, closure of the set $S$ will follow from an a priori estimate in some function space and the application of the Arzela-Ascoli Theorem. Hence just as in the Linear case we need to establish a priori estimates for solutions with sufficient regularity. Once these estimates have been shown a simple application of the Continuity Method gives us a solution to the Nonlinear PDE. As a precursor for what is to come, we mention that the establishment of a priori estimates for $\varphi$ is Yau’s primary task. He then successfully applies a variant of the Nonlinear Continuity Method to prove existence of a solution to the nonlinear elliptic PDE under consideration. Hence most of the labor Yau undertakes is in establishing a priori estimates for solutions of the Complex Monge Ampère Equation.

4. Proof of Calabi Conjecture

In this section we present a proof of the Calabi Conjecture. Before providing the details of the proof let us take a moment to mention how our results of the previous section generalize to general compact manifolds. The interior estimates we obtained were for general open domains in $\mathbb{R}^n$. To transfer these estimates onto a compact manifold we simply use our compactness assumption to find a finite open cover for our manifold. Since each set in this cover is diffeomorphic to an open set in $\mathbb{R}^n$, one can simply use coordinate transformations to transfer the estimate onto our manifold.

A tool that will be useful in our computations is the fact that around each point of our Kähler Manifold one can find a coordinate system which can simultaneously diagonalize the Kähler Metric $g_{ij}$ and the Hessian of $\varphi$. The utility of this representation can hardly be overestimated. More specifically:

**Theorem 10.** (Existence of Holomorphic Normal Coordinates) Let $(M, \omega)$ be a Kähler Manifold. In local coordinates $\omega = \sum g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. At each $z_0 \in M$ holomorphic normal coordinates can be introduced i.e.

(i) $g_{0\bar{j}}(z_0) = \delta_{i\bar{j}}$

(ii) $\partial_k g_{i\bar{j}}(z_0) = \partial_k g_{\bar{j}i}(z_0) = 0$

(iii) $g'_{ij}(z_0) = (1 + \varphi_{\bar{i}}) \delta_{ij}$ where $g'_{ij} = g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}$

We start by stating the main estimates which are used to provide a solution to the Calabi Conjecture.

**Theorem 11.** (Yau’s Second Order Estimates) Let $M$ be a compact Kähler Manifold with metric tensor $2 \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Let $\varphi$ be a real valued function in $C^4(M)$ such that $\int_M \varphi = 0$ and $\sum g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}$ defines another metric tensor on $M$. Suppose $\det \left( g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) = \exp(F) \det(g_{i\bar{j}})$. Then there exist positive constants $C_1, C_2, C_3$, and $C_4$ depending on $\inf_M F$, $\sup_M F$, $\inf \Delta F$, and $M$ such that $\sup_M |\varphi| \leq C_1$, $\sup_M |\nabla \varphi| \leq C_2$, $0 < C_3 \leq 1 + \varphi_{\bar{i}} \leq C_4$ for all $i$.

**Theorem 12.** (Yau’s Third Order Estimate) Let $M$ be a compact Kähler Manifold with metric tensor $2 \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Let $\varphi$ be a real valued function in $C^5(M)$ such that $\int_M \varphi = 0$ and $\sum g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j$ defines another metric tensor on $M$. Suppose $\det \left( g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) = \exp(F) \det(g_{i\bar{j}})$. Then there is an estimate of
the derivatives $\varphi_{i\bar{j}k}$ in terms of $\sum_{i,j} g_{ij} dz^i \otimes d\bar{z}^j$, sup $|F|$, sup $|\nabla F|$, sup$_{M} \sup_{i} |F_{ii}|$ and sup$_{M} \sup_{i,j,k} |F_{ijk}|$.

We consider

$$
\det \left( g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) = C \exp (F) \det (g_{ij})
$$

(71)

Recall our assumptions: $\varphi \in C^5(M)$, $F \in C^k(M)$ for $k \geq 3$ and $\int_M \exp \{ F \} = \text{Vol}(M)$. We set our constant $C = 1$. Our goal is to show the existence of a unique $C^\infty$ function $\varphi$ solving the Monge Ampère equation and satisfying the compatibility condition $\int_M \varphi = 0$. Recall that we have already established that if such a smooth function $\varphi$ does exist then $\tilde{g} = \sum_{i,j} g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j$ indeed defines a Kähler metric on our manifold. Moreover we showed that the metric is unique. Hence establishing the existence of a smooth solution to the Complex Monge Ampère equation will lead us to a solution of the Calabi Conjecture. We now demonstrate how Yau’s estimates and an application of a variant of the continuity method will lead us to a solution of the problem.

In the first step we show that under the stated assumptions we can find a solution $\varphi \in C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$ to the Complex Monge Ampère Equation. Consider the set:

$$
S = \{ t \in [0,1] \mid \text{the equation } \det \left( g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) \text{det}(g_{ij})^{-1} = \text{Vol}(M) \left[ \int_M \exp \{ tF \} \right]^{-1} \exp (tF) \text{ has a solution in } C^{k+1,\alpha}(M) \}
$$

Note: $0 \in S \Rightarrow S \neq \emptyset$.

If we can show that $S$ is both open and closed this will imply that $S = [0,1]$. Hence $\Rightarrow 1 \in S \Rightarrow$ our equation has a solution in $C^{k+1,\alpha}(M)$. This is an application of the Nonlinear Continuity Method. Define the following sets:

$$
\Theta = \{ \varphi \in C^{k+1,\alpha}(M) \mid 1 + \varphi_{ii} > 0 \forall i \text{ and } \int_M \varphi = 0 \}
$$

(72)

$$
B = \{ f \in C^{k-1,\alpha}(M) \mid \int_M f = \text{Vol}(M) \}
$$

(73)

We note that $\Theta \subseteq C^{k+1,\alpha}$ open and $B \subseteq C^{k-1,\alpha}$ is a hyperplane. Furthermore $C^{k+1,\alpha}$ and $C^{k-1,\alpha}$ are Banach Spaces.

We define the Monge Ampère map $G : \Theta \to B$:

$$
G(\varphi) = \det \left( g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) (\text{det}(g_{ij})^{-1})
$$

(74)

This is a nonlinear Map between Banach Spaces. We compute its Fréchet Derivative. We let $A$ denote the matrix $\left( g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right)$ and $X$ the matrix $\left( \frac{\partial^2 \psi}{\partial z^i \partial \bar{z}^j} \right)$ where
\( \psi \in \Theta \). We recall a fact from linear algebra that to first order the derivative of the determinant of an invertible matrix is given by the trace:

\[
\det(A + \epsilon X) - \det(A) = \det(A) \text{Tr}(A^{-1} X) \epsilon + o(\epsilon^2)
\]

(75)

Furthermore on a Kähler Manifold the Laplace-Beltrami Operator is locally represented by:

\[
\Delta = -\sum_{i,j} g^{ij} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}
\]

(76)

Where \((g^{ij})\) represents the inverse matrix of the metric coefficient matrix \(g_{ij}\).

Hence the differential of \(G\) at the point \(\varphi_0\) is given by:

\[
G'(\varphi_0) = \det \left( g^{ij} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j} \right) (\det(g_{ij}))^{-1} \Delta \varphi_0
\]

(77)

Where \(G' : T_\psi \Theta \to T_{G(\psi)} \mathcal{B}\) is a map between the respective tangent spaces and \(\Delta \varphi_0\) is the Laplace-Beltrami Operator with respect to the metric \(g_{ij} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j}\).

Furthermore:

\[
T_\gamma \mathcal{B} \subseteq \{ f \in C^{k-1,\alpha}(M) \mid \int_M f = 0 \}
\]

(78)

We now state a lemma about the Laplace-Beltrami Operator on compact Riemannian Manifolds.

**Lemma 4.** Let \(\Delta\) be the Laplace-Beltrami operator on a compact Riemannian manifold \((M, g)\). Assume \(f : M \to \mathbb{R}\) is a smooth function. Then there exists unique solution (in the weak sense) \(u \in \mathcal{H}\) to the Poisson equation \(\Delta u = f\) where \(\mathcal{H} = \{ u \in H^{1,2}(M) \mid \int_M u = 0 \text{ and } \int_M f u = 1 \} \iff \int_M f = 0\).

This lemma implies that the Laplace Beltrami Operator is a bijection on the space of mean zero functions. Hence the condition we need to ensure that:

\[
\det \left( g^{ij} + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j} \right) (\det(g_{ij}))^{-1} \Delta \varphi_0 \varphi = f
\]

(79)

has a solution is that \(\int_M f = 0\). This equation is a Linear Second Order Elliptic PDE. Hence by Schauder Theory we know that \(u \in C^{k+1,\alpha}\). Furthermore by requiring that \(\int_M \varphi = 0\) we know that our solution is unique. Hence the differential of \(G, G'\) at \(\varphi_0\) is an isomorphism. By the Implicit Function Theorem for Banach Spaces \(G\) maps and open neighborhood of \(\varphi_0\) to an open neighborhood of \(G(\varphi_0)\). This shows that \(S\) is an open set.
We now show that $S$ is also a closed set. Let \( \{t_n\} \) be an arbitrary sequence in $S$. This gives rise to a sequence \( \{\varphi_n\} \in C^{k+1,\alpha} \) such that:

\[
\det \left( \frac{\partial^2 \varphi_n}{\partial z^i \partial \bar{z}^j} \right) \det (g_{ij})^{-1} = \operatorname{Vol}(M) \left[ \int_M \exp \{t_n F\} \right]^{-1} \exp (t_n F)
\]

Differentiating this equation with respect to $z$ we have:

\[
\det \left( \frac{\partial^2 \varphi_n}{\partial z^i \partial \bar{z}^j} \right) \sum_{i,j} g_{ij}^{\bar{n}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \frac{\partial \varphi_n}{\partial \bar{z}^p} \right) = \operatorname{Vol}(M) \left[ \int_M \exp \{t_n F\} \right]^{-1} \frac{\partial}{\partial \bar{z}^p} \left[ \exp (t_n F) \det (g_{ij}) \right]
\]

Where \( \left( g_{ij}^{\bar{n}} \right) \) is inverse matrix of \( \left( g_{ij} + \frac{\partial^2 \varphi_n}{\partial z^i \partial \bar{z}^j} \right) \) for all $n$.

We notice that the left hand side of this equation is a Linear Second Order Elliptic PDE with variable coefficients. In fact the coefficient matrix \( [a_{ij}] = [g_{ij}] \) = \( [g_{ij} + \frac{\partial^2 \varphi_n}{\partial z^i \partial \bar{z}^j}] \). By using Holomorphic Normal Coordinates and applying Yau’s 2nd Order Estimates we have $0 < C_3 \leq 1 + \varphi_{n, i} \leq C_4$ for all $i$. Hence the eigenvalues of the inverse matrix are bounded and the left hand side of (81) is uniformly elliptic. Yau’s Third Order Estimates imply that \( \varphi_n \in C^{2,\alpha} \). This implies that the coefficients of the operator on the left hand side and the functions on the right hand side are in fact Hölder continuous for every exponent $0 \leq \alpha \leq 1$. Hence by the Schauder Estimate we know that we have a priori $C^{2,\alpha}$ estimate for \( \frac{\partial \varphi_n}{\partial \bar{z}^p} \forall n \). Arguing in a similar fashion one shows that we have a priori $C^{2,\alpha}$ estimate for \( \frac{\partial^2 \varphi_n}{\partial z^i \partial \bar{z}^j} \forall n$.

Furthermore this implies that \( \varphi_n \in C^{3,\alpha} \). This gives us better differentiability properties for the coefficients of our Linear Second Order Elliptic operator. Appealing to Schauder Estimates we find a priori $C^{3,\alpha}$ estimate for \( \frac{\partial^2 \varphi_n}{\partial z^i \partial \bar{z}^j} \forall n \). We bootstrap this argument and iterate to find $C^{k+1,\alpha}$ estimates for \( \varphi_n \forall n \) where the constant is independent of $n$. Hence our sequence \( \{\varphi_n\} \) is uniformly bounded. One also readily sees that our functions are equicontinuous. Hence by the Arzela-Ascoli Theorem our sequence \( \{\varphi_n\} \) has a convergent subsequence in $S$. This implies that our set $S$ is closed. We have now shown the existence of a $\varphi \in C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$ to the Complex Monge Ampère Equation.

We are now in a position to prove the existence of a smooth solution $\varphi$ to the Complex Monge Ampère Equation. Recall that in our formulation of the Calabi Conjecture we had a compact complex manifold $M$ and a Kähler metric $g$ on $M$ with Kähler form $\omega$. Furthermore we assumed $F$ to be a smooth function on $M$. This implies $F \in C^k(M) \forall k$. Hence by our argument above we have that $\varphi \in C^\infty$. We have now followed Yau’s footsteps and provided an affirmative answer to the Conjecture of Calabi.

5. CALABI-YAU MANIFOLDS AND HOLonomy

Let us begin by first defining some necessary differential geometric concepts we will need, and proceed to mathematically define a Calabi-Yau Manifold and
give a proof of their existence. Since a Kähler Manifold is really a Riemannian Manifold with a compatible Complex Structure, we will focus our discussion of differential geometric concepts in the context of Riemannian Geometry. We assume \((M,g)\) is a Riemannian Manifold with Riemannian Metric \(g\). The notion of a connection is introduced on a Riemannian Manifold so that one has a suitable notion of differentiating vector fields. In fact it can be seen as a kind of covariant derivative (a natural generalization of a directional derivative from vector calculus) on the tangent bundle \(TM\).

**Theorem 13. (Existence of Unique Torsion-Free Connection)** Let \((M,g)\) be a Riemannian Manifold with Riemannian Metric \(g\). Then there exists a unique, torsion-free Connection \(\nabla\) on \(TM\) with \(\nabla g = 0\) called the Levi-Civita Connection.

We next consider the holonomy of a connection. Intuitively holonomy is a local representation of the curvature of our space. To understand the global geometry of an object one can send vectors around closed loops in a space (formal notion of parallel transport) and quantitatively measure how the initial vector and final vector differ. This failure to preserve geometric data around closed loops is what we mean by the holonomy of the connection. More specifically we choose points \(x, y \in M\). Let

\[
\gamma : [0, 1] \to M
\]

be a smooth curve with \(\gamma(0) = x\) and \(\gamma(1) = y\). The connection on our manifold allows us to transport vectors along this curve so that they remain parallel with respect to the connection. We define \(P_{\gamma}\) to be the parallel transport map. It is a linear and invertible map. Parallel transport is a way to locally move the geometry of our manifold along a curve. Fixing a point \(x \in M\) one defines the holonomy of our connection,

\[
\text{Hol}_{x}(\nabla) = \{P_{\gamma} \mid \gamma \text{ is a loop based at } x\} \subseteq GL(n, \mathbb{R})
\]

Furthermore one can easily see that \(\text{Hol}_{x}(\nabla)\) has a group structure and is independent of the basepoint \(x \in M\) “up to conjugation” if \(M\) is simply-connected: \(P_{x}\text{Hol}_{x}(\nabla)P_{y}^{-1} = \text{Hol}_{y}(\nabla)\) for any piecewise smooth map \(\gamma : [0, 1] \to M\) with \(\gamma(0) = x\) and \(\gamma(1) = y\).

If one considers \(x \in M\) and \(T_{x}M\) the tangent space at \(x\), one can show that the constant tensor \(g\) i.e. \(\nabla g = 0\) is preserved under the action of \(\text{Hol}_{x}(\nabla)\) on \(T_{x}M\) (acting on \(g|_{x}\)). Since \(O(n)\) are the group of transformations of \(T_{x}M\) preserving \(g|_{x}\), we know that,

\[
\text{Hol}_{x}(\nabla) \subseteq O(n)
\]

The holonomy group of a Riemannian Manifold is referred to as the Riemannian Holonomy Group.

If one instead considers only null-homotopic curves on our manifold (that is curves that are homotopic to the constant curve) then one can suitably define what is known as the Restricted Holonomy Group,

\[
\text{Hol}_{x}^{0}(\nabla) = \{P_{\gamma} \mid \gamma \text{ is a null-homotopic loop based at } x\} \subseteq GL(n, \mathbb{R})
\]
All the properties described above of the Holonomy group carry through. The two notions of Holonomy are equivalent on simply-connected Manifolds.

There is a classification theorem due to Marcel Berger [MB] that answers the question which subgroups of $O(n)$ can be the Holonomy group of some Riemannian Manifold of dimension $n$. Berger found that for generic Riemannian Manifolds $\text{Hol}(\nabla) = \text{SO}(n)$. Riemannian Metrics with $\text{Hol}(\nabla) \subseteq U(m)$ where $n = 2m$ are called Kähler Metrics. Riemannian Metrics with $\text{Hol}(\nabla) \subseteq SU(m)$ where $n = 2m$ are called Calabi-Yau Metrics. Hence for our purpose we now have a mathematical definition of a Calabi-Yau Manifold:

**Definition 9. (Calabi-Yau Manifold)** A Calabi-Yau Manifold is a compact Kähler Manifold $(M, J, g)$ of dimension $m \geq 2$ with $\text{Hol}(\nabla) \subseteq SU(m)$.

A consequence of $\text{Hol}(\nabla) \subseteq SU(m)$ is that the first Chern Class of our Manifold vanishes. We now state a lemma which combined with the proof of the Calabi-Conjecture will prove the existence of Calabi-Yau Manifolds,

**Lemma 5.** Let $(M, J, g)$ be a Kähler Manifold. Then $\text{Hol}^0(\nabla) \subseteq SU(m) \iff g$ is Ricci-flat.

Assuming that the first Chern Class of our Manifold vanishes, by the Calabi Conjecture we can find a unique metric in each Kähler Class with vanishing Ricci Form. Hence $\text{Hol}(\nabla) \subseteq SU(m)$ and we can prove the existence of Ricci-flat metrics.

6. **Concluding Remarks**

Having looked at the proof of the Calabi Conjecture we conclude this paper with a remark on what we feel it is important to look to in the future. We hope to provide a detailed explanation of the estimates obtained by Yau in the course of his proof, as well as explore the exciting world of Theoretical Physics and understand more fully the relationship between Calabi-Yau Manifolds and String Theory. We would also like to explore further the concept of Holonomy and topics related to the Uniformization of Kähler Manifolds. We hope to combine exploration of these topics with tools from PDE theory and see how much mileage one can get from a combination of these varying topics.

**References**