BONUS 8

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0.1. Let f be a differentiable vector-valued function and assume that $\|f(t)\| \neq 0$ for all t. Show

$$\frac{d}{dt}\|f(t)\| = \frac{f(t)\cdot f'(t)}{\|f(t)\|}.$$

[Hint: Write the norm of a vector in terms of the dot product and differentiate.]

0.2. Prove that if f is continuous on $[a,b]$ and vector-valued, then

$$\int_a^b \Big[\mathbf{v}\cdot f(t)\Big]dt = \mathbf{v}\cdot\int_a^b f(t)dt,$$

and

$$\int_a^b \Big[\mathbf{v}\times f(t)\Big]dt = \mathbf{v}\times\int_a^b f(t)dt.$$

0.3. Let a,b,c be vectors and assume that $a\neq 0$. Further, assume that

(1)
$$a\times b = a\times c,$$

(2)
$$a\cdot b = a\cdot c.$$

Show that $b=c$.

0.4. Show that a, b, and c are coplanar if and only if

$$(a\times b)\cdot c = 0.$$

[Hint: If and only if means that there are two parts. First, assume that they are coplanar, and show that $(a\times b)\cdot c=0$. Second, assume that $(a\times b)\cdot c = 0$, and show that they are coplanar.]

0.5. Let u be a unit vector and a an arbitrary vector. Show the following:

(a) $(a\cdot u)u$ is parallel to u.
(b) $(u\times a)\times u$ is orthogonal to u.
(c) $a = (a\cdot u)u + (u\times a)\times u$.

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(d) From part (c), we see that every vector can be written as a vector that is parallel to a unit vector and a piece that is orthogonal. Let i, j, and k be the standard unit vectors in $\mathbb{R}^3$. Let a be an arbitrary vector in $\mathbb{R}^3$. That is

$$a = a_0 i + a_1 j + a_2 k,$$

where a_0, a_1, and a_2 are real numbers. Write a as in part (c), where you let $u = i$. Do the same for $u = j$ and $u = k$.

Remark. If you are in linear algebra, here is another problem related to the last problem? Well, take three units vectors u_1, u_2, and u_3, so that each is orthogonal to the other two. Then

$$\{u_1, u_2, u_3\}$$

forms an orthonormal basis. What would the matrix transforming the basis i, j, and k to the basis u_1, u_2, and u_3 look like? It would be an orthogonal matrix A, that is

$$A^T = A^{-1}.$$

Of course, the word orthogonal here is suggestive, huh? Try and show that the column vectors (or row vectors) of A form an orthonormal basis for $\mathbb{R}^3$. This is much easier than you think. From this, you can deduce that the change of basis matrix for any two orthonormal bases is orthogonal. To see this, show that if A and B are orthogonal matrices, then so is AB^{-1}. Then, we have

$$\{i, j, k\} \longrightarrow \{u_1, u_2, u_3\}$$

by a orthogonal matrix A, and

$$\{i, j, k\} \longrightarrow \{v_1, v_2, v_3\}$$

by an orthogonal matrix B. Then

$$\{v_1, v_2, v_3\} \longrightarrow \{u_1, u_2, u_3\}$$

by an orthogonal matrix AB^{-1}. In some fancy language, you have shown the orthogonal group O(3) acts transitively on the set of all orthonormal basis.

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