Research Statement

Xavier Ros-Oton

November 12, 2013

Overview

My research mainly focuses on elliptic Partial Differential Equations and related problems. In particular, I have worked on integro-differential equations, reaction-diffusion problems, and weighted Sobolev and isoperimetric inequalities.

I started my PhD thesis working on reaction-diffusion problems, more precisely on the regularity of stable solutions \[4\]. Then, while studying this, we were led to some weighted isoperimetric and Sobolev inequalities that we studied in \[5\] and \[7\]-\[6\]. After that, I got interested in integro-differential equations. In joint works with J. Serra we established the Pohozaev identity for the fractional Laplacian \[23\]-\[22\], as well as some regularity results for this operator \[24, 25\]. For more general integro-differential operators, we have obtained nonexistence results for supercritical nonlinearities \[26\], and in a forthcoming paper \[27\] we are planning to extend our boundary regularity results from \[24\] to a class of fully nonlinear equations.

In the first section of this research statement, I describe my research on integro-differential equations. In section 2, I talk about my work on stable solutions to reaction-diffusion problems. In section 3, I discuss my research related to weighted isoperimetric and Sobolev inequalities. Finally, in section 4 I describe briefly some works that I made as an undergraduate student in the fields of Ordinary Differential Equations, Control Theory, and Number Theory.

1 Integro-differential equations

Integro-differential equations arise naturally in the study of stochastic processes with jumps. This type of processes are of particular interest in finance, population dynamics, and in some physical and biological models. Moreover, all partial differential equations are a limit case of integro-differential equations. Thus, a good understanding of these equations can provide a better understanding of their limit case: the PDEs.

The study of integro-differential equations is attracting an increasing level of interest, from the point of view of both analysis and probability. The operators under study usually have the form

\[ Lu(x) = \sum a_{ij} \partial_{ij} u + \text{PV} \int_{\mathbb{R}^n} (u(x + y) - u(x)) K(y) dy, \]  

(1)
where \((a_{ij})\) is a nonnegative definite matrix and \(K\) is a nonnegative symmetric kernel (i.e., \(K(y) = K(-y)\)) that satisfies \(\int_{\mathbb{R}^n} \min(1, |y|^2)K(y)dy < \infty\). These operators are infinitesimal generators of symmetric Levy processes.

The prime example of an elliptic integro-differential operator is the fractional Laplacian

\[
(-\Delta)^s u(x) = c_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x + y)}{|y|^{n+2s}} dy, \quad s \in (0, 1).
\]

It is the infinitesimal generator of the radially symmetric and \(2s\)-stable Levy process. Much of the literature on integro-differential equations is centered on the study of the fractional Laplacian. For this operator, one has some particular properties such as the extension of Caffarelli-Silvestre or the inverse operator \((-\Delta)^{-s}\) given by the Riesz potential.

In the following paragraphs I describe my different works on integro-differential equations.

- First, in a joint work with Joaquim Serra [23]-[22], we established the Pohozaev identity for the fractional Laplacian.

  In the classical case of the Laplacian operator, the Pohozaev identity applies to any solution to \(-\Delta u = f(x, u)\) in \(\Omega\), \(u = 0\) on \(\partial\Omega\). Its first immediate consequence is the nonexistence of solutions for critical and supercritical nonlinearities. For example, it gives nonexistence of nontrivial solutions to the critical problem \(-\Delta u = u^{n+2\over n-2}\), which appear in some geometrical contexts such as the Yamabe problem. Moreover, Pohozaev-type identities have been used in many different contexts, and lead to monotonicity properties, radial symmetry of solutions, uniqueness results, and partial regularity of stable solutions. Furthermore, it is also commonly used in nonlinear wave and heat equations, and in harmonic maps.

  For the fractional Laplacian [2], the Pohozaev identity was not known, and was not even known which form should it have. As said before, we established in [23]-[22] the Pohozaev identity for the fractional Laplacian, which reads as follows. If \(u\) is any bounded solution of \((-\Delta)^s u = f(x, u)\) in \(\Omega\), \(u \equiv 0\) in \(\mathbb{R}^n \setminus \Omega\), then \(u/\delta^s \in C^\alpha(\Omega)\), and it holds the identity

\[
\int_{\Omega} (x \cdot \nabla u)(-\Delta)^s u \, dx = \frac{2s - n}{2} \int_{\Omega} u(-\Delta)^s u \, dx - \frac{\Gamma(1 + s)^2}{2} \int_{\partial\Omega} \left( \frac{u}{\delta^s} \right)^2 (x \cdot \nu) d\sigma,
\]

where \(\delta(x) = \text{dist}(x, \partial\Omega)\). The boundary term \(u/\delta^s|_{\partial\Omega}\) has to be understood in the limit sense —recall that \(u/\delta^s \in C^\alpha(\overline{\Omega})\).

  As a consequence of [3], we also obtained the completely new integration-by-parts-type identity

\[
\int_{\Omega} u_x (-\Delta)^s v \, dx = -\int_{\Omega} (-\Delta)^s u_v \, dx + \Gamma(1 + s)^2 \int_{\partial\Omega} \left( \frac{u}{\delta^s} \right) \frac{u}{\delta^s} \nu \, d\sigma,
\]

where \(u\) and \(v\) are any two solutions to different linear or semilinear problems that satisfy \(u \equiv v \equiv 0\) in \(\mathbb{R}^n \setminus \Omega\).

  For the Laplacian \(-\Delta\), the Pohozaev identity follows quite easily from integration by parts or the divergence theorem. However, in this nonlocal framework these tools are not
available. The only known integration by parts for the fractional Laplacian was the one for the whole \( \mathbb{R}^n \), which has no boundary terms. In fact, the constant \( \Gamma(1+s)^2 \) in our identity seems to indicate that there is no trivial way to prove this identity without some work. To our knowledge, these identities are new even in dimension \( n = 1 \).

In our proof we do not made any use of the extension or other particular properties of the fractional Laplacian, but only the scale invariance of the operator. In future works we are planning to establish Pohozaev identities for more general integro-differential operators: for the ones corresponding to (anisotropic) \( \alpha \)-stable Levy processes (see the definition of the class \( \mathcal{L}_s \) below), and also for higher order fractional Laplacians with \( s > 1 \).

To prove (3) we needed, among other things, the precise boundary regularity of solutions to the Dirichlet problem \((-\Delta)^s u = g \) in \( \Omega \), with \( u \equiv 0 \) outside. Here, \( \Omega \) is a \( C^{1,1} \) domain. This was done in \([24]\), where we proved that \( u \in C^s(\mathbb{R}^n) \) and \( u/\delta^s \in C^\alpha(\Omega) \) whenever \( g \in L^\infty(\Omega) \). Moreover, when \( g \) has better regularity, we obtained higher order \( C^\beta \) interior estimates for \( u \) and \( u/\delta^s \), also needed in our proof of the Pohozaev identity.

The Hölder regularity for the quotient \( u/\delta^s \) is the main result of that paper. This was already known for \( s \)-harmonic functions (that is, for \( g \equiv 0 \)) that vanish in a neighborhood of \( \partial \Omega \) outside \( \Omega \) \([1]\). However, the method in \([1]\) does not seem to apply to the case in which one has a nonzero right hand side \( g \). Instead, to prove our result we adapted the method of Krylov for second order elliptic equations in nondivergence form with bounded measurable coefficients \([14]\) to this nonlocal framework.

In a forthcoming paper \([27]\) we will study the boundary regularity for fully nonlinear integro-differential equations. We are interested in the regularity up to the boundary of the quotient \( u/\delta^s \). As said before, the quantity \( \frac{u}{\delta^s}|_{\partial\Omega} \) plays the role that \( \frac{\partial u}{\partial \nu} \) plays in second order PDEs.

First, we will show that the class \( \mathcal{L}_0 \) of Caffarelli-Silvestre is “too big” to obtain boundary regularity results. Namely, we prove that there exist two different numbers \( \beta_1 \) and \( \beta_2 \) such that

\[
M_{\mathcal{L}_0}^+(x_+)^{\beta_1} \equiv 0, \quad M_{\mathcal{L}_0}^-(x_+)^{\beta_2} \equiv 0, \quad x \in \mathbb{R}_+.
\]

Here, \( M_{\mathcal{L}_0}^+ \) and \( M_{\mathcal{L}_0}^- \) are the extremal operators associated to the class \( \mathcal{L}_0 \). In fact, we see that the same result holds true for the classes \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \).

Instead, we will obtain boundary regularity results for the following class of kernels

\[
\mathcal{L}_s = \left\{ L u = \int_{\mathbb{R}^n} \left( u(x+y) - u(x) \right) K(y) dy; \quad K(y) = \frac{a(y/|y|)}{|y|^{n+2s}}, \quad \lambda \leq a \leq \Lambda \right\}.
\]

Our main result for this class of kernels reads as follows. Let \( u \) be any bounded solution of the Isaacs equation \( \inf_{\beta} \sup_{\alpha} L_{\alpha\beta} u = g \) in \( \Omega \), \( u = 0 \) in \( \mathbb{R}^n \setminus \Omega \), with \( g \in L^\infty(\Omega) \) and \( L_{\alpha\beta} \in \mathcal{L}_s \). Then, \( u \in C^s(\mathbb{R}^n) \) and \( u/\delta^s \in C^\alpha(\Omega) \) for all \( \alpha < s \). The result is new even for linear equations with operators in \( \mathcal{L}_s \).

We have also obtained some nonexistence results for nonlocal problems with critical and supercritical nonlinearities \([26]\). Of course, for the fractional Laplacian, these results follow from the Pohozaev identity (3). Our results in \([26]\) apply to more general operators of the form (1), either with \( a_{ij} \equiv 0 \) or with \( (a_{ij}) \) positive definite. Note that in case
that $a_{ij} \equiv 0$, the operator may not have a defined order. In this case, our result applies to kernels $K$ satisfying that $K(y)|y|^{n+\sigma}$ is nondecreasing along rays from the origin, for some $\sigma \in (0,2)$. For example, it applies to sums of fractional Laplacians, or to anisotropic operators. Moreover, our method also yield nonexistence results for higher order fractional Laplacians (i.e., $(-\Delta)^s$ with $s > 1$) and also for nonlinear operators such as the fractional $p$-Laplacian.

Finally, we have also studied the regularity of stable solutions to reaction-diffusion problems with fractional diffusion [25]. For it, we needed estimates for solutions to the linear Dirichlet problem with $L^p$ data, i.e., $(-\Delta)^s u = g$ in $\Omega$, with $g \in L^p(\Omega)$. We proved optimal $L^q$ and $C^\beta$ estimates for $u$, depending on the value of $p$. In addition, we needed also an $L^\infty$ estimate near the boundary of convex domains, which we obtained via the moving planes method. The results of [25] on the regularity of stable solutions are explained in the next section.

2 Stable solutions to reaction-diffusion problems

Reaction-diffusion equations play a central role in PDE theory and its applications to other sciences. They model many problems, running from Physics, Biology and Ecology, to Financial Mathematics and Economy. They also play an important role in some geometric problems: the problem of prescribing a curvature on a manifold, conformal classification of varieties, and parabolic flows on manifolds.

My research on this field concerns the regularity of minimizers to some elliptic equations, a classical problem in the Calculus of Variations. An important example in Geometry is the regularity of minimal hypersurfaces of $\mathbb{R}^n$ which are minimizers of the area functional. A deep result from the seventies states that these hypersurfaces are smooth if $n \leq 7$, while in $\mathbb{R}^8$ the Simons cone is a minimizing minimal hypersurface with a singularity at 0. The same phenomenon—the fact that regularity holds in low dimensions—happens for other nonlinear equations in bounded domains. For instance, let $u$ be a solution of

\[
\begin{cases}
-\Delta u &= f(u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{cases}
\] (4)

It is still an open question whether local minimizers of this equation have or not singularities if $n \leq 9$. In dimensions $n \geq 10$ there are examples of singular solutions to this problem which are local minimizers.

One can consider a more general class of solutions to (4): stable solutions. These are the solutions with nonnegative second variation of energy. In particular, any local minimizer is a stable solution.

A typical example is the semilinear elliptic equation $-\Delta u = \lambda f(u)$ in $\Omega \subset \mathbb{R}^n$, with $\lambda > 0$, posed in a smooth bounded domain $\Omega$ of $\mathbb{R}^n$. The nonlinearity $f$ is a continuous, positive, and increasing function, with $f(0) > 0$ and $\lim_{t \to +\infty} f(t)/t = +\infty$. Under these conditions, there exists an extremal value $\lambda^* \in (0, +\infty)$ such that for each $0 < \lambda < \lambda^*$ there exists a positive minimal solution $u_{\lambda}$, while for $\lambda > \lambda^*$ the problem has no solution; see [3]. Here, minimal means the smallest positive solution. For $\lambda = \lambda^*$, there exists a
weak solution, given by \( u^*(x) = \lim_{\lambda \to \lambda^*} u_\lambda(x) \), which is called the extremal solution. The extremal solution is always a stable solution. The existence of other non-minimal solutions for \( \lambda < \lambda^* \) depends strongly on the regularity of the extremal solution.

The regularity of stable solutions was studied in the nineties for different nonlinearities \( f \), essentially exponential and power nonlinearities. In both cases a similar result holds: if \( n \leq 9 \) then stable solutions are bounded (and hence classical) for every domain \( \Omega \), while for \( n \geq 10 \) there are examples of unbounded stable solutions even in the unit ball; see [11].

For general nonlinearities \( f \) and general domains \( \Omega \), it is known that when \( n \leq 4 \) any stable solution is bounded. The problem is still open in dimensions \( 5 \leq n \leq 9 \). A partial result in this direction is that all stable solutions are bounded in dimensions \( n \leq 9 \) when the domain \( \Omega \) is a ball.

• In [4] we studied the regularity of stable solutions in the class of domains that we call of double revolution. These are those domains which are invariant under rotations of the first \( m \) variables and of the last \( n-m \) variables, that is,

\[
\Omega = \{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} : (s = |x^1|, t = |x^2|) \in \Omega_2\},
\]

where \( \Omega_2 \subset \mathbb{R}^2 \) is a bounded domain even (or symmetric) with respect to each coordinate. Our main result is the following:

If \( \Omega \) is any convex domain of double revolution and \( n \leq 7 \), then all stable solutions to (4) are bounded.

Except for the radial case, our result is the first partial answer valid for all nonlinearities \( f \) in dimensions \( 5 \leq n \leq 9 \). Moreover, in all dimensions \( n \geq 8 \) we proved that stable solutions belong to \( H^1_0(\Omega) \), and obtained also \( L^q \) estimates, with \( q > 2 \) depending on the dimension \( n \).

The proofs of our results use the stability property to obtain control on some integrals of the form

\[
\int_{\Omega_2} \left( s^{-\alpha} u_s^2 + t^{-\beta} u_t^2 \right) dsdt.
\]

Then, after a change of variables of the form \( \sigma = s^\gamma, \tau = t^\nu \), we were led to the following problem: given \( a > -1 \) and \( b > -1 \), find the greatest exponent \( q \) for which

\[
\left( \int_{\Omega_2} \sigma^a \tau^b |u|^q d\sigma d\tau \right)^{1/q} \leq C \left( \int_{\Omega_2} \sigma^a \tau^b |
abla u|^2 d\sigma d\tau \right)^{1/2}
\]

holds for all smooth functions \( u \) vanishing on the boundary of \( \Omega_2 \).

• Thus, we were led to study weighted Sobolev inequalities with monomial weights \( w(x) = x_1^{A_1} \cdots x_n^{A_n} \).

These weighted Sobolev inequalities, and their corresponding isoperimetric inequalities with weights, were extensively studied later in our works [5] and [6]-[7], as detailed in the next section.

• Finally, as mentioned in the previous section, we have also studied stable solutions to reaction-diffusion problems with fractional diffusion [25]. This corresponds to (4) with the
Laplacian $-\Delta$ replaced by the fractional Laplacian (2). For exponential and power-like nonlinearities, we showed that stable solutions are bounded whenever $n < 10$. Note that in the limit $s \uparrow 1$, $n < 10$ is optimal. In addition, we showed that any stable solution belongs to $H^s(\mathbb{R}^n)$ whenever the domain is convex.

Although our boundedness result is optimal for $s \uparrow 1$, we do not expect this result to be optimal for any $s \in (0, 1)$. In fact, we expect stable solutions to be bounded in dimensions $n \leq 7$ for all $s \in (0, 1)$. In a future work we are planning to study radial stable solutions to $(-\Delta)^s u = f(u)$ in $B_1$, $u \equiv 0$ in $\mathbb{R}^n \setminus B_1$. This should be done with similar ideas to the case of the Laplacian. However, the generalization does not seem straightforward but instead presents interesting mathematical questions.

3 Isoperimetric inequalities with weights

Sobolev-type inequalities play a key role in Analysis and in the study of solutions to partial differential equations. They are flexible tools and are useful in many different settings.

As explained in the previous section, while studying reaction-diffusion equations (4) we were led to some Sobolev inequalities with monomial weights. These Sobolev inequalities, and the corresponding isoperimetric inequalities with weights, were studied later in our works [5] and [6]-[7].

- The main result in [5] is the following Sobolev inequality with monomial weights of the form $x^A = |x_1|^{A_1} \cdot |x_n|^{A_n}$ with every $A_i \geq 0$:

\[
\left( \int_{\mathbb{R}^n} x^A |u|^{p_*} \, dx \right)^{1/p_*} \leq C_p \left( \int_{\mathbb{R}^n} x^A |\nabla u|^p \, dx \right)^{1/p},
\]

(5)

Here, $1 \leq p < \infty$, $p_* = \frac{pD}{D-p}$ and $D = n + A_1 + \cdots + A_n$. Moreover, we obtain an explicit expression of the best constant $C_p$ in inequality (5), as well as extremal functions for which the best constant is attained. For $p > D$ and $p = D$, we proved weighted versions of the classical Morrey and Trudinger inequalities, respectively.

Note that the expression of $p_*$ is exactly the one from the classical Sobolev inequality, but in this case the ‘dimension’ is given by $D$. This ‘dimension’ $D$ has a clear interpretation in case of integer exponents $A_i$. Indeed, in this integer case the inequality (5) is just the classical Sobolev inequality in $\mathbb{R}^D = \mathbb{R}^{A_1+1} \times \cdots \times \mathbb{R}^{A_n+1}$ for functions depending on the radial variables of each $\mathbb{R}^{A_i+1}$.

The proof of the inequality (5) is based on a new weighted isoperimetric inequality,

\[
\left( \int_{E} x^A \, dx \right)^{\frac{D-1}{D}} \leq C \int_{\partial E} x^A \sigma(x),
\]

with the optimal constant $C$ depending on $n, A_1, ..., A_n$. We establish it by adapting a proof of the classical Euclidean isoperimetric inequality due to X. Cabré. Our proof uses a linear Neumann problem for the operator $x^{-A} \text{div}(x^A \nabla \cdot)$ combined with the Alexandroff contact set method (or ABP method). As before, when all the exponents $A_i$ are nonnegative integers, the operator $x^{-A} \text{div}(x^A \nabla \cdot)$ has an interpretation in terms of the Laplacian in
written in appropriate radial coordinates. The best constant is attained, for example when $A_i > 0$ for each $i$, by domains of the form $E = B_R(0) \cap (\mathbb{R}_+)^n$.

This type of isoperimetric inequalities with weights (also called densities) have attracted much attention recently, and there are many results available on existence, regularity, or boundedness of minimizers—see for example [13, 16, 12].

However, the solution to the isoperimetric problem in $\mathbb{R}^n$ with a weight $w$ is known only for very few weights, even in the case $n = 2$. For example, in $\mathbb{R}^n$ with the Gaussian weight $w(x) = e^{-|x|^2}$ all the minimizers are half-spaces [2], and with $w(x) = e^{\|x\|^2}$ all the minimizers are balls centered at the origin [28]. For the radial homogeneous weight $|x|\alpha$ in the plane $\mathbb{R}^2$, with $\alpha > 0$, all minimizers are circles passing through the origin (not centered at the origin). In particular, we see that even radial homogeneous weights may lead to nonradial minimizers.

- In [7]-[6] we obtained a family of sharp isoperimetric inequalities with homogeneous weights in convex cones $\Sigma \subset \mathbb{R}^n$. We proved that Euclidean balls centered at the origin solve the isoperimetric problem in any open convex cone $\Sigma$ of $\mathbb{R}^n$ (with vertex at the origin) for a certain class of nonradial homogeneous weights. More precisely, our result applies to all nonnegative continuous weights $w$ which are positively homogeneous of degree $\alpha \geq 0$ and such that $w^{1/\alpha}$ is concave in the cone $\Sigma$. It states that if $w$ satisfies these conditions, then

$$\frac{P_w(E; \Sigma)}{w(E \cap \Sigma)^{n-1/D}} \geq \frac{P_w(B_1; \Sigma)}{w(B_1 \cap \Sigma)^{n-1/D}}$$

for all sets $E$ with finite measure, where $D = n + \alpha$.

We also solved weighted anisotropic isoperimetric problems in cones for the same class of weights. In these weighted anisotropic problems, the perimeter of a smooth domain $E$ is given by

$$\int_{\partial E \cap \Sigma} H(\nu(x))w(x)dS,$$

where $\nu(x)$ is the unit outward normal to $\partial E$ at $x$, and $H$ is a positive, positively homogeneous of degree one, and convex function.

The proof of (6) consists of applying the ABP method to a linear Neumann problem involving the operator $w^{-1}\text{div}(w\nabla u)$, where $w$ is the weight. When $w \equiv 1$, the idea goes back to 2000 in a work of X. Cabré, where the classical isoperimetric inequality in all of $\mathbb{R}^n$ was proved with a new method.

Moreover, as a particular case of our results, we provide with new proofs of classical results such as the Wulff inequality and the isoperimetric inequality in convex cones of Lions and Pacella.

Equality in (6) holds whenever $E \cap \Sigma = B_r \cap \Sigma$, where $r$ is any positive number. However, in [7] we did not prove that $B_r \cap \Sigma$ is the unique minimizer of (6). The reason is that our proof involves the solution of an elliptic equation and, due to an important issue on its regularity, we need to approximate the given set $E$ by smooth sets. In a future work with E. Cinti and A. Pratelli we are planning to refine the analysis in the proof of [3]-[7] and obtain a quantitative version of our isoperimetric inequality in cones. In particular, we will deduce uniqueness of minimizers (up to sets of measure zero). The quantitative
version will be proved using the techniques of [5]-[7] (the ABP method applied to a linear Neumann problem) together with the ideas of Figalli-Maggi-Pratelli developed in [13].

4 Other works

As an undergraduate student I worked in other fields, not related to elliptic PDEs.

On the one hand, during the years 2009 and 2010 I collaborated with Prof. J.M. Olm and we obtained some results in the fields of control theory [20] [17] and of ordinary differential equations [19] [18].

On the other hand, in 2009 I also worked in number theory under the supervision of Prof. J.C. Lario, and I obtained some results that were published in [21].

References


