TEST 3 REVIEW

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1. Vectors

1.1. Form the vector starting at the point P and ending at the point Q:

(a) $P = (0, 0, 0)$, $Q = (1, 2, 3)$.
(b) $P = (1, -5, 3)$, $Q = (8, 18, 0)$.
(c) $P = (-3, 1, 1)$, $Q = (-2, 4, 12)$.
(d) $P = (0, 1, -1)$, $Q = (0, 1, -1)$.

1.2. For the following two vectors, do the following: Draw each vector having it start at the origin. Add the two vectors. Compute the dot product of the two vectors. Find the norm of each vector. Find the angle between the two vectors. Compute the cross product of the two vectors. Find the area of the parallelogram spanned by the two vectors. Find the volume of the parallelepiped of the two vectors and their cross product:

(a) $v = (1, 0, 0)$, $w = (0, 1, 0)$.
(b) $v = \mathbf{k}$, $w = \mathbf{i} + j$.
(c) $v = (1, 3, -5)$, $w = (1, -3, -5)$.
(d) $v = 6\mathbf{i} + 2\mathbf{j} + -\mathbf{k}$, $w = (0, 0, 1)$.
(e) $v = (5, 5, 5)$, $w = (0, 0, 1)$.
(f) $v = (-2, 4, 0)$, $w = \mathbf{k}$.
(g) $v = (1, 2, 3)$, $w = (5, 6, -10)$.
(h) $v = (4, -2, 1)$, $w = (1, 1, 1)$.
(i) $v = (1, -1, 3)$, $w = (2, -2, 6)$.

1.3. Find a vector parallel to the given vector and having norm 6.

(a) $v = (1, 2, 3)$.
(b) $v = (-1, 4, 6)$
(c) $v = (0, 0, 0)$.
(d) $v = (1, 2)$.
(e) $v = (-7, 4, 2, 3)$.

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1.4. Compute the projection of v in the direction of w. Find the component.

(a) $v = (1, 0, 0)$, $w = (0, 1, 0)$.
(b) $v = \mathbf{k}$, $w = \mathbf{i} + \mathbf{j}$.
(c) $v = (1, 3, -5)$, $w = (1, -3, -5)$.
(d) $v = 6\mathbf{i} + 2\mathbf{j} + -\mathbf{k}$, $w = (0, 0, 1)$.
(e) $v = (5, 5, 5)$, $w = (0, 0, 1)$.
(f) $v = (-2, 4, 0)$, $w = \mathbf{k}$.
(g) $v = (1, 2, 3)$, $w = (5, 6, -10)$.
(h) $v = (4, -2, 1)$, $w = (1, 1, 1)$.
(i) $v = (1, -1, 3)$, $w = (2, -2, 6)$.

2. Lines and Planes

2.1. Find the equation of the line given:

(a) Passing through the origin and parallel to $v = (1, 2, 3)$.
(b) Passing through the point $(0, 2, 1)$ and parallel to $v = (-2, \frac{5}{2}, 1)$.
(c) Passing through the point $(-2, 0, 3)$ and parallel to

$$v = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}.$$

(d) Passing through the points $(5, -3, -2)$ and $(-2, 2, 1)$.
(e) Passing through the point $(1, 0, 1)$ are parallel to the line

$$\begin{aligned} x &= 3 + 3t \\ y &= 5 - 2t \\ z &= -7 + t. \end{aligned}$$

(f) Passing through the point $(2, 3, 4)$ and parallel to the xz-plane and yz-plane.
(g) Passing through the point $(2, 3, 4)$ and perpendicular to the plane given by

$$3x + 2y - z = 6.$$

(h) Passing through the origin and parallel to the planes:

$$\begin{aligned} x + y + z &= 8 \\ x - y - z &= 0. \end{aligned}$$

(i) Passing through the origin and perpendicular to the lines:

$$\begin{aligned} r_1(t) &= (1, 2, 3)t \\ r_2(t) &= (2 + 4t)\mathbf{i} - (1 + t)\mathbf{j} + (7 - 9t)\mathbf{k}. \end{aligned}$$

2.2. Find the equation of the line given:

(a) Perpendicular to the line

$$r(t) = (1, -1, 1)t,$$

parallel to the plane given by

$$3x + y - 2z = 0,$$

and containing the point of intersection of the above line and plane.

(b) Passing through the points $(1, 2, -1)$, $(0, 0, 0)$, and $(-2, -4, 2)$.

(c) Parallel to the lines

$$r_1(t) = (1, 0, -4)t$$
$$r_2(t) = (1 - 2t)\mathbf{i} + 6\mathbf{j} - (2 + 8t)\mathbf{k},$$

and passing through the origin.

(d) Passing through the origin, and perpendicular to the planes

$$2x + 3y - z = 0$$
$$\frac{1}{2}x + \frac{3}{4}y - \frac{1}{4}z = 0.$$

(e) Contained in the planes

$$x + 3y - 2z = 0$$
$$-2x + 3y - 7z = 0.$$

(f) Passing through the point $(5, 5, -6)$ and parallel to the tangent line of the curve

$$r(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t^2\mathbf{k}$$

at $t = \frac{\pi}{4}$.

(g) Orthogonal to the plane

$$2x - y - z = 0,$$

and containing the point of intersection of the above plane with the line

$$r(t) = t\mathbf{k}.$$

(h) Contained in the plane

$$x - y - z = 0,$$

and the plane

$$x + y + z = 0.$$

2.3. Find the equation of the plane given:

(a) Passing through the point $(2, 1, 2)$ with normal vector $\mathbf{i}$.

(b) Passing through the point $(1, 0, -3)$ with normal vector $(2, 3, -1)$.

(c) Passing through the points $(0,0,0)$, $(1,2,3)$, and $(3,2,2)$.
(d) Passing through the points $(1,2,-3)$, $(2,3,1)$, and $(0,-2,-1)$.
(e) Passing through the point $(1,2,3)$ and parallel to the yz-plane.
(f) Containing the y-axis, and makes an angle of $\pi/6$ with the positive x-axis.
(g) Contains the lines

$$r_1(t) = (1,2,3)t$$
$$r_2(t) = (-1,-2,-3)t.$$

(h) Passing through the points $(2,2,1)$ and $(-1,1,-1)$ and perpendicular to the plane

$$2x - 3y + z = 3.$$

(i) Passing through the points $(1,-2,-1)$ and $(2,5,6)$ and parallel to the x-axis.

2.4. Find the equation of the plane given:
(a) Intersecting the xy-plane at the line

$$r(t) = (1,3,0)t.$$

(b) Intersecting the yz-plane at the line

$$r(t) = (1+4t)\mathbf{j} + (2-4t)\mathbf{k}.$$

(c) Passing through the origin and making an angle of $\pi/4$ with the positive x-axis and positive y-axis.
(d) Containing the tangent line of the curve

$$r(t) = 2t\mathbf{i} - 4t^3\mathbf{j} + \sec t\mathbf{k}$$

at $t = \pi/4$, and parallel to the line of intersection of the planes

$$x - y + z = 0$$
$$x + y + 3z = 0.$$

(e) Containing the zeros of the function

$$f(x,y,z) = x - y + z.$$

(f) Parallel to the plane

$$x + 3z = 0,$$

and containing the origin.
(g) Perpendicular to the planes

$$x - 4y + z = 0$$
$$-2x + 8y - 2z = 16,$$

and containing the origin.

2.5. Determine if the following lines intersect. If so, find the angle.

(a)

$$\begin{aligned} x &= 4t+2 & x &= 2s+2 \\ y &= 3 & y &= 2s+3 \\ z &= -t+1 & z &= s+1. \end{aligned}$$

(b)

$$\begin{aligned} x &= -3t+1 & x &= 3s+1 \\ y &= 4t+1 & y &= 2s+4 \\ z &= 2t+4 & z &= -s+1. \end{aligned}$$

(c) The line containing the points $(0,0,0)$ and $(1,2,3)$, and the line containing the point $(0,0,0)$ and the point $(1,-1,2)$.

3. Vector-valued Functions

3.1. Sketch the vector-valued curve.

(a)

$$r(t) = 3t\mathbf{i} + (t-1)\mathbf{j}.$$

(b)

$$r(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}.$$

(c)

$$r(t) = (-t+1)\mathbf{i} + (4t+2)\mathbf{j} + (2t+3)\mathbf{k}.$$

(d)

$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}.$$

(e)

$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}.$$

(f)

$$r(t) = \left(\frac{e^t + e^{-t}}{2}\right)\mathbf{i} + \left(\frac{e^t - e^{-t}}{2}\right)\mathbf{j}.$$

(g)

$$r(t) = t^4\mathbf{i} - 8t^8\mathbf{j}.$$

3.2. Find the derivative of the vector-valued function. Locate a point on the curve (your choice). Compute the derivative at this point. What does it mean geometrically?

(a)

$$r(t) = 3t\mathbf{i} + (t-1)\mathbf{j}.$$

(b)

$$r(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}.$$

(c)
$$r(t) = (-t+1)\mathbf{i} + (4t+2)\mathbf{j} + (2t+3)\mathbf{k}.$$
(d)
$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}.$$
(e)
$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}.$$
(f)
$$r(t) = \left(\frac{e^t + e^{-t}}{2}\right)\mathbf{i} + \left(\frac{e^t - e^{-t}}{2}\right)\mathbf{j}.$$
(g)
$$r(t) = t^4\mathbf{i} - 8t^8\mathbf{j}.$$

3.3. Evaluate the indefinite integrals.

(a)
$$\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k})dt.$$
(b)
$$\int \left[(2t-1)\mathbf{i} + 4t^2\mathbf{j} + 3\sqrt{t}\mathbf{k}\right]dt.$$
(c)
$$\int (e^t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k})dt.$$
(d)
$$\int \left[\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right] dt.$$
(e)
$$\int (e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j})dt.$$
(f)
$$\int \left[e^t\sin(at)\cos(bt)\mathbf{i} + t^3\mathbf{j} + \frac{1}{t^2+1}\mathbf{k}\right]dt.$$

3.4. Find the tangent line at the given point.

(a)
$$r(t) = 3t\mathbf{i} + (t-1)\mathbf{j}, \quad t = 1.$$
(b)
$$r(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}, \quad t = \pi.$$
(c)
$$r(t) = (-t+1)\mathbf{i} + (4t+2)\mathbf{j} + (2t+3)\mathbf{k}, \quad t = 0.$$

(d)
$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad t = \frac{\pi}{4}.$$
(e)
$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}, \quad t = \frac{\pi}{4}.$$
(f)
$$r(t) = \left(\frac{e^t + e^{-t}}{2}\right)\mathbf{i} + \left(\frac{e^t - e^{-t}}{2}\right)\mathbf{j}, \quad t = 3.$$
(g)
$$r(t) = t^4\mathbf{i} - 8t^8\mathbf{j}, \quad t = 3.$$

3.5. Find the given limit.

(a)
$$\lim_{t \longrightarrow 2} \left(t\mathbf{i} + \frac{t^2 - 4}{t^2 - 2t}\mathbf{j} + \frac{1}{t}\mathbf{k}\right).$$
(b)
$$\lim_{t \longrightarrow 0} \left(e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right).$$
(c)
$$\lim_{t \longrightarrow 0} \left(t^2\mathbf{i} + 3t\mathbf{j} + \frac{1 - \cos t}{t}\mathbf{k}\right).$$
(d)
$$\lim_{t \longrightarrow 1} \left(\sqrt{t}\mathbf{i} + \frac{\ln t}{t^2 - 1}\mathbf{j} + 2t^2\mathbf{k}\right).$$

3.6. Find the length of the space curve over the given interval:

(a)
$$r(t) = \frac{1}{2}t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}, \quad 0 \leq t \leq \pi.$$
(b)
$$r(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{k}, \quad 0 \leq t \leq \pi.$$

4. Multivariable Functions

4.1. Sketch the trace curves for the following:

(a)
$$z = 3.$$
(b)
$$x = 4.$$
(c)
$$y^2 + z^2 = 9.$$

(d)
$$x^2 + z^2 = 16.$$
(e)
$$x^2 - y = 0.$$
(f)
$$y^2 + z = 4.$$
(g)
$$z - \sin y = 0.$$
(h)
$$y^2 - z^2 = 0.$$
(i)
$$4x^2 + y^2 + 4z^2 = 4.$$

4.2. Sketch the trace curves for the following:

(a)
$$16x^2 + 9y^2 + 9z^2 = 144.$$
(b)
$$x^2 - y + z^2 = 0.$$
(c)
$$z = 4x^2 + y^2.$$
(d)
$$4x^2 + y^2 - 4z^2 = -16.$$
(e)
$$4x^2 - y + z = 0.$$
(f)
$$x^2 + y + z = 0.$$
(g)
$$x^2 + y^2 + 4z^2 = 0.$$
(h)
$$x^2 + 9y^2 + 4z^2 = 36.$$

4.3. Sketch the level sets for the following functions:

(a)
$$f(x, y) = \sqrt{25 - x^2 - y^2}.$$
(b)
$$f(x, y) = x^2 + y^2.$$
(c)
$$f(x, y) = xy.$$

(d)
$$f(x, y) = \frac{x}{x^2 + y^2}.$$
(e)
$$f(x, y) = \ln(x - y).$$
(f)
$$f(x, y) = e^{xy}.$$
(g)
$$f(x, y) = \cos(x + y).$$

4.4. Compute the first order partial derivatives and second order partial derivatives of the following functions:

(a)
$$f(x, y) = 2x - 3y + 5.$$
(b)
$$f(x, y) = xy.$$
(c)
$$f(x, y) = x^2 - 3y^2 + 7.$$
(d)
$$f(x, y) = \frac{x}{y}.$$
(e)
$$f(x, y) = e^{-(x^2+y^2)}.$$
(f)
$$g(x, y) = \cos(x^2 + y^2).$$
(g)
$$h(x, y) = e^y \sin xy.$$
(h)
$$\alpha(x, y, z) = e^{xyz}.$$
(i)
$$\theta(x, y) = e^x \tan y.$$

4.5. Compute the first order partial derivatives and second order partial derivatives of the following functions:

(a)
$$\gamma = x^2 + xy + 3y^2.$$
(b)
$$\eta(x, y) = \frac{xy}{x - y}.$$
(c)
$$\xi(x, y) = xe^y + ye^x.$$

(d)
$$z = x \sec y.$$
(e)
$$z = x^3 + 3x^2 y.$$
(f)
$$\Gamma(x, y, z) = xyz.$$
(g)
$$\Gamma(x, y, z) = \frac{x}{y+z}.$$
(h)
$$\Lambda(x, y, z) = x^2 + y^2 + z^2.$$
(i)
$$\eta(x, y, z) = e^{-x} \sin yz.$$

4.6. Compute the limit in the following directions: (1) Along the y-axis. (2) along the x-axis. (3) along the line $y = mx$.

(a)
$$\lim_{(x,y)\longrightarrow(0,0)} e^{xy}.$$
(b)
$$\lim_{(x,y)\longrightarrow(0,0)} 5x + 3xy + y + 1.$$
(c)
$$\lim_{(x,y)\longrightarrow(0,0)} \frac{\cos x \sin y}{y}.$$
(d)
$$\lim_{(x,y)\longrightarrow(0,0)} \frac{\sin xy}{xy}.$$
(e)
$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^3 + y^3}{x^2 + y^2}.$$
(f)
$$\lim_{(x,y)\longrightarrow(0,0)} \frac{xy^2}{x^2 + y^2}.$$
(g)
$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2 y^2}{x^2 + y^2}.$$

4.7. Compute the gradient. Find the directional derivative for the given vector and at the given point P. Find the direction of maximum increase.

(a)
$$f(x,y) = 3x - 4xy + 5y, \quad P = (1,2),\ v = (1, \sqrt{3}).$$
(b)
$$f(x,y) = xy, \quad P = (2,3),\ v(1,1).$$
(c)
$$f(x,y) = \frac{x}{y}, \quad P = (1,1),\ v = (0,-1,0).$$
(d)
$$f(x,y,z) = xyz, \quad P = (4,1,1),\ v = (1,1,-1).$$
(e)
$$h(x,y) = e^{-(x^2+y^2)}, \quad P = (0,0),\ v = (1,1).$$
(f)
$$\Gamma(x,y) = e^x \sin y, \quad P = (1, \pi/2),\ v = (-1,0,0).$$
(g)
$$\eta(x,y,z) = x^2 + y^2 + z^2, \quad P = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}),\ v = (1,1,1).$$
(h)
$$\lambda(x,y,z) = x^2 - y^2 - z^2, \quad P = (1,0,0),\ v = (1,-1,0).$$
(i)
$$\eta(x,y) = e^{\sin xy}, \quad P = (\pi, \pi/2),\ v = (1,1).$$

4.8. Find the gradient of the given function and the maximum value of of the directional derivative at the given point:

(a)
$$f(x,y) = y\sqrt{x}, \quad P = (4,2).$$
(b)
$$\eta(\alpha,\beta) = \alpha \tan\beta, \quad (2, \pi/4).$$
(c)
$$\gamma(x,y) = y\cos(x-y), \quad (0, \pi/3).$$
(d)
$$\mu(x,y) = \ln\sqrt[3]{x^2+y^2}, \quad P = (1,2).$$
(e)
$$f(x,y,z) = xe^{yz}, \quad P = (2,0,-4).$$

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