A strong form of the quantitative Wulff inequality

Robin Neumayer

University of Texas at Austin

6th Symposium on Analysis and PDE
Purdue University
June 2, 2015
1 Previous stability results

2 The anisotropic case

3 Strong stability for the Wulff inequality
Isoperimetric inequality:

\[ P(E) \geq P(B) \text{ when } |E| = |B|, \]

with equality iff \( E + x_0 = B \).
The isoperimetric inequality

- **Isoperimetric inequality:**
  \[ P(E) \geq P(B) \quad \text{when} \quad |E| = |B|, \]
  with equality iff \( E + x_0 = B \).

- **Stability:** If \( E \) almost attains equality in the isoperimetric inequality, then how much does \( E \) “look like” \( B \)?
The isoperimetric inequality

- Isoperimetric inequality:

\[ P(E) \geq P(B) \text{ when } |E| = |B|, \]

with equality iff \( E + x_0 = B \).

- Stability: If \( E \) almost attains equality in the isoperimetric inequality, then how much does \( E \) “look like” \( B \)?
The isoperimetric inequality

- **Isoperimetric inequality:**

  \[ P(E) \geq P(B) \text{ when } |E| = |B|, \]

  with equality iff \( E + x_0 = B \).

- **Stability:** If \( E \) almost attains equality in the isoperimetric inequality, then how much does \( E \) “look like” \( B \)?
Deficit:

\[ \delta(E) = P(E) - P(B) \geq 0. \]
Stability for the isoperimetric inequality

Deficit:
\[ \delta(E) = P(E) - P(B) \geq 0. \]

Asymmetry:
\[ A(E) = \inf_{y \in \mathbb{R}^n} |(E + y) \Delta B|. \]
Stability for the isoperimetric inequality

Deficit:
\[ \delta(E) = P(E) - P(B) \geq 0. \]

Asymmetry:
\[ A(E) = \inf_{y \in \mathbb{R}^n} |(E + y) \Delta B|. \]
Deficit:
\[ \delta(E) = P(E) - P(B) \geq 0. \]

Asymmetry:
\[ A(E) = \inf_{y \in \mathbb{R}^n} |(E + y) \Delta B|. \]
Stability for the isoperimetric inequality

Deficit:

\[ \delta(E) = P(E) - P(B) \geq 0. \]

Asymmetry:

\[ A(E) = \inf_{y \in \mathbb{R}^n} |(E + y) \Delta B|. \]
Stability for the isoperimetric inequality

Theorem (Fusco-Maggi-Pratelli ’08; Figalli-Maggi-Pratelli ’10; Cicalese-Leonardi ’12)

There exists a constant \( C = C(n) \) such that

\[
A(E) \leq C \delta(E)^{1/2}
\]

for all sets \( E \) of finite perimeter with \( 0 < |E| < \infty \).

Recall: \( \delta(E) = P(E) - P(B) \), \( A(E) = |E \Delta B| \) up to translation.
Fuglede’s earlier, stronger stability result for perturbations

**Definition**

A set $E$ is a **nearly spherical set** if $|E| = |B|$, $\text{bar}(E) = \text{bar}(B)$, and

$$\partial E = \{x + u(x)x : x \in \partial B\}$$

for $u : \partial B \to \mathbb{R}$ with $\|u\|_{C^1} < \epsilon$. 

Robin Neumayer (UT Austin)  
Quantitative Wulff inequality  
June 2, 2015  6 / 25
Fuglede’s earlier, stronger stability result for perturbations

**Definition**

A set $E$ is a **nearly spherical set** if $|E| = |B|$, $\text{bar}(E) = \text{bar}(B)$, and

$$\partial E = \{ x + u(x)x : x \in \partial B \}$$

for $u : \partial B \to \mathbb{R}$ with $\|u\|_{C^1} < \epsilon$. 

---

![Diagram showing a nearly spherical set $E$ with $|E| = |B|$, $\text{bar}(E) = \text{bar}(B)$, and $\partial E = \{ x + u(x)x : x \in \partial B \}$ for a function $u : \partial B \to \mathbb{R}$ with $\|u\|_{C^1} < \epsilon$.](image)
Definition

A set $E$ is a nearly spherical set if $|E| = |B|$, $\text{bar}(E) = \text{bar}(B)$, and

$$\partial E = \{x + u(x)x : x \in \partial B\}$$

for $u : \partial B \to \mathbb{R}$ with $\|u\|_{C^1} < \epsilon$. 
Fuglede’s earlier, stronger stability result for perturbations

Theorem (Fuglede ’89)

There exists a constant $C = C(n)$ such that if $E$ is a nearly spherical set, then

$$\|u\|_{H^1} \leq C \delta(E)^{1/2}.$$

Robin Neumayer (UT Austin)
Fuglede’s earlier, stronger stability result for perturbations

For nearly spherical sets:

Fuglede:  $\|u\|_{H^1} \leq C \delta(E)^{1/2}$,

Fusco-Maggi-Pratelli:  $\|u\|_{L^1} \leq C \delta(E)^{1/2}$. 
Proof idea:

- Taylor expansion + volume constraint:

\[ \delta(E) = \int_{\partial B} |\nabla u|^2 - (n - 1)u^2 \, d\mathcal{H}^{n-1} + \epsilon O(\|u\|_{H^1}^2) , \]
Proof idea:

- Taylor expansion + volume constraint:

\[ \delta(E) = \int_{\partial B} |\nabla u|^2 - (n-1)u^2 \, d\mathcal{H}^{n-1} + \epsilon O(\|u\|_{H^1}^2), \]

- Expand \( u = \sum_{i=0}^{\infty} a_i Y_i \) in basis of spherical harmonics:

\[ \delta(E) = \sum_{i=0}^{\infty} \lambda_i a_i^2 - (n-1) \sum_{i=0}^{\infty} a_i^2 + \epsilon O(\|u\|_{H^1}^2), \]
Fuglede's earlier, stronger stability result for perturbations

Proof idea:

- Taylor expansion + volume constraint:

\[ \delta(E) = \int_{\partial B} |\nabla u|^2 - (n - 1)u^2 d\mathcal{H}^{n-1} + \epsilon O(\|u\|_{H^1}^2), \]

- Expand \( u = \sum_{i=0}^{\infty} a_i Y_i \) in basis of spherical harmonics:

\[ \delta(E) = \sum_{i=0}^{\infty} \lambda_i a_i^2 - (n - 1) \sum_{i=0}^{\infty} a_i^2 + \epsilon O(\|u\|_{H^1}^2), \]

- \( |E| = |B| \implies u \) orthogonal to dilations,
Fuglede’s earlier, stronger stability result for perturbations

Proof idea:

- **Taylor expansion + volume constraint:**

  \[ \delta(E) = \int_{\partial B} |\nabla u|^2 - (n - 1)u^2 \, d\mathcal{H}^{n-1} + \epsilon O(\|u\|_{H^1}^2), \]

- **Expand** \( u = \sum_{i=0}^{\infty} a_i Y_i \) in basis of spherical harmonics:

  \[ \delta(E) = \sum_{i=0}^{\infty} \lambda_i a_i^2 - (n - 1)\sum_{i=0}^{\infty} a_i^2 + \epsilon O(\|u\|_{H^1}^2), \]

- \( |E| = |B| \implies u \text{ orthogonal to dilations,} \)
- \( \text{bar}(E) = \text{bar}(B) \implies u \text{ orthogonal to translations,} \)
Fuglede’s earlier, stronger stability result for perturbations

Proof idea:

- Taylor expansion + volume constraint:

\[
\delta(E) = \int_{\partial B} |\nabla u|^2 - (n-1)u^2 \, d\mathcal{H}^{n-1} + \epsilon \, O(\|u\|_{H^1}^2),
\]

- Expand \( u = \sum_{i=0}^{\infty} a_i Y_i \) in basis of spherical harmonics:

\[
\delta(E) = \sum_{i=0}^{\infty} \lambda_i a_i^2 - (n-1) \sum_{i=0}^{\infty} a_i^2 + \epsilon \, O(\|u\|_{H^1}^2),
\]

- \( |E| = |B| \implies u \) orthogonal to dilations,
- \( \text{bar}(E) = \text{bar}(B) \implies u \) orthogonal to translations,
- Spectral gap: \( \lambda_2 > (n-1) \).
The question of Fusco and Julin:

Can the deficit control a stronger $H^1$-type quantity for sets that are not nearly spherical?
Fusco and Julin defined

\[
\beta(E) = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} |\nu_{E+y}(x) - \nu_B(x)\|_2^2 \, d\mathcal{H}^{n-1} \right)^{1/2}
\]
Fusco and Julin defined

$$\beta(E) = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} \left| \nu_{E+y}(x) - \nu_B \left( \frac{x}{|x|} \right) \right|^2 d\mathcal{H}^{n-1} \right)^{1/2}$$

$$= \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} 1 - \nu_{E+y}(x) \cdot \frac{x}{|x|} d\mathcal{H}^{n-1} \right)^{1/2}$$
Fusco and Julin defined

\[ \beta(E) = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} |\nu_{E+y} (x) - \nu_B (\frac{x}{|x|})|^2 d\mathcal{H}^{n-1} \right)^{1/2} \]

\[ = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} 1 - \nu_{E+y} (x) \cdot \frac{x}{|x|} d\mathcal{H}^{n-1} \right)^{1/2} \]
Fusco and Julin defined

$$\beta(E) = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} |\nu_{E+y}(x) - \nu_B(\frac{x}{|x|})|^2 \, d\mathcal{H}^{n-1} \right)^{1/2}$$

$$= \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} 1 - \nu_{E+y}(x) \cdot \frac{x}{|x|} \, d\mathcal{H}^{n-1} \right)^{1/2}$$
Fusco and Julin defined

\[
\beta(E) = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} |\nu_{E+y}(x) - \nu_{B}(\frac{x}{|x|})|^2 d\mathcal{H}^{n-1} \right)^{1/2}
\]

\[
= \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} 1 - \nu_{E+y}(x) \cdot \frac{x}{|x|} d\mathcal{H}^{n-1} \right)^{1/2}
\]
Using Fuglede to improve Fusco-Maggi-Pratelli

Theorem (Fusco-Julin ’14)

There exists $C = C(n)$ such that

$$\beta(E) \leq C\delta(E)^{1/2}$$

for all sets $E$ of finite perimeter such that $0 < |E| < \infty$.

Recall: $\delta(E) = P(E) - P(B)$, $\beta(E) = H^1$-type quantity.
Fix a bounded convex set $K \ni 0$ (Wulff shape).
The anisotropic perimeter

- Fix a bounded convex set $K \ni 0$ (Wulff shape).
- $f(\nu) = \sup\{x \cdot \nu : x \in K\}$

$$f^*(x) = \inf\{\lambda \in \mathbb{R} : x \lambda \in K\}$$

$$K = \{f^*(x) < 1\}.$$
The anisotropic perimeter

- Fix a bounded convex set $K \ni 0$ (Wulff shape).
- $f(\nu) = \sup \{ x \cdot \nu : x \in K \}$
The anisotropic perimeter

- Fix a bounded convex set $K \ni 0$ (Wulff shape).
- $f(\nu) = \sup\{x \cdot \nu : x \in K\}$
The anisotropic perimeter

- Fix a bounded convex set $K \ni 0$ (Wulff shape).
- $f(\nu) = \sup \{ x \cdot \nu : x \in K \}$
Fix a bounded convex set $K \ni 0$ (Wulff shape).

- $f(\nu) = \sup\{x \cdot \nu : x \in K\}$
- $f^*(x) = \inf\{\lambda \in \mathbb{R} : \frac{x}{\lambda} \in K\}$
The anisotropic perimeter

- Fix a bounded convex set $K \ni 0$ (Wulff shape).
- $f(\nu) = \sup\{x \cdot \nu : x \in K\}$
- $f_*(x) = \inf\{\lambda \in \mathbb{R} : \frac{x}{\lambda} \in K\}$
- $K = \{f_*(x) < 1\}$. 

**Diagram:**

- A convex set $K$.
- A vector $\nu$.
- A point $x$.
- The anisotropic perimeter $f_*(x)$.
- The Wulff shape $K$. 

Robin Neumayer (UT Austin)

Quantitative Wulff inequality

June 2, 2015 13 / 25
Fix a bounded convex set $K \ni 0$ (Wulff shape).

- $f(\nu) = \sup\{x \cdot \nu : x \in K\}$
- $f_*(x) = \inf\{\lambda \in \mathbb{R} : \frac{x}{\lambda} \in K\}$
- $K = \{f_*(x) < 1\}$.
- $x \cdot \nu \leq f_*(x)f(\nu)$
The anisotropic perimeter

- Fix a bounded convex set $K \ni 0$ (Wulff shape).

- $f(\nu) = \sup\{x \cdot \nu : x \in K\}$

- $f_*(x) = \inf\{\lambda \in \mathbb{R} : \frac{x}{\lambda} \in K\}$

- $K = \{f_*(x) < 1\}$.

- $x \cdot \nu \leq f_*(x)f(\nu)$
Anisotropic perimeter:

\[ P_K(E) = \int_{\partial E} f(\nu_E) \, d\mathcal{H}^{n-1}, \]

with \( f(\nu) = \sup \{ x \cdot \nu : x \in K \} \).
### The Wulff inequality

**Anisotropic perimeter:**

\[ P_K(E) = \int_{\partial E} f(\nu_E) \, d\mathcal{H}^{n-1}, \]

with \( f(\nu) = \sup\{x \cdot \nu : x \in K\} \).

**Wulff inequality:**

\[ P_K(E) \geq P_K(K) \text{ when } |E| = |K|, \]

with equality iff \( E + x_0 = K \).
The Wulff inequality

Anisotropic perimeter:

\[ P_K(E) = \int_{\partial E} f(\nu_E) \, d\mathcal{H}^{n-1}, \]

with \( f(\nu) = \sup \{ x \cdot \nu : x \in K \} \).

Wulff inequality:

\[ P_K(E) \geq P_K(K) \text{ when } |E| = |K|, \]

with equality iff \( E + x_0 = K \).

Stability: (Figalli-Maggi-Pratelli)

\[ A(E) \leq C \, \delta(E)^{1/2} \]

where now \( \delta(E) = P_K(E) - P_K(K) \), \( A(E) = \inf_{y \in \mathbb{R}^n} |(E + y) \Delta K| \).
Can we obtain an analog of the Fuglede result in the case of the Wulff inequality? Should it depend on regularity or convexity properties of $K$?
Can we obtain an **analog of the Fuglede result** in the case of the Wulff inequality? Should it depend on regularity or convexity properties of $K$?

Can we obtain an **analog of the Fusco-Julin result** in this case? What form should such a result take?
Question 1: Can we obtain an analog of the Fuglede result in the case of the Wulff inequality?

**Definition**

Suppose $K$ is $C^2$. A set $E$ is a **nearly-$K$ set** if $|E| = |K|$, $\overline{E} = \overline{K}$, and

$$\partial E = \{x + u(x)\nu_K(x) : x \in \partial K\}$$

where $u : \partial K \to \mathbb{R}$ has $\|u\|_{C^1(\partial K)} < \epsilon$. 
Question 1: Can we obtain an analog of the Fuglede result in the case of the Wulff inequality?

**Definition**

Suppose $K$ is $C^2$. A set $E$ is a nearly-$K$ set if $|E| = |K|$, $\text{bar}(E) = \text{bar}(K)$, and

$$\partial E = \{ x + u(x)\nu_K(x) : x \in \partial K \}$$

where $u : \partial K \to \mathbb{R}$ has $\|u\|_{C^1(\partial K)} < \epsilon$. 

Question 1: Can we obtain an analog of the Fuglede result in the case of the Wulff inequality?

**Definition**

Suppose $K$ is $C^2$. A set $E$ is a nearly-$K$ set if $|E| = |K|$, $\mathrm{bar}(E) = \mathrm{bar}(K)$, and

$$
\partial E = \{ x + u(x)\nu_K(x) : x \in \partial K \}
$$

where $u : \partial K \to \mathbb{R}$ has $\|u\|_{C^1(\partial K)} < \epsilon$. 

Robin Neumayer (UT Austin)  
Quantitative Wulff inequality  
June 2, 2015 16 / 25
Question 1: Can we obtain an analog of the Fuglede result in the case of the Wulff inequality?

**Definition**

Suppose $K$ is $C^2$. A set $E$ is a nearly-$K$ set if $|E| = |K|$, $\text{bar}(E) = \text{bar}(K)$, and

$$\partial E = \{ x + u(x)\nu_K(x) : x \in \partial K \}$$

where $u : \partial K \to \mathbb{R}$ has $\|u\|_{C^1(\partial K)} < \epsilon$. 
Question 1: Can we obtain an analog of the Fuglede result in the case of the Wulff inequality?

Theorem (N., 2015)

Suppose $K$ is $C^2$ and uniformly convex. Then there exists $C = C(n, K)$ such that if $E$ is a nearly-$K$ set, then

$$\|u\|_{H^1} \leq C \delta(E)^{1/2}.$$
Question 1: Can we obtain an analog of the Fuglede result in the case of the Wulff inequality?

Proof Idea:

- Taylor expansion + volume constraint:

\[
\delta(E) = \int_{\partial K} (\nabla u)^T \nabla^2 f(\nu_K) \nabla u \, d\mathcal{H}^{n-1} \\
- \int_{\partial K} H_K u^2 \, d\mathcal{H}^{n-1} + \epsilon O(\|u\|_{H^1(\partial K)}).
\]
Question 1: Can we obtain an analog of the Fuglede result in the case of the Wulff inequality?

Proof Idea:
- Taylor expansion + volume constraint:
  \[
  \delta(E) = \int_{\partial K} (\nabla u)^T \nabla^2 f(\nu_K) \nabla u \, d\mathcal{H}^{n-1} - \int_{\partial K} H_K u^2 \, d\mathcal{H}^{n-1} + \epsilon O(\|u\|_{H^1(\partial K)}).
  \]

- Biggest challenge: No explicit information about spectrum.
Question 1: Can we obtain an analog of the Fuglede result in the case of the Wulff inequality?

- We cannot expect to understand the spectrum of

\[ \mathcal{L}u = -\text{div}(\nabla^2 f(\nu_K) \nabla u) \]

explicitly.
Question 1: Can we obtain an analog of the Fuglede result in the case of the Wulff inequality?

- We cannot expect to understand the spectrum of
  \[ \mathcal{L}u = -\text{div}(\nabla^2 f(\nu_K) \nabla u) \]
  explicitly.

- However, Figalli-Maggi-Pratelli \( \implies \) a spectral gap exists:
  \[
  \delta(E) = \int_{\partial K} (\nabla u)^T \nabla^2 f(\nu_K) \nabla u \, d\mathcal{H}^{n-1} - \int_{\partial K} H_K u^2 \, d\mathcal{H}^{n-1} \\
  \geq c \left( \int_{\partial K} |u| \, d\mathcal{H}^{n-1} \right)^2.
  \]
Question 2: Can we obtain an analog of the Fusco-Julin result in this case?

How to define the anisotropic oscillation index $\beta$?

$$\beta(E) = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} f(\nu_E) - \nu_E(x) \cdot \frac{x}{f_*(x)} d\mathcal{H}^{n-1} \right)^{1/2}$$
Question 2: Can we obtain an analog of the Fusco-Julin result in this case?

How to define the anisotropic oscillation index $\beta$?

$$\beta(E) = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} f(\nu_E) - \nu_E(x) \cdot \frac{x}{f_*(x)} d\mathcal{H}^{n-1} \right)^{1/2}$$

Recall $f(\nu) = \sup \{ x \cdot \nu : x \in K \}, \quad \frac{x}{f_*(x)} \cdot \nu \leq f(\nu)$, with equality iff $\nu = \nu_K(\frac{x}{f_*(x)})$. 
Question 2: Can we obtain an analog of the Fusco-Julin result in this case?

How to define the anisotropic oscillation index $\beta$?

$$\beta(E) = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} f(\nu_E) - \nu_E(x) \cdot \frac{x}{f_*(x)} d\mathcal{H}^{n-1} \right)^{1/2}$$

Recall $f(\nu) = \sup \{ x \cdot \nu : x \in K \}$, $\frac{x}{f_*(x)} \cdot \nu \leq f(\nu)$, with equality iff $\nu = \nu_K(\frac{x}{f_*(x)})$. 

Robin Neumayer  (UT Austin)
Question 2: Can we obtain an analog of the Fusco-Julin result in this case?

How to define the anisotropic oscillation index $\beta$?

$$\beta(E) = \inf_{y \in \mathbb{R}^n} \left( \int_{\partial E} f(\nu_E) - \nu_E(x) \cdot \frac{x}{f^*(x)} \, d\mathcal{H}^{n-1} \right)^{1/2}$$

Recall $f(\nu) = \sup \{ x \cdot \nu : x \in K \}$, $\frac{x}{f^*(x)} \cdot \nu \leq f(\nu)$, with equality iff $\nu = \nu_K(\frac{x}{f^*(x)})$. 

![Diagram of E, K, and ν_E(y), ν_E(x), f*(x), y]
Question 2: Can we obtain an analog of the Fusco-Julin result in this case?

**Theorem (N., 2015)**

Suppose $K$ is a uniformly convex, $C^2$ Wulff shape. Then there exists $C = C(n, K)$ such that

$$\beta(E) \leq C \delta(E)^{1/2}$$

for any set of finite perimeter $E$ with $0 < |E| < \infty$.

Recall: $\delta(E) = P_K(E) - P_K(K)$, $\beta(E) =$ anisotropic $H^1$-type quantity.
Strong Stability in the Smooth Case

Proof idea:

- Selection Principle in the spirit of Cicalese-Leonardi and Fusco-Julin to reduce to the case of almost-minimizers of $K$-perimeter.
Proof idea:

- **Selection Principle** in the spirit of Cicallese-Leonardi and Fusco-Julin to reduce to the case of almost-minimizers of $K$-perimeter.

- **Regularity theory** for almost-minimizers plus $L^1$ closeness to $K \in C^2$ allows us to reduce to sets that are small normal $C^1$ perturbations of $K$. 
Strong Stability in the Smooth Case

Proof idea:

- Selection Principle in the spirit of Cicalese-Leonardi and Fusco-Julin to reduce to the case of almost-minimizers of $K$-perimeter.

- Regularity theory for almost-minimizers plus $L^1$ closeness to $K \in C^2$ allows us to reduce to sets that are small normal $C^1$ perturbations of $K$.

- Apply the Fuglede-type theorem.
Theorem (N., 2015)

There exists a constant \( C = C(n) \) such that

\[
\beta(E) \leq C \delta(E)^{\gamma}
\]

\[\gamma = 1/(4 + 4n), \text{ for all sets of finite perimeter } E \text{ such that } 0 < |E| < \infty.\]
Strong stability results

Theorem (N., 2015)

There exists a constant $C = C(n)$ such that

$$\beta(E) \leq C \delta(E)^\gamma$$

$$\gamma = 1/(4 + 4n), \text{ for all sets of finite perimeter } E \text{ such that } 0 < |E| < \infty.$$ 

Theorem (N., 2015)

Let $K$ be a convex polygon in $\mathbb{R}^2$. There exists a constant $C = C(K)$ such that

$$\beta(E) \leq C \delta(E)^{1/2}$$
Proof idea:

- Again, we use a Selection Principle as in Cicalese-Leonardi and Fusco-Julin to reduce to the case of almost-minimizers of $K$-perimeter.
Proof idea:

- Again, we use a Selection Principle as in Cicalese-Leonardi and Fusco-Julin to reduce to the case of almost-minimizers of $K$-perimeter.
- With no assumptions on $K$, we only get **very weak regularity properties**.
Proof idea:

- Again, we use a Selection Principle as in Cicalese-Leonardi and Fusco-Julin to reduce to the case of almost-minimizers of $K$-perimeter.
- With no assumptions on $K$, we only get very weak regularity properties.
- For 2-d polygonal case, Figalli-Maggi rigidity result $\implies$ Coarea formula, explicit computation.
Thank you for your attention!