

# HOMWORK ASSIGNMENT 1 NOTES

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## 1. BEFORE WE START

When we solve problems, one important aspect of problem solving that is often overlooked is believing that you can solve the problem. If you do not think that you can do these problems, then you need to convince yourself otherwise. You **must** be confident in your ability.

## 2. SECTION 10.2

This section deals primarily with the question of whether or not a sequence of real numbers is bounded and/or monotone. Recall that a sequence is **bounded** if you can find real numbers  $M_1$  and  $M_2$  such that

$$M_1 \leq a_n \leq M_2$$

A sequence is monotone if it satisfies one of the following:

(1)

$$a_n \leq a_{n+1}$$

(2)

$$a_{n+1} \leq a_n$$

In the first case, we say the sequence  $a_n$  is **nondecreasing**. If the inequality is strict, we say the sequence is **increasing**.

In the second case, we say the sequence  $a_n$  is **nonincreasing**. If the inequality is strict, we say the sequence is **decreasing**.

Lastly, we say a sequence  $a_n$  is **eventually** monotone if there exists  $N$ , a positive integer, such that the sequence  $a_n$  is monotone when  $n \geq N$ .

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*Date:* January 20, 2002.

2.1. **Problems 1-8.** These problems are pattern recognition coupled with testing your ability to construct a formula.

**Example 2.1.** Consider the sequence

$$2, 5, 10, 17, 26, \dots$$

We see that

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 2 + 3 \\ a_3 &= 2 + 3 + 5 \\ a_4 &= 2 + 3 + 5 + 7 \\ a_5 &= 2 + 3 + 5 + 7 + 9 \\ &\vdots \\ a_n &= 2 + 3 + 5 + \dots + (2n - 1) \end{aligned}$$

We will see later that we can write this in a more compact notation, by

$$a_n = 2 + \sum_{i=2}^n (2i - 1)$$

2.2. **Problems 9-40.** These problems comprise the bulk of the first assignment. We work a few examples:

**Example 2.2.** Consider the sequence

$$a_n = \frac{1}{n!}$$

Now, since the denominator is growing and the numerator is fixed, we would conjecture that the numbers in the sequence would get small. This tells us that the sequence should be bounded. We show this as follows:

$$\begin{aligned} 1 &\leq n!, & \text{if } n \geq 1 \\ 1 &\geq \frac{1}{n!}, & \text{if } n \geq 1 \end{aligned}$$

So we have shown that the sequence is bounded above by 1. As for below, since the sequence is positive, we have

$$0 < \frac{1}{n!}$$

So that we have shown the sequence is bounded.

For monotone, we have two possible techniques that we could try. We could take derivatives of the function we get by replacing  $n$  with  $x$ , or we can look at ratios.

**Important:** Any time you see the factorial,  $n!$ , **YOU SHOULD ALWAYS THINK RATIO!**

So, we consider the ratio

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}$$

This equals

$$\frac{1}{(n+1)!} \cdot \frac{n!}{1}$$

Now, the reason we think of ratios with the factorial is the following:

$$\boxed{(n+1)! = (n+1)n!}$$

So, using this, we get

$$\frac{1}{(n+1)n!} \cdot \frac{n!}{1} = \frac{1}{n+1}$$

Since  $n \geq 1$ , we know that  $n+1 > 1$ . Thus

$$\frac{1}{n+1} < 1$$

So, our sequence is monotone decreasing.

**2.3. Problems 41 and 42.** Here is a hint to these problems. Do Problem 41 first. Try using the ideas we used in the above example involving the factorial. Then try doing problem 42 like problem 41.

### 3. SECTION 10.3

You need to read section 10.3 if you have not. Some important results are the following:

**Theorem 3.1.** *Every convergent sequence is bounded.*

A very important result follows by taking the contrapositive of the above theorem.

**Theorem 3.2.** *Every sequence that is unbounded is not convergent. That is, unbounded sequence are divergent.*

This is quite useful in showing a sequence is not convergent. The **Pinching Theorem** or **Squeeze Theorem** is important as well. In fact, you can use this to do almost every problem in 1-36.

3.1. **Problems 1-36.** Try using the Squeeze Theorem and Theorem 3.2.

3.2. **Problems 51-58.** Try rewriting the sequences in a more familiar form.

**Example 3.1** (Problem 51). We are given that  $a_1 = 1$  and

$$a_{n+1} = \frac{1}{e}a_n$$

Writing out the first few terms,

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{1}{e} \cdot 1 = \frac{1}{e} = \frac{1}{e^{2-1}} \\ a_3 &= \frac{1}{e}(a_2) = \frac{1}{e} \cdot \frac{1}{e} = \frac{1}{e^2} = \frac{1}{e^{3-1}} \\ a_4 &= \frac{1}{e}(a_3) = \frac{1}{e} \cdot \frac{1}{e^2} = \frac{1}{e^3} = \frac{1}{e^{4-1}} \\ &\vdots \\ a_n &= \frac{1}{e^{n-1}} \end{aligned}$$

So that the sequence is really

$$a_n = \frac{1}{e^{n-1}}$$

We know that

$$1 \leq e^{n-1}$$

when  $n \geq 1$ . Recall that

$$e \approx 2.71818 > 1$$

So,

$$\frac{1}{e^{n-1}} \leq 1$$

Also, we obviously have

$$0 < \frac{1}{e^{n-1}}$$

So that our sequence is bounded.

To so that the sequence is monotone, consider the ratio of consecutive terms.

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{1}{e^n}}{\frac{1}{e^{n-1}}} \\ &= \frac{1}{e^n} \cdot \frac{e^{n-1}}{1} \\ &= \frac{1}{e e^{n-1}} \cdot \frac{e^{n-1}}{1} \\ &= \frac{1}{e} < 1 \end{aligned}$$

So that our sequence is monotone decreasing. Thus, our sequence converges.

**Remark 1.** Notice, when we have a sequence of the form

$$\frac{1}{c^n}$$

where  $c$  is a real number, that looking at ratios works quite well. Keep this as a general philosophy for these and later problems.

Now, the question wants us to find the limit. If you notice, we did a lot of work just to show it was convergent.

**Moral:** Monotone and bounded does not give you the limit.

We could have just noticed that the sequence

$$\frac{1}{e^{n-1}}$$

has constant numerator and growing denominator. Thus, the sequence converges to zero. We could also use the squeeze theorem (though I think it is a bit artificial), and

$$0 < \frac{1}{e^{n-1}} \leq \frac{1}{n}$$

You would need to verify that

$$e^{n-1} \geq n$$

for all positive integers  $n$ .

Also, notice that from the definition of our sequence (at the beginning of the problem), we can see that it is decreasing, since

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{e} a_n}{a_n} = \frac{1}{e} < 1$$

## 4. BONUS PROBLEMS

I would like the bonus to be turned in before the first exam. When all the bonus problems are turned in, I will send out solutions.

**Bonus.** An important asymptotic relation for the factorial is the following, which is a special case of **Stirling's Formula**:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

where  $\sim$  means

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1$$

Use this to show the sequence

$$a_n = \frac{n!}{n^{2n}}$$

is bounded. Does this sequence converge?

**Bonus.** Here is an easier question. Give an example for each of the following:

- (a) A sequence of real numbers that is bounded but not convergent.
- (b) A sequence of real numbers,  $a_n$  such that

$$a_n = 1$$

an infinite number of times, but the sequence is not bounded.

- (c) A sequence of real numbers,  $a_n$  such that  $a_n^2$  is a convergent sequence but  $a_n$  is not.
- (d) A sequence of real numbers,  $a_n$  such that

$$b_n = a_{n+1} - a_n$$

is a convergent sequence but  $a_n$  is not.

- (e) Sequences of real numbers  $a_n$  and  $b_n$  such that  $a_n$  and  $b_n$  are both convergent sequences but

$$c_n = \frac{a_n}{b_n}$$

is not.

## REFERENCES

[Nickalls] Salas, Hille, and Etgen. *Calculus: One and Several Variables*. Wiley and Sons. 1999.

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