

M 408D SERIES PROBLEMS

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Problem 1 (Easy-Medium). Determine whether the following series converge or diverge.

$$\begin{array}{lll}
 \sum_{n=1}^{\infty} \frac{n^2 + 3}{3^n} & \sum_{n=1}^{\infty} \frac{n^n}{(2n + 1)!} & \sum_{n=1}^{\infty} \frac{(\pi - e)^n n!}{n^n} \\
 \sum_{n=1}^{\infty} \frac{1}{n^2 - 1} & \sum_{n=1}^{\infty} \frac{\sin n}{5^n} & \sum_{n=1}^{\infty} \frac{3^{n+1}}{7^n} \\
 \sum_{n=1}^{\infty} \frac{1}{\sin\left(\frac{1}{n}\right)} & \sum_{n=2}^{\infty} \frac{n^2 + 1}{n^2 \ln n} & \sum_{n=1}^{\infty} \frac{\pi + n^5}{(3n + 1)!} \\
 \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) & \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} & \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{3}} \\
 \sum_{n=1}^{\infty} e^{-n} & \sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right) & \sum_{n=1}^{\infty} \frac{3}{n(n + 3)}
 \end{array}$$

Problem 2 (Easy-Medium). Evaluate the following limits.

$$\begin{array}{lll}
 \lim_{x \rightarrow \frac{\pi^-}{2}} \frac{\cos x}{\ln(\sin x)} & \lim_{x \rightarrow 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) & \lim_{x \rightarrow \infty} \sqrt{\left(\frac{k^2 + 1}{k^2} \right)^x} \\
 \lim_{x \rightarrow \infty} 10^{\frac{1}{x}} & \lim_{x \rightarrow 0} \frac{6^x - 3^x}{4x} & \lim_{x \rightarrow 0} \frac{a^x - b^x}{x} \\
 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x^2}} & \lim_{x \rightarrow \infty} \frac{x - \sin x}{x} & \lim_{x \rightarrow \infty} \left(e^x + e^{2x} \right)^{\frac{1}{x}}
 \end{array}$$

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Problem 3 (Medium). Determine whether the following series converge or diverge. Some of these are ridiculously hard!

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}} \quad \sum_{n=1}^{\infty} \frac{2n+1}{n^2+n} \quad \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{n!}} \\ & \sum_{n=1}^{\infty} \frac{(n^n)!}{n^{n^n}} \quad \sum_{n=1}^{\infty} \frac{2^{2n}}{4^{n+1}} \quad \sum_{n=1}^{\infty} \frac{1}{2^{\ln n}} \\ & \sum_{n=1}^{\infty} \frac{\cos(2n\pi)}{10^n} \quad \sum_{n=1}^{\infty} \frac{n^3+3n^2+3n+1}{(n+1)^6} \quad \sum_{n=1}^{\infty} \frac{3^n n!}{n^{2n}} \end{aligned}$$

Problem 4 (Hard). Suppose that we have the following for functions $f(x)$ and $g(x)$:

$$\begin{aligned} f(0) &= 1 & g(0) &= \infty \\ \frac{f'(0)}{f(0)} &= 1 & \frac{-g'(0)}{(g(0))^2} &= 1 \end{aligned}$$

Then find

$$\lim_{x \rightarrow 0} \left(f(x) \right)^{g(x)}$$

Problem 5 (Easy). Determine whether the following series converge or diverge.

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{n \ln n + 1}{n^3 \ln n} \quad \sum_{n=1}^{\infty} \frac{1+2^n}{3^n} \quad \sum_{n=1}^{\infty} \frac{n!}{n^n} \\ & \sum_{n=1}^{\infty} \frac{1+n^3}{\pi^n} \quad \sum_{n=1}^{\infty} \frac{6^n+1}{4^{n+10}} \quad \sum_{n=1}^{\infty} \frac{n^n}{119^{2n}} \\ & \sum_{n=1}^{\infty} \frac{n+3}{2n+n^2} \quad \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1.001}} \end{aligned}$$

Problem 6 (Medium). Suppose that a_n is a sequence of numbers and

$$\lim_{n \rightarrow \infty} n a_n = L$$

where $0 < L < \infty$, then what can you say about

$$\sum_{n=1}^{\infty} a_n$$

Problem 7 (Easy). Determine whether the following series converge or diverge.

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \quad \sum_{n=1}^{\infty} \frac{2^n}{4n^2 - 1} \quad \sum_{n=1}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!} \\ & \sum_{n=1}^{\infty} \frac{n7^n}{n!} \quad \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n \quad \sum_{n=1}^{\infty} n \left(\frac{3}{2}\right)^n \\ & \sum_{n=1}^{\infty} (-1)^n \ln \left(\frac{n+2}{n}\right) \quad \sum_{n=1}^{\infty} \frac{10}{3\sqrt[3]{n}} \quad \sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1\right)^n \end{aligned}$$

Problem 8 (Easy). Determine whether the following series converge or diverge.

$$\begin{aligned} & \sum_{n=1}^{\infty} n e^{-n} \quad \sum_{n=1}^{\infty} \frac{1}{n^\pi} \quad \sum_{n=1}^{\infty} n e^{-n^2} \\ & \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n \quad \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2-1}} \quad \sum_{n=1}^{\infty} \frac{n+10}{10n+1} \\ & \sum_{n=1}^{\infty} \frac{1}{n(n+3)} \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad \sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}} \end{aligned}$$

Problem 9 (Easy-Medium). Determine whether the following series converge or diverge.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$$

Problem 10 (Easy-Medium). Determine whether the following series converges conditionally, absolutely, or diverges.

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{4n^4+1} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \sum_{n=1}^{\infty} \frac{(-1)^n(n+2)}{3n^2+5} \\ & \sum_{n=1}^{\infty} \frac{3+4(-1)^n}{n^3+8} \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \quad \sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n} \\ & \sum_{n=1}^{\infty} \frac{(-1)^n(2n+3)^3 2^n}{(n+1)!} \quad \sum_{n=1}^{\infty} (-1)^n e^{-3n} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \\ & \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}} \quad \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+1} \quad \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 2^n} \\ & \sum_{n=1}^{\infty} (-1)^n \cos(n\pi) \quad \sum_{n=1}^{\infty} (-1)^n \sin(n\pi) \quad \sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n+2}{n}\right) \end{aligned}$$

Problem 11 (Limit Comparison). Use the Limit Comparison Test to determine the convergence or divergence of the series.

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{n^2+1} \quad \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}} \quad \sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1} \\ & \sum_{n=1}^{\infty} \frac{n+3}{n(n+2)} \quad \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n^2+1}} \quad \sum_{n=1}^{\infty} \frac{1}{2^n-5} \\ & \sum_{n=1}^{\infty} \frac{1}{n!+6n^3} \quad \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \quad \sum_{n=1}^{\infty} \frac{n^3}{5n^4+3} \end{aligned}$$

Problem 12 (Alternating Series Test). Use the Alternating Series test to determine the convergence or divergence of the series.

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1} \\ & \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n+2} \quad \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{(2n-1)\pi}{2}\right) \\ & \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n - e^{-n}} \quad \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n+1)}{n+1} \end{aligned}$$

Problem 13 (Ratio Test). Use the Ratio Test to test for convergence or divergence of the series.

$$\begin{array}{ccc} \sum_{n=1}^{\infty} \frac{n!}{3^n} & \sum_{n=1}^{\infty} \frac{(n!)^2}{(3n)!} & \sum_{n=1}^{\infty} \frac{4^n}{n!} \\ \sum_{n=1}^{\infty} \frac{n^2}{2^n} & \sum_{n=1}^{\infty} \frac{(2n)!}{n^5} & \sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n} \end{array}$$

Problem 14 (Root Test). Use the Root Test to test for convergence or divergence of the series.

$$\begin{array}{ccc} \sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n & \sum_{n=1}^{\infty} \left(2\sqrt[n]{n} + 1\right)^n & \sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1\right)^n \\ \sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n & \sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n & \sum_{n=2}^{\infty} \left(\frac{1}{\ln n}\right)^n \end{array}$$

Problem 15 (Integral Test). Use the Integral Test to test for convergence or divergence of the given series.

$$\begin{array}{ccc} \sum_{n=1}^{\infty} \frac{1}{n+1} & \sum_{n=1}^{\infty} e^{-n} & \sum_{n=1}^{\infty} ne^{-n} \\ \sum_{n=1}^{\infty} e^{-n} \cos n & \sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + c} & \sum_{n=1}^{\infty} \frac{n}{n^2 + 3} \end{array}$$

Problem 16. Do the following series converge or diverge? Why? Show all work!

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{n}{n^3 + 2^n} & \sum_{n=1}^{\infty} \frac{n!}{n^{2n}} & \sum_{n=1}^{\infty} \frac{n^2 - n}{n^4 + 1} \\ & \sum_{n=1}^{\infty} \frac{1}{n! + \sqrt{n}} & \sum_{n=1}^{\infty} \frac{n^n}{(n+1)^n} & \sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n-1)^n} \\ & \sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \ln n} & \sum_{n=1}^{\infty} \frac{n - \sqrt{n}}{n^3 + \sqrt{n}} & \sum_{n=1}^{\infty} \frac{n}{n^2 + \frac{1}{n}} \\ & \sum_{n=2}^{\infty} \frac{n}{(\ln n)^n} & \sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{(n^2+1)^n} & \sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n} \\ & \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + n}} & \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1} & \sum_{n=1}^{\infty} \frac{1}{n + \ln n} \\ & \sum_{n=1}^{\infty} \frac{n!}{n^{2n}} \end{aligned}$$

Problem 17. Compute the following limits. Show work.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} & \lim_{x \rightarrow 0^+} x^{\ln x} & \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^{2n} \\ & \lim_{n \rightarrow \infty} \left(n^3\right)^{\frac{1}{2n}} & \lim_{n \rightarrow \infty} n^{\frac{1}{\sqrt{n}}} & \lim_{x \rightarrow 0^+} (\ln x)^x \\ & \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n}}\right)^{n^2} & \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n+1}\right)^n & \lim_{n \rightarrow \infty} (2n)^{\frac{1}{n}} \end{aligned}$$

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