Gradient stability for the Sobolev inequality: the case $p \geq 2$

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Let $n \geq 2$ and $1 \leq p < n$. Then for all $u \in W^{1,p}$,

$$\| \nabla u \|_{L^p} \geq S \| u \|_{L^{p^*}}$$

where $p^* = \frac{np}{n-p}$. 
Extremal functions: If $p > 1$, then $\| \nabla v \|_{L^p} = S \| v \|_{L^{p^*}}$ for

$$v(x) = (1 + |x|^{p'})^{-(n-p)/p}.$$
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All equality cases are

$$\mathcal{M} = \{cv(\lambda(x-x_0)) : c \in \mathbb{R}, \lambda \in \mathbb{R}_+, x_0 \in \mathbb{R}^n\}.$$
Stability for the Sobolev inequality

- **Stability:** If $u \in W^{1,p}$ *almost* attains equality in the Sobolev inequality, then how close is $u$ to some $v \in \mathcal{M}$?
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\delta(u) = \|\nabla u\|_{L^p} - S_p \|u\|_{L^p}^* \geq 0.
\]
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- **Deficit**: \[
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\]

- The strongest distance we expect to control in this setting is \[
\inf_{v \in \mathcal{M}} \|\nabla (u - v)\|_{L^p}.
\]
Past stability results

- Bianchi, Egnell (1991), $p = 2$:
  \[ C\delta(u) \geq \inf_{v \in M} \|\nabla(u - v)\|_{L^2}^2. \]
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  \[ C \delta(u) \geq (\text{optimal gradient distance})^2. \]
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Gradient stability: $p \geq 2$

**Theorem (Figalli, N., 2015)**

Let $2 \leq p < n$. There exist $C$ and $\alpha$ such that

$$C \delta(u) \geq \inf_{v \in \mathcal{M}} \|\nabla(u - v)\|_{L^p}^\alpha.$$

The proof has two main steps:
- Positivity of the second variation
- Controlling higher order terms

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Gradient stability: \( p \geq 2 \)

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Positivity of the second variation

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d(u, \mathcal{M}) = \inf_{v \in \mathcal{M}} \left( \int | \nabla (u - v) |^2 | \nabla v |^{p-2} \right)^{1/2}.
\]
Positivity of the second variation

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We want to introduce a **Hilbert space structure**.

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Given \( u \in W^{1,p} \), suppose the infimum is attained at \( v \in \mathcal{M} \).

![Diagram showing a manifold \( \mathcal{M} \) with points \( v \) and \( u \).]
Positivity of the second variation

We want to introduce a Hilbert space structure.

\[ d(u, M) = \inf_{v \in M} \left( \int |\nabla (u - v)|^2 |\nabla v|^{p-2} \right)^{1/2} = \varepsilon \left( \int |\nabla \varphi|^2 |\nabla v|^{p-2} \right)^{1/2}. \]

- Given \( u \in W^{1,p} \), suppose the infimum is attained at \( v \in M \).
- Let \( \varepsilon \varphi = u - v \), with the normalization \( \int |\nabla \varphi|^p = 1 \).
Part 1: Positivity of the second variation

Expand $\delta(u)$ around $v$:

$$\delta(u) = \delta(v) + \varepsilon \text{ first variation} + \varepsilon^2 \text{ second variation} + o(\varepsilon^2).$$
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The second variation is:

$$\text{second variation} = \int |\nabla \varphi|^2 |\nabla v|^{p-2} - S \int |\varphi|^2 v^{p^* - 2}$$
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Expand this expression in a basis of eigenfunctions.
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$$\text{second variation} = \int |\nabla \varphi|^2 |\nabla v|^{p-2} - S \int |\varphi|^2 v^{p^* - 2}$$

Expand this expression in a basis of eigenfunctions. $\varphi$ is orthogonal to $T_v \mathcal{M}$, which allows us to exploit a spectral gap.

$$\varepsilon^2 \text{ second variation} \geq d(u, \mathcal{M})^2.$$
So

\[ \delta(u) \geq c d(u, M)^2 + o(\varepsilon^2) \]
The problem

So

$$\delta(u) \geq cd(u, \mathcal{M})^2 + o(\varepsilon^2)$$

Problem: Our higher order terms are not really higher order.

$$d(u, \mathcal{M})^2 = \int |\nabla (u - \nu)|^2 |\nabla \nu|^{p-2}, \quad \varepsilon^2 = \|\nabla (u - \nu)\|_{L^p}^2$$
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Problem: Our higher order terms are not really higher order.

\[ d(u, M)^2 = \int |\nabla(u - v)|^2|\nabla v|^{p-2}, \quad \varepsilon^2 = \|\nabla(u - v)\|_{L^p}^2 \]

Recall: \( \varepsilon \varphi = u - v, \int |\nabla \varphi|^p = 1. \)
A Taylor expansion of $\delta(u)$ cannot work, as we can never absorb the higher order terms.
Part 2: Handling the higher order terms

A Taylor expansion of $\delta(u)$ cannot work, as we can never absorb the higher order terms.

Two inequalities for numbers/vectors:

$$|a + \varepsilon b|^p \geq a^p + \varepsilon p a^{p-1} b + c \varepsilon^2 a^{p-2} b^2 - C \varepsilon^p b^p,$$
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Two inequalities

Apply these for $a = \nabla \nu$, $\varepsilon b = \varepsilon \nabla \varphi = \nabla (u - \nu)$, recalling the fact that second variation controls $d(u, \mathcal{M})^2$.

$$
\delta(u) \geq c d(u, \mathcal{M})^2 - C \varepsilon^p,
$$

$$
\delta(u) \geq -C d(u, \mathcal{M})^2 + c \varepsilon^p.
$$

Recall: $d(u, \mathcal{M})^2 = \int |\nabla (u - \nu)|^2 |\nabla \nu|^{p-2}$, $\varepsilon^p = \int |\nabla (u - \nu)|^p$. 
Case 1: $L^p$ norm dominates

If we are in the case

$$\varepsilon^p \gg d(u, M)^2,$$

Recall:

$$d(u, M)^2 = \int |\nabla (u - v)|^2 - 2 \varepsilon^p = \int |\nabla (u - v)|^p.$$
Case 1: $L^p$ norm dominates

If we are in the case

$$\varepsilon^p \gg d(u, \mathcal{M})^2,$$

then $\delta(u) \geq -Cd(u, \mathcal{M})^2 + c\varepsilon^p$ gives

$$\delta(u) \geq c \int |\nabla(u - v)|^p.$$

Recall: $d(u, \mathcal{M})^2 = \int |\nabla(u - v)|^2|\nabla v|^{p-2}$, $\varepsilon^p = \int |\nabla(u - v)|^p$. 
Case 2: $L^2$ norm dominates

On the other hand, if we are in the case

$$\varepsilon^p \ll d(u, \mathcal{M})^2,$$

then

$$\delta(u) \geq c d(u, \mathcal{M})^2 - C \varepsilon^p$$

implies that then

$$\delta(u) \geq c d(u, \mathcal{M})^2.$$

Then from

$$\delta(u) \geq -C d(u, \mathcal{M})^2 + c \varepsilon^p,$$

we have

$$C d(u, \mathcal{M})^2 + \delta(u) \geq c \varepsilon^p,$$

and so

$$\delta(u) \geq c \varepsilon^p = c \int |\nabla (u - v)|^p.$$

Recall:

$$d(u, \mathcal{M})^2 = \int |\nabla (u - v)|^2 |\nabla v|^p - 2,$$

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$$\delta(u) \geq c\varepsilon^p = c \int |\nabla(u - v)|^p.$$

Recall: $d(u, M)^2 = \int |\nabla(u - v)|^2|\nabla v|^{p-2}$, $\varepsilon^p = \int |\nabla(u - v)|^p$.  

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Stability is shown in two cases.

To summarize, if
\[ \int |\nabla (u - v)|^p \leq c_0 \int |\nabla (u - v)|^2 |\nabla v|^{p-2} \]
or\[ \int |\nabla (u - v)|^p \geq C_0 \int |\nabla (u - v)|^2 |\nabla v|^{p-2} \]
then
\[ \delta(u) \geq c \int |\nabla (u - v)|^p. \]
In other words, if $R(u) = \frac{\int |\nabla (u-v)^2| |\nabla v|^{p-2}}{\int |\nabla (u-v)|^p}$ is the ratio,
Interpolation

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R(u) \geq C_0 \quad \text{or} \quad R(u) \leq c_0.
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To handle the regime \( c_0 < R(u) < C_0 \), we consider the following linear interpolation:

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u_t = tu + (1-t)v.
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Interpolation

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To handle the regime

$$c_0 < R(u) < C_0,$$

we consider the following linear interpolation:

$$u_t = tu + (1 - t)v.$$
Then, since

\[ u_t - v = t(u - v), \]

we easily see that

\[ R(u_t) = \frac{1}{2} \int |\nabla(u - v)|^2 |\nabla v|^p - \frac{2}{p} \int |\nabla(u - v)|^p = t^2 - \frac{2}{p} R(u). \]

Therefore, if \( c_0 < R(u) < C_0 \),

\[ R(u_t) = t^2 - \frac{2}{p} R(u) \geq t^2 - \frac{2}{p} c_0 > C_0 \]

for \( t \) small enough.
Then, since
\[ u_t - v = t(u - v), \]
we easily see that
\[ R(u_t) = \int \frac{|t \nabla (u - v)|^2 |\nabla v|^{p-2}}{\int |t \nabla (u - v)|^p} = t^{2-p} R(u). \]
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for \( t \) small enough.
Recovering information about $u$

Therefore

$$\delta(u_t) \geq c \int |\nabla (u_t - v)|^p$$
We would like to show something like

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It's easy to recover information about the distance:

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For the deficit, it’s not clear that an estimate of the form

\[ C \delta(u) \geq \delta(u_t) \]

should hold,
Recovering information about \( u \)

We would like to show something like

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For the deficit, it’s not clear that an estimate of the form

\[
\delta(u_t) \leq C \delta(u)
\]

should hold, but we can show

\[
\delta(u_t) \leq C \delta(u) + C \|u - v\|_{L^p*}.
\]
Recovering information about $u$

So we have

$$C\|u - v\|_{L^p} + C\delta(u) \geq \delta(u_t) \geq c \int |\nabla (u_t - v)|^p \geq c \int |\nabla (u - v)|^p.$$  

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Recovering information about $u$

Thus we have

$$C\|u - v\|_{L^{p^*}} + C\delta(u) \geq \int |\nabla (u - v)|^p.$$

Pairing this with the result of Cianchi, Fusco, Maggi and Pratelli, we have

$$\|\nabla (u - v)\|_{L^p} \leq C\delta(u)^\alpha$$

concluding the proof.
Thank you for your attention!