RADIUS OF CONVERGENCE OF A POWER SERIES

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1. Review

Two results that are useful in computing the radius of convergence of a power series will be given here. One is a corollary to the Cauchy-Hadamard theorem. The other can be shown in a manner similar to the proof of Cauchy-Hadamard.

Before stating the two results, let us recall that a real power series in $x - a$ is

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

We say that the power series converges for x_0 if

$$\sum_{n=0}^{\infty} a_n (x_0-a)^n$$

converges as a series. It is a trivial observation that for $x_0 = a$, the series

$$\sum_{n=0}^{\infty} a_n (a-a)^n$$

converges (it is the zero series). The **radius of convergence** of a power series

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

is the largest number R such that if

$$a - R < x < a + R$$

then the series

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

converges. The **interval of convergence** is

$$(a - R, a + R)$$

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2. The Results

With this said, here are two useful results in computing R for a power series.

Theorem 2.1 (Root Test). *Let*

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

be a power series. Then if

$$\lim_{n\longrightarrow\infty} \frac{1}{\sqrt[n]{|a_n|}} = L$$

then

$$L = R$$

and the interval of convergence is $(a-R, a+R)$.

Remark 1. If

$$\frac{1}{\sqrt[n]{|a_n|}}$$

is unbounded, then $R = \infty$.

The second result is the Absolute Ratio test.

Theorem 2.2 (Ratio Test). *Let*

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

be a power series. Then if

$$\lim_{n\longrightarrow\infty} \left|\frac{a_n}{a_{n+1}}\right| = L$$

then

$$R = L$$

and the interval of convergence is $(a-R, a+R)$.

The same remark proceeding the Root test is valid here.

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