

## ASSIGNMENT 6 SOLUTIONS

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**Solution** (11.2, # 10). Use the integral test (check that the function satisfies the three conditions). To integrate

$$f(x) = \frac{\ln x}{x^3}$$

use integration by parts (ILATE says  $u = \ln x$ ).

From this, you will discover that the series converges.

On the other hand, we know that

$$\ln x < x$$

for all  $x \in [1, \infty)$ . So that

$$\begin{aligned} \ln k &< k \\ \frac{\ln k}{k^3} &< \frac{k}{k^3} = \frac{1}{k^2} \end{aligned}$$

Then, since the series

$$\sum \frac{1}{k^2}$$

converges (and both series are nonnegative), the comparison test says our series is convergent.

**Solution** (11.2, # 16). Use the integral test (check that the function satisfies the three conditions). To integrate, make the  $u$ -substitution,  $u = \ln x$ . Then, from this, you will discover that the series converges.

You can use the comparison test here, but since this sequence is *very close to diverging*, you will have to work the same to show that the thing you could compare it with converges. For instance, if  $\varepsilon > 0$  is any number (thinking of it being small), you will observe that

$$\lim_{x \rightarrow \infty} \frac{x \ln x}{x^{1+\varepsilon}} = 0$$

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For this, we have

$$\begin{aligned}\lim \frac{\ln x}{\ln x} &= * \lim \frac{x^{-1}}{x^{1-\varepsilon}} \\ &= \lim \frac{1}{x^\varepsilon} = 0\end{aligned}$$

Think about what this says, this says that our series is closer to the Harmonic series than any series of the form

$$\sum \frac{1}{k^{1+\varepsilon}}$$

This should tell you that a delicate argument must be made.

**Solution.** We see that asymptotically, the terms that we are summing are order  $O(n^{-4})$ . So we suspect the series will converge. Use the limit comparison test, where we compare it with the terms

$$\frac{7}{2k^4}$$

Then, since

$$\lim_{k \rightarrow \infty} \frac{\frac{7k+2}{2k^5+7}}{\frac{7}{2k^4}} = 1$$

and

$$\sum \frac{7}{2k^4}$$

is convergent, being a scalar multiple of a  $p$ -series with  $p = 4$ , the limit comparison test tells us that our series converges.

**Solution** (11.2, # 30). Our series is very convergent. The terms are

$$\frac{k^2}{2k^3}$$

Now, we have

$$2^{k^3} = 2^{k^k k} = \left( (2^k)^k \right)^k$$

This is certainly bigger than  $2^k$ . Likewise, we know that

$$k^2 \leq \left( \frac{3}{2} \right)^k$$

eventually, so that, we get

$$\frac{k^2}{2k^3} \leq \frac{\frac{3^k}{2^k}}{2^k} = \left( \frac{3}{4} \right)^k$$

This is a geometric series, which is convergent. Thus by the comparison test, we get that our series converges.

I suppose that you could use the integral test. It appears that a  $u$ -substitution can be made, since

$$\frac{d}{dx}x^3 = 3x^2$$

so that we can remove the  $x^2$  and  $x^3$  that occur. I will let you have all the fun in doing that.

**Important:**

Now, we discussed the  $(p, q)$ -series in discussion. Recall, that a  $(p, q)$ -series is of the form

$$\sum \frac{1}{k^p (\ln k)^q}$$

I stated the following:

$$\sum \frac{1}{k^p (\ln k)^q} = \begin{cases} p > 1, & \text{convergent} \\ p < 1, & \text{divergent} \\ p = 1, q > 1, & \text{convergent} \\ p = 1, q \leq 1, & \text{divergent} \end{cases}$$

Now, since this was not covered in the textbook or in Dr. Rodin's lectures, you cannot use this to show that a series is convergent. I simply gave this to you for intuitive purposes, since often it is useful to know that answer before you begin.

As an extra problem, you might consider trying to show that the above is true. That is, prove the  $(p, q)$ -series converges and diverges when I said it does.

**Problem 1** (Extra problems).

- (1) Find the limit

$$\frac{(n!)^{n^n}}{(n^n)^{n!}}$$

- (2) Find the limit of

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$$

and show that

$$\int_{-\infty}^{\infty} \left| \frac{\sin x}{x} dx \right|$$

diverges. (Recall that this is quite difficult. Technically, what you have shown is that the above function has a convergent improper Riemann integral, but it is not Lebesgue integrable. On the other hand, and this should not be clear, but you can

change how you count up the areas of the above function, and get the improper integral to converge to any real number. So, in some sense, the first part of the question is misleading.)

- (3) Prove the  $(p, q)$ -series converges when  $p > 1$  and  $p = 1, q > 1$  and diverges when  $p < 1$  and  $p = 1, q \leq 1$ .
- (4) Find

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^2}$$

Recall, this is our twenty dollar problem, and you cannot tell me that after 100,000 terms, it is close to some number. I need an exact number.

These are not bonus problems, so please do not turn them in.

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