408 D, SPRING 2002
ASSIGNMENT 10

DAVID BEN MCREYNOLDS

1. Chapter 11, §7

Solution (Problem # 8)**.** Here

$$a_k = \frac{(-1)^k}{\sqrt{k}}$$

and so

$$\begin{aligned}\lim_{k\longrightarrow\infty} \sqrt[k]{|a_k|}\,|x| &= \lim_{k\longrightarrow\infty} \frac{1}{k^{1/2k}}\,|x| \\ &= |x| \lim_{k\longrightarrow\infty} \left(k^{1/k}\right)^{-2} \\ &= |x|\,(1)^{-2} \\ &= |x|\end{aligned}$$

Since the Root test says that we want this to be smaller than one, we see the that

$$|x| < 1$$

Now, it remains only to check the end points $x = \pm 1$. If $x = 1$, then our series is

$$\sum \frac{(-1)^k}{\sqrt{k}}$$

which converges by the Alternating Series Test. If $x = -1$, then our series is

$$\sum \frac{(-1)^{2k}}{\sqrt{k}} = \sum \frac{1}{\sqrt{k}}$$

which diverges by the direct comparison test with the harmonic series. Thus, our interval of convergence is $(-1, 1]$.

Solution (Problem # 28)**.** Here

$$a_k = \frac{\ln k}{k}$$

Date: March 3, 2002.

Now, observe that

$$1 < \ln k < k$$

$$1^{1/k} = 1 < \Big(\ln k\Big)^{1/k} < k^{1/k}$$

So that by the squeeze theorem,

$$\lim_{k \longrightarrow \infty} \Big(\ln k\Big)^{1/k} = 1$$

Using this, we see that

$$\begin{aligned} \lim_{k \longrightarrow \infty} \sqrt[k]{|a_k|}\,|x| &= |x+1| \lim_{k \longrightarrow \infty} \frac{\Big(\ln k\Big)^{1/k}}{k^{1/k}} \\ &= |x+1| \quad \text{(The top and bottom go to 1)} \end{aligned}$$

Thus, we want

$$\begin{aligned} |x+1| &< 1 \\ -1 < x+1 &< 1 \\ -2 < x &< 0 \end{aligned}$$

We are left to consider the end points, $x = -2$ and $x = 0$. If $x = 0$, we have the series

$$\sum \frac{\ln k}{k}$$

which diverges from the direct comparison test with the harmonic series. If $x = -2$, we have

$$\sum \frac{(-1)^k \ln k}{k}$$

which converges by the Alternating Series Test. So the interval of convergence is $(-2, 0]$.

2. Chapter 11, §8

Solution (Problem # 12)**.** We are asked to expand

$$f(x) = x^2 \tan^{-1} x$$

in powers of x. We could derive the power series of $\tan^{-1} x$ using the fact that

$$\tan^{-1} x + C = \int \frac{dx}{1+x^2}$$

However, since the textbook does this, we omit such. We are given

$$\tan^{-1} x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{2k-1}$$

Then, we have

$$\begin{aligned} x^2 \tan^{-1} x &= x^2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{2k-1} \\ &= \sum_{k=1}^{\infty} x^2 \frac{(-1)^{k+1} x^{2k-1}}{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2k-1} \end{aligned}$$

We conclude with a remark. The above expansion of $\tan^{-1} x$ is valid on the interval $[-1, 1]$. Where is the power series of $x^2 \tan^{-1} x$ valid?

Solution. We are asked to estimate

$$\int_0^1 x^4 e^{-x^2} dx$$

within 0.01.

First, recall that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

so that

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}$$

and

$$x^4 e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+4}}{k!}$$

Now, this expansion is valid on $(-\infty, \infty)$. Then, we have

$$\begin{aligned} \int_0^1 x^4 e^{-x^2} dx &= \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+4}}{k!} \right) dx \\ &= \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k x^{2k+4}}{k!} dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+5}}{(2k+5)k!} \Bigg]_0^1 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+5)k!} \end{aligned}$$

Now, the last series is an alternating series which is convergent, further

$$\int_0^1 x^4 e^{-x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+5)k!}$$

So that we have

$$\left| S_n - \int_0^1 x^4 e^{-x^2} dx \right| \leq a_{n+1}$$

$$\begin{aligned} \left| \sum_{k=0}^{n} \frac{(-1)^k}{(2k+5)k!} - \int_0^1 x^4 e^{-x^2} dx \right| &\leq \frac{1}{(2(n+1)+5)(n+1)!} \\ &= \frac{1}{(2n+7)(n+1)!} \end{aligned}$$

So that if

$$\frac{1}{(2n+7)(n+1)!} < \frac{1}{100}$$

then S_n estimates the integral in question to within 0.01. So that we consider

$$\begin{gathered} \frac{1}{(2n+7)(n+1)!} < \frac{1}{100} \\ (2n+7)(n+1)! > 100 \end{gathered}$$

If $n = 2$, the righthand side is only 66. But, if $n = 3$, we win. So that

$$S_3 = \frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{66}$$

estimates the integral to within 0.01.

University of Texas at Austin, Department of Mathematics
E-mail address: `dmcreyn@math.utexas.edu`