ASSIGNMENT 4 SOLUTIONS

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Solution (Exercise 4, 10.7)**.** We want to evaluate the integral

$$\int_0^\infty e^{-px}dx, \quad p > 0$$

So that we see

$$\begin{aligned}\lim_{b\longrightarrow\infty}\int_0^b e^{-px}dx &= \lim_{b\longrightarrow\infty}\left[-\frac{1}{p}(e^{-pb}-1)\right]\\ &= \frac{1}{p}\end{aligned}$$

Thus,

$$\boxed{\int_0^\infty e^{-px}dx = \frac{1}{p}}$$

Solution (Exercise 10, 10.7)**.** We want to evaluate the integral

$$\int_0^1 \frac{dx}{\sqrt{1-x}}$$

Now, observe that the integrand is undefined when $x = 1$, since we have division by zero. So we must take a limit. Then

$$\begin{aligned}\lim_{b\longrightarrow 1^+}\int_0^b \frac{dx}{\sqrt{1-x}} &= \lim_{b\longrightarrow 1^+}\left[-2(1-x)^{1/2}\right]_0^b\\ &= \lim_{b\longrightarrow 1^+} -2\left(\sqrt{1-b}-1\right)\\ &= 2\end{aligned}$$

Thus,

$$\boxed{\int_0^1 \frac{dx}{\sqrt{1-x}} = 2}$$

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Solution (Exercise 14, 10.7)**.** We want to evaluate the integral

$$\int_e^\infty \frac{dx}{x\ln x}$$

The issue here is the standard, off to infinity, business. That is, there are no undefined points. Then

$$\lim_{b\longrightarrow\infty}\int_e^b \frac{dx}{x\ln x}$$

Now, letting $u = \ln x$, we have

$$du = \frac{dx}{x}$$

So that our integral becomes

$$\int \frac{du}{u} = \ln|u|$$

Substituting for u, we get

$$\lim_{b\longrightarrow\infty}\Bigg[\ln\ln b - \ln\ln e\Bigg]$$

which diverges, since

$$\lim_{x\longrightarrow\infty}\ln\ln x = \infty$$

Thus

$$\boxed{\int_e^\infty \frac{dx}{x\ln x} \quad \text{diverges}}$$

Solution (Exercise 22, 10.7)**.** We want to evaluate the integral

$$\int_{-\infty}^0 xe^x$$

Standard deal, except the problem is at the left endpoint. let us compute the indefinite integral

$$\int xe^x dx$$

Now, using integration by parts, we let

$$u = x \quad dv = e^x dx$$

Then

$$du = dx \quad v = e^x$$

So that

$$\int xe^x dx = xe^x - \int e^x dx$$
$$= xe^x - e^x$$

Then

$$\lim_{a \longrightarrow -\infty} \int_a^0 xe^x dx = \lim_{a \longrightarrow -\infty} \left[xe^x - e^x \right]_a^0$$
$$= \lim_{a \longrightarrow -\infty} -1 - ae^a - e^a$$
$$= \lim_{b \longrightarrow \infty} -1 + \frac{b}{e^b} - \frac{1}{e^b}$$

Now, the first and last term are no problem, with the last going to zero. The middle term can be shown to go to zero using L'hopital's rule. Thus, we obtain

$$\boxed{\int_{-\infty}^0 xe^x dx = -1}$$

Notice, that since

$$xe^x < 0, \quad \text{when} \quad x < 0$$

You should expect this integral to be negative.

Solution (Exercise 34, 10.7)**.** We want to evaluate the integral

$$\int_0^{\pi/2} \frac{\cos x dx}{\sqrt{\sin x}}$$

Now, letting $u = \sin x$, we get

$$du = \cos x dx$$

Thus, the integral becomes

$$\int \frac{du}{\sqrt{u}} = 2\sqrt{u}$$

Substituting for u, we get

$$\int \frac{\cos x dx}{\sqrt{\sin x}} = 2\sqrt{\sin x}$$

Then

$$\lim_{a \longrightarrow 0^+} \int_a^{\pi/2} \frac{\cos x dx}{\sqrt{\sin x}} = \lim_{a \longrightarrow 0^+} \left[2\sqrt{\sin x} \right]_a^{\pi/2}$$
$$= \lim_{a \longrightarrow 0^+} 2 - 2\sqrt{\sin a}$$
$$= 2$$

Thus,

$$\boxed{\int_0^{\pi/2} \frac{\cos x dx}{\sqrt{\sin x}} = 2}$$

Notice that

$$\frac{\cos x}{\sqrt{\sin x}} > 0, \quad 0 < x < \pi/2$$

You would expect the integral to be positive.

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