M 408D SERIES PROBLEMS

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1. Series Test

1.1. **Divergence Test.** The divergence test is always the first test you do (mentally at least). As an example of a series, consider

$$\sum \left(\frac{n}{n+1}\right)^n$$

This series is divergent, but you cannot use the root test (try it, it fails). You see that

$$\lim_{n \longrightarrow \infty} \left(\frac{n}{n+1}\right)^n = e^{-1}$$

Thus, the divergence test tells you that it diverges.

Again, as just a mental check, you **never use** the divergence test to show something converges.

1.2. **Basic Series.** We know some basic series. To start, we know all about the **geometric series** which has the form

$$\sum_{k=0}^{\infty} ar^k$$

where $a, r \in \mathbb{R}$, that is, a and r are real numbers. We know that this series converges if $|r| < 1$, and diverges otherwise. If it converges, then we know the limit as well:

$$\sum ar^k = \frac{a}{1-r}$$

We also know the **p-series**, which has the form

$$\sum \frac{1}{k^p}$$

where $p > 0$. We know that this series converges if $p > 1$ and diverges otherwise. We **do not**, in general, know the limit. For the special case of $p = 1$, this series is called the **harmonic series**, and diverges. This is useful when you want to show a series diverges, because most series diverge "more" than the harmonic series.

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Lastly, we have **telescoping series**. These are series where we can readily compute the sequence of partial sums. An example of a telescoping series is

$$\sum \frac{1}{k(k+1)}$$

this series converges to 1 (can you show this).

1.3. **Integral Test.** The integral test is based on controlling a series whose terms are positive and decreasing, with an improper Riemann integral. I do not often use this test, but for series that a very delicate, like

$$\sum \frac{1}{k \ln k}$$

or

$$\sum \frac{1}{k\,(\ln k)^2}$$

is is quite useful. If the terms of the sequence satisfy the hypothesis, and you can integrate the corresponding function, then it is always an option.

1.4. **Comparison and Limit Comparison.** The comparison and limit comparison tests are very powerful tools, theoretically and in application. Any time you think that something does not matter, then the comparison test is your tool. For instance

$$\sum \frac{n+1}{n^3-n-3}$$

Here, you think that this should act like

$$\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$$

Then you can compare these two. These test work well at eliminating "small" terms in a sum. However, we could not remove n in the series

$$\sum \frac{1}{n2^n}$$

because it does "matter". This is an important thing to keep in mind. The ratio and roots tests do, however, remove this term.

The comparison test and limit comparison test are the weapon of choice, if you will. When used in conjunction with the other tests, you will find it hard not to succeed.

1.5. **Ratio.** The ratio test is a powerful tool for handling factorials. One thing about the ratio test, which is not clear, is that it only really works when the series is "very" convergent. That is, when it is obviously convergent, the ratio test will work. As an example of this, consider the following example:

Example 1.1. Consider the p-series,

$$\sum \frac{1}{k^p}$$

Then taking successive ratios, we have

$$\frac{k^p}{(k+1)^p}$$

The limit of this, regardless of p is 1. Thus, the test fails for all p. Now, in some sense, the series

$$\sum \frac{1}{k^{1000}}$$

converges quickly, but the ratio test does not see that this converges. So the "speed" of convergence the ratio test sees is faster than any polynomial decay.

As another example, consider the following:

Example 1.2. Consider the series

$$\sum \frac{1}{a^k}$$

Then taking successive ratios, we get

$$\frac{1}{a}$$

The limit of this is $\frac{1}{a}$, since it is constant. Now, we see that the larger a is, then the closer $\frac{1}{a}$ is. So that the ratio test does see that the series

$$\sum \frac{1}{2^k}$$

converges, since the limit of the ratios is $\frac{1}{2}$.

We observe that the closer the limit of the ratios is to zero, the better the convergence. All convergent geometric series converge "fast", but none of the limits of the ratios is zero.

Example 1.3. As a last example, consider

$$\sum \frac{1}{n!}$$

Then the ratio of successive terms is

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}$$

The limit of this, as n tends to infinity is zero. So the convergence is "very fast".

The ratio test works well when we see factors like

$n!$

a^n

when you have sums that the root test doesn't like

when you want to kill polynomial factors

The ratio test does not see polynomials. So if you have a series like

$$\sum \frac{k^5 + k^2 + 1}{k^8 + 7k^4 - 92k + 9}$$

the ratio test would be a bad choice (likewise the root test).

Sometimes, it is useful to use the comparison test and the ratio test together. For instance, if we have

$$\sum \frac{2^n + n^4 - \ln n}{n! + 2}$$

You see here that the dominant terms are 2^n and $n!$. So that, you really want to know about the series

$$\sum \frac{2^n}{n!}$$

Then, we see that ratio test is the "best" test for this. The moral here is do not rule out using the comparison test and another test together.

1.6. **Root.** The root test, like the ratio test, does not see polynomials. This is because

$$\lim_{n \longrightarrow \infty} \sqrt[n]{P(n)} = 1$$

for any polynomial $P(x)$ (recall what a polynomial is). For this reason, the root test only works when the answer is obvious.

The root test works well for things that have:

n^n

a^n

when you want to kill polynomials in products

The root test is very bad for sums, like

$$\frac{1}{n^n + 2^n + n^3 - 1}$$

Since there are no good formulas for roots of sums.

Again, this can be dealt with using the comparison test, by eliminating the lesser terms in sums. Notice that the comparison test is sort of bad for products, but good of sums, and the ratio and root tests are good for products but bad for sums.

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