FINAL REVIEW

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1. Normal Lines and Tangent Planes

1. Find the unit normal vector to the given surface at the given point.

(a) The surface
$$x + y + z = 4,$$
at the point $(2, 0, 2)$.

(b) The surface
$$x^2 + y^2 + z^2 = 11,$$
at the point $(3, 1, 1)$.

(c) The surface
$$z = \sqrt{x^2 + y^2},$$
at the point $(3, 4, 5)$.

(d) The surface
$$z = x^3,$$
at the point $(2, 1, 8)$.

(e) The surface
$$x^2 y^4 - z = 0,$$
at the point $(1, 2, 16)$.

(f) The surface
$$x^2 + 3y + z^3 = 9,$$
at the point $(2, -1, 2)$.

(g) The surface
$$z - x \sin y = 4,$$
at the point $(6, \pi/6, 7)$.

(h) The surface
$$ze^{x^2 - y^2} - 3 = 0,$$
at the point $(2, 2, 3)$.

(i) The surface
$$\ln\left(\frac{x}{y - z}\right) = 0,$$
at the point $(1, 4, 3)$.

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2. Find the unit normal vector and normal line to the given surface at the given point.

(a) The function
$$f(x, y) = x^2 + y^2,$$
at the point $(1, 1, 2)$.

(b) The function
$$f(x, y) = \sin(xy),$$
at the point $(0, 1, 0)$.

(c) The function
$$g(x, y) = e^{-x^2-y^2},$$
at the point $(1, 1, e^{-2})$.

(d) The function
$$h(x, y) = e^{x+y} - e^{x-y},$$
at the point $(1, 0, 0)$.

(e) The function
$$V(r, h) = \pi r^2 h,$$
at the point $(1, 1, \pi)$.

(f) The function
$$f(x, y) = \frac{x^2 - y^2}{xy^2},$$
at the point $(1, 1, 0)$.

(g) The function
$$d(x, y) = \sqrt{(x-2)^2 + (y-3)^2}$$
at the point $(2, 3, 0)$.

(h) The function
$$H(x, y) = x^2 + y^2 + \frac{1}{xy},$$
at the point $(1, 1, 3)$.

(i) The function
$$\Gamma(x, y) = \frac{\sin(x+y)}{\cos(x+y)},$$
at the point $(0, 0, 0)$.

3. Find the tangent plane at the given point on the given surface.

(a) The surface
$$f(x, y) = 25 - x^2 - y^2,$$
at the point $(3, 1, 15)$.

(b) The surface
$$f(x, y) = \sqrt{x^2 + y^2},$$
at the point $(3, 4, 5)$.

(c) The surface
$$f(x, y) = \frac{y}{x},$$
at the point $(1, 2, 2)$.

(d) The surface
$$f(x, y) = 2 - \frac{2}{3}x - y,$$
at the point $(3, -1, 1)$.

(e) The surface
$$g(x, y) = x^2 - y^2,$$
at the point $(5, 4, 9)$.

(f) The surface
$$z = e^x(\sin y + 1),$$
at the point $(0, \pi/2, 2)$.

(g) The surface
$$z = x^3 - 3xy + y^3,$$
at the point $(1, 2, 3)$.

(h) The surface
$$h(x, y) = \ln \sqrt{x^2 + y^2},$$
at the point $(3, 4, \ln 5)$.

(i) The surface
$$x^2 + 4y^2 + z^2 = 36,$$
at the point $(2, -2, 4)$.

4. Find the tangent plane and normal line for the given surface at the given point.

(a) The function
$$f(x, y) = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$
at the point $(1, 1, 16)$.

(b) The function
$$f(x, y) = g_{xx}(x, y) + g_{yy}(x, y) - \frac{1}{x^2 + y^2},$$
where
$$g(x, y) = e^x \sin(y),$$
and at the point $(0, 0, f(0, 0))$.

(c) The function
$$h(x,y) = e^{\sin(x+y)},$$
at the point $(0, \pi/2, e)$.

(d) The surface
$$x + y + z = 0,$$
at the point $(1, 1, -2)$.

(e) The surface
$$\Lambda(x,t) = \frac{1}{2}x^2t^2,$$
at the point $(2, 1, 2)$.

(f) The surface
$$\mu(x,y) = \frac{1}{10}\tan(e^x - e^{-x}) + \ln(y^2 - x),$$
at the point $(0, 1, 0)$.

(g) The surface
$$x^3 + y^3 = z^2,$$
at the point $(2, 2, 4)$.

(h) The surface
$$\frac{z}{e^{x+y}} - 10 = 0,$$
at the point $(0, 0, 10)$.

(i) The surface given by
$$f(x,y) = x^2 - 2xy + y^2,$$
at the point $(0, 1, -1)$.

2. Chain Rule

5. Find dw/dt.

(a)
$$w = x^2 + y^2,$$
and
$$x = e^t, \quad y = e^{-t}.$$

(b)
$$w = \sqrt{x^2 + y^2},$$
and
$$x = \sin t, \quad y = e^t.$$

(c)
$$w = x \sec y,$$
and
$$x = e^t, \quad y = \pi - t.$$

(d)

$$w = \ln \frac{y}{x},$$

and

$$x = \cos t, \quad y = \sin t.$$

3. Extreme Value Problems

6. Find all the stationary (critical points) or the given function, and classify them.

(a)

$$f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3.$$

(b)

$$f(x, y) = -x^2 - 5y^2 + 8x - 10y - 13.$$

(c)

$$f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10.$$

(d)

$$f(x, y) = x^2 6xy + 10y^2 - 4y + 4.$$

(e)

$$z = 2x^2 + 3y^2 - 4x - 12y + 13.$$

(f)

$$g(x, y) = 120x + 120y - xy - x^2 - y^2.$$

(g)

$$f(x, y) = xy.$$

(h)

$$z = -3x^2 - y^2 + 3x - 4y + 5.$$

(i)

$$f(x, y) = x^3 - y^3.$$

7. Find all the stationary (critical points) or the given function, and classify them.

(a)

$$f(x, y) = x^3 - 3xy + y^3.$$

(b)

$$f(x, y) = 4xy - x^4 - y^4.$$

(c)

$$f(x, y) = \frac{3x^2 + 1}{2} - x(x^2 + y^2).$$

(d)

$$y^3 - 3yx^2 - 3y^2 - 3x^2 + 1.$$

(e)
$$z = (x^2 + 4y^2)e^{1-x^2-y^2}.$$
(f)
$$z = e^{-x}\sin y.$$
(g)
$$z = \arctan\frac{1}{x^2+y^2}.$$
(h)
$$z = e^{-(x^2+y^2)}.$$
(i)
$$z = \frac{-4x}{x^2+y^2+1}.$$

8. Find the absolute extreme for the given function over the given region.

(a)
$$f(x, y) = x^2 + xy$$
on
$$R = \{(x, y) \; : \; |x| \leq 2, \; |y| \leq 1\}.$$
(b)
$$f(x, y) = x^2 + 2xy + y^2,$$
on
$$R = \{(x, y) \; : \; |x| \leq 2, \; |y| \leq 1\}.$$
(c)
$$f(x, y) = x^2 + 2xy + y^2,$$
on
$$R = \left\{(x, y) \; : \; x^2 + y^2 \leq 8\right\}.$$
(d)
$$f(x, y) = x^2 - 4xy,$$
on
$$R = \left\{(x, y) \; : \; 0 \leq x \leq 4, \; 0 \leq y \leq \sqrt{x}\right\}.$$

9. Find x, y, and z positive numbers that sum to 30 and whose product is maximum.

10. Find x, y, and z positive numbers that sum to 1 and whose product is minimum.

11. Find the minimum distance from the paraboloid
$$z = x^2 + y^2,$$
and the point $(5, 5, 0)$. How about the point $(5, 0, 0)$.

12. Find the equation of a line (here m and b are the constants that you want to find)

$$y = mx + b,$$

such that

$$g(m, b) = (m + b - 5)^2 + (3m + b - 12)^2 + (-m + b)^2$$

is minimized. Note that the line

$$y = mx + b$$

minimizes the sum of the square of the distance from the line to the points $(1, 5)$, $(3, 12)$, and $(-1, 0)$.

13. Solve the given optimization problem. [Hint: Use Lagrange Multipliers.]

(a) Maximize

$$f(x, y)$$

with constraint

$$x + y = 10.$$

(b) Maximize

$$f(x, y) = xy$$

with constraint

$$2x + y = 4.$$

(c) Minimize

$$f(x, y) = x^2 + y^2$$

with constraint

$$x + y - 4 = 0.$$

(d) Minimize

$$f(x, y) = x^2 + y^2$$

with constraint

$$2x - 4y + 5 = 0.$$

(e) Maximize

$$f(x, y) = x^2 - y^2$$

with constraint

$$y - x^2 = 0.$$

(f) Maximize

$$f(x, y) = 2x + 2xy + y$$

with constraint

$$2x + y = 100.$$

(g) Maximize

$$f(x, y) = 3x = y + 10$$

with constraint

$$x^2 y = 6.$$

(h) Maximize

$$f(x, y) = \sqrt{6 - x^2 - y^2}$$

with constraint

$$x + y - 2 = 0.$$

(i) Maximize

$$f(x, y) = e^{xy}$$

with constraint

$$x^2 + y^2 - 8 = 0.$$

14. Solve the given optimization problem. [Hint: Use Lagrange Multipliers.]

(a) Minimize

$$f(x, y) = 2x + y$$

with constraint

$$xy = 32.$$

(b) Minimize

$$f(x, y, z)m = x^2 + y^2 + z^2$$

with constraint

$$x + y + z - 6 = 0.$$

(c) Maximize

$$f(x, y, z) = xyz$$

with constraint

$$x + y + z - 6 = 0.$$

4. Double Integrals

15. Evaluate each of the following iterated integrals.

(a)

$$\int_0^2 \int_0^{x^2} y dy dx.$$

(b)

$$\int_1^2 \int_y^{y^2} (x + 2y) dx dy.$$

(c)

$$\int_0^1 \int_0^{x^2} e^{\frac{y}{x}} dy dx.$$

(d)

$$\int_0^{\pi} \int_0^{\cos\theta} r \sin\theta dr d\theta.$$

(e)

$$\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} dx dy.$$

(f)

$$\int_0^1 \int_y^{\sqrt{y}} x^2 y^2 dx dy.$$

(g)

$$\int_0^1 \int_y^{\sqrt{y}} \frac{\sin x}{x} dx dy,$$

where if $x = 0$, assume that $\frac{\sin x}{x} = 1$.

(h)

$$\int_0^1 \int_{y-1}^0 e^{x+y} dx dy.$$

(i)

$$\int_0^1 \int_0^x e^{y^2} dy dx.$$

16. Compute the following double integrals.

(a)

$$\iint_R \frac{y}{x^2 + y^2} dA,$$

where R is the triangle bounded by $y = x$, $y = 2x$, and $x = 2$.

(b)

$$\iint_R x dA,$$

where R is the sector of a circle in the first quadrant bounded by

$$y = \sqrt{25 - x^2}, \quad 3x - 4y = 0, \quad y = 0.$$

(c)

$$\int\int_R (x+y)dA,$$

where R is bounded by the graphs

$$y = x^2, \quad y^2 = x.$$

(d)

$$\int\int_R y^2 dA,$$

where R is bounded by

$$x + y = 2, \quad y = x^2.$$

(e)

$$\int\int_R (2-x)dA,$$

where R is bounded by

$$x^2 + y^2 = 4.$$

(f)

$$\int\int_R (25 - x^2 - y^2)dA,$$

where R is bounded by

$$x + y = 5, \quad x = 0, \quad y = 0.$$

(g)

$$\int\int_R (4 - x^2 - y^2)dA,$$

where R is bounded by

$$y = \sqrt{1-x^2}, \quad y = 0.$$

(h)

$$\int\int_R \sin(x^2)dA,$$

where R is bounded by

$$y = x, \quad y = 0, \quad x = 1.$$

(i)

$$\iint_R dA,$$

where R is bounded by

$$xy = 6, \quad x + y = 5.$$

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