Integro-differential equations:
Regularity theory and Pohozaev identities

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PhD Thesis
Advisor: Xavier Cabré
Structure of the thesis

- **PART I**: Integro-differential equations

\[ Lu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x+y)) K(y) dy \]

- **PART II**: Regularity of stable solutions to elliptic equations

\[ -\Delta u = \lambda f(u) \quad \text{in} \quad \Omega \subset \mathbb{R}^n \]

- **PART III**: Isoperimetric inequalities with densities

\[ \frac{|\partial \Omega|}{|\Omega|^{\frac{n-1}{n}}} \geq \frac{|\partial B_1|}{|B_1|^{\frac{n-1}{n}}} \]
PART I

1. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, \([J. Math. Pures Appl. '14]\)

2. The Pohozaev identity for the fractional Laplacian, \([ARMA '14]\)

3. Nonexistence results for nonlocal equations with critical and supercritical nonlinearities, \([Comm. PDE '14]\)

4. Boundary regularity for fully nonlinear integro-differential equations, \(Preprint\).
PART II

5. Regularity of stable solutions up to dimension 7 in domains of double revolution, [Comm. PDE ’13]

6. The extremal solution for the fractional Laplacian, [Calc. Var. PDE ’14]

PART III

8. Sobolev and isoperimetric inequalities with monomial weights, 
   \[ J. \text{ Differential Equations '13} \]

PART I:

Integro-differential equations
Nonlocal equations

Linear **elliptic** integro-differential operators:

\[ Lu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x + y)) K(y) dy, \]

with \( K \geq 0 \), \( K(y) = K(-y) \), and

\[ \int_{\mathbb{R}^n} \min(1, |y|^2) K(y) dy < \infty. \]

- Brownian motion \( \rightarrow \) 2nd order PDEs
- Lévy processes \( \rightarrow \) Integro-Differential Equations
Expected payoff

Brownian motion

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega \\
u &= \phi \quad \text{on } \partial \Omega
\end{aligned}
\]

\[u(x) = \mathbb{E}(\phi(X_\tau)) \quad \text{(expected payoff)}\]

\[X_t = \text{Random process, } X_0 = x\]

\[\tau = \text{first time } X_t \text{ exits } \Omega\]
Expected payoff

Brownian motion

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega \\
\u &= \phi \quad \text{on } \partial \Omega 
\end{aligned}
\]

Lévy processes

\[
\begin{aligned}
\mathcal{L} u &= 0 \quad \text{in } \Omega \\
u &= \phi \quad \text{in } \mathbb{R}^n \setminus \Omega 
\end{aligned}
\]

\[
\begin{aligned}
u(x) &= \mathbb{E}(\phi(X_\tau)) \quad \text{(expected payoff)} \\
X_t &= \text{Random process}, \quad X_0 = x \\
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More equations from Probability

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| Optimal stopping time | Obstacle problem |
The fractional Laplacian

- Most canonical example of elliptic integro-differential operator:

\[ (-\Delta)^s u(x) = c_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \, dy, \quad s \in (0, 1). \]

- Notation justified by

\[ \hat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi), \quad \rightarrow \quad (-\Delta)^s \circ (-\Delta)^t = (-\Delta)^{s+t}. \]

- It corresponds to **stable** and radially symmetric Lévy process.
Stable Lévy processes

Special class of Lévy processes: stable processes

\[ Lu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x + y)) \frac{a(y/|y|)}{|y|^{n+2s}} \, dy \]

- Very important and well studied in Probability
- These are processes with self-similarity properties \( X_t \approx t^{-1/\alpha} X_1 \)
- Central Limit Theorems \( \leftrightarrow \) stable Lévy processes
- \( a(\theta) \) is called the spectral measure (defined on \( S^{n-1} \)).
Why studying nonlocal equations?

Nonlocal equations are used to model (among others):

- Prices in Finance (since the 1990’s)
- Anomalous diffusions (Physics, Ecology, Biology): $u_t + Lu = f(x, u)$

Also, they arise naturally when long-range interactions occur:

- Image Processing
- Relativistic Quantum Mechanics $\sqrt{-\Delta + m}$
- Boltzmann equation
Why studying nonlocal equations?

Still, these operators appear in:

- Fluid Mechanics (surface quasi-geostrophic equation)
- Conformal Geometry

Finally, all PDEs are limits of nonlocal equations (as $s \uparrow 1$).
Important works

- Works in Probability 1950-2014 (Kac, Getoor, Bogdan, Bass, Chen, ...)
- Fully nonlinear equations: Caffarelli-Silvestre ’07-10 [CPAM, Annals, ARMA]
- Reaction-diffusion equations $u_t + Lu = f(x, u)$
- Obstacle problem, free boundaries
- Nonlocal minimal surfaces, fractional perimeters
- Fluid Mech.: Caffarelli-Vasseur [Annals’10], [JAMS’11]
The classical Pohozaev identity

\[
\begin{aligned}
-\Delta u &= f(u) \quad \text{in } \Omega \\
 u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

Theorem (Pohozaev, 1965)

\[
\int_{\Omega} \left\{ n F(u) - \frac{n-2}{2} uf(u) \right\} = \frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) d\sigma
\]

Follows from: For any function \( u \) with \( u = 0 \) on \( \partial \Omega \),

\[
\int_{\Omega} (x \cdot \nabla u) \Delta u = \frac{2-n}{2} \int_{\Omega} u \Delta u + \frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) d\sigma
\]

And this follows from the divergence theorem.
The classical Pohozaev identity

Applications of the identity:

- Nonexistence of solutions: critical exponent $-\Delta u = u^{\frac{n+2}{n-2}}$
- Unique continuation “from the boundary”
- Monotonicity formulas
- Concentration-compactness phenomena
- Radial symmetry
- Stable solutions: uniqueness results, $H^1$ interior regularity
- Other: Geometry, control theory, wave equation, harmonic maps, etc.
Pohozaev identities for \((-\Delta)^s\)

Assume

\[
\left| (-\Delta)^s u \right| \leq C \quad \text{in } \Omega \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\]

(+ some interior regularity on \(u\))

**Theorem (R-Serra’12; ARMA)**

*If \(\Omega\) is \(C^{1,1}\),*

\[
\int_{\Omega} (x \cdot \nabla u) (-\Delta)^s u = \frac{2s - n}{2} \int_{\Omega} u (-\Delta)^s u - \frac{\Gamma(1 + s)^2}{2} \int_{\partial\Omega} \left( \frac{u}{d^s(x)} \right)^2 (x \cdot \nu)
\]

Here, \(\Gamma\) is the gamma function.
Remark

\[
\frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) \quad \sim \quad \frac{\Gamma(1 + s)^2}{2} \int_{\partial \Omega} \left( \frac{u}{d^s} \right)^2 (x \cdot \nu)
\]

\[\left. \frac{u}{d^s} \right|_{\partial \Omega} \quad \text{plays the role that} \quad \frac{\partial u}{\partial \nu} \quad \text{plays in 2nd order PDEs}\]
Pohozaev identities for \((-\Delta)^s\)

Changing the origin in our identity, we find

\[
\int_{\Omega} u_{xi} (-\Delta)^s u = \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left( \frac{u}{d^s} \right)^2 \nu_i
\]

Thus,

**Corollary**

*Under the same hypotheses as before*

\[
\int_{\Omega} (-\Delta)^s u \, \nu_{xi} = - \int_{\Omega} u_{xi} (-\Delta)^s v + \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{d^s} \frac{v}{d^s} \nu_i
\]

Note the contrast with the nonlocal flux in the formula

\[
\int_{\Omega} (-\Delta)^s w = \int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} \ldots
\]
Ideas of the proof

1. \( u_\lambda(x) = u(\lambda x), \lambda > 1, \Rightarrow \)

\[
\int_\Omega (x \cdot \nabla u)(-\Delta)^s u = \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_\Omega u_\lambda(-\Delta)^s u
\]

2. \( \Omega \) star-shaped \( \Rightarrow \)

\[
\int_\Omega (x \cdot \nabla u)(-\Delta)^s u = \frac{2s-n}{2} \int_\Omega u(-\Delta)^s u + \frac{1}{2} \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda},
\]

\( w = (-\Delta)^{\frac{s}{2}} u \)

3. Analyze very precisely the singularity of \( (-\Delta)^{\frac{s}{2}} u \) along \( \partial \Omega \), and compute.

4. Deduce the result for general \( C^{1,1} \) domains.
1. \( u(x) = (x_+)^s \) satisfies \( (-\Delta)^s u = 0 \) in \( (0, +\infty) \).

2. Explicit solution by [Getoor, 1961]:

\[
(\Delta)^s u = 1 \quad \text{in} \quad B_1 \quad \text{and} \quad u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_1
\]

\[\implies u(x) = c(1 - |x|^2)^s\]

- They are \( C^\infty \) inside \( \Omega \), but \( C^s(\overline{\Omega}) \) and not better!
- In both cases, they are comparable to \( d^s \), where \( d(x) = \text{dist}(x, \mathbb{R}^n \setminus \Omega) \).
Boundary regularity: First results

\[ \begin{cases} (-\Delta)^s u = g \quad \text{in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \]

Then, \( \|u\|_{C^s(\Omega)} \leq C\|g\|_{L^\infty}. \) Moreover,

**Theorem (R-Serra’12; J. Math. Pures Appl.)**

\( \Omega \) bounded and \( C^{1,1} \) domain. Then,

- \( \|u/d^s\|_{C^\gamma(\partial \Omega)} \leq C\|g\|_{L^\infty} \) for some small \( \gamma > 0 \),

where \( d \) is the distance to \( \partial \Omega \).

Proof: Can not do odd reflection! (boundary behavior different from interior!)
Boundary for integro-differential operators?

\[
\begin{aligned}
(-\Delta)^s u &= g(x) \quad \text{in } \Omega \\
 u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

\[\Rightarrow\quad u/d^s \in C^\gamma(\overline{\Omega}).\]

We answer an open question: What about boundary regularity for more general operators of “order” 2s?

\[Lu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x + y)) K(y) dy\]

Is it true that \(u/d^s\) is Hölder continuous? At least bounded?

- We answer this for linear and also for fully nonlinear equations.
Fully nonlinear integro-differential equations

Let us consider solutions to

\[
\begin{cases}
  lu = f & \text{in } \Omega \\
  u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

where \( I \) is a fully nonlinear operator like

\[
lu(x) = \sup_{\alpha} L_\alpha u(x) \quad \text{(controlled diffusion)}
\]

Here, all \( L_\alpha \in \mathcal{L} \) for some class of linear operators \( \mathcal{L} \).

- The class \( \mathcal{L} \) is called the **ellipticity class**.

(When \( L_\alpha \) are 2nd order operators, we have \( F(D^2 u) = f(x) \) in \( \Omega \))
**Interior regularity:**

- Was developed by Caffarelli and Silvestre in 2007-2010 (CPAM, Annals, ARMA)
- They established: Krylov-Safonov, Evans-Krylov, perturbative theory, etc.
- The reference ellipticity class of Caffarelli-Silvestre is $\mathcal{L}_0$, with kernels
  \[
  \frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}
  \]

**Boundary regularity:**

We establish boundary regularity for the class $\mathcal{L}_*$, with kernels

\[
K(y) = \frac{a(y/|y|)}{|y|^{n+2s}}, \quad \lambda \leq a(\theta) \leq \Lambda
\]

$\mathcal{L}_* = \text{Subclass of } \mathcal{L}_0$, corresponding to *stable* Lévy processes
Let \( I(u, x) \) be a fully nonlinear operator elliptic w.r.t. \( \mathcal{L}_* \), and

\[
\begin{align*}
I(u, x) &= f(x) \quad \text{in } \Omega \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{align*}
\]

**Theorem (R-Serra; preprint’14)**

*If \( \Omega \) is \( C^{1,1} \), then any viscosity solution satisfies*

\[
\|u/d^s\|_{C^{s-\epsilon}(\Omega)} \leq C\|f\|_{L^\infty(\Omega)} \quad \text{for all } \epsilon > 0
\]

Even for \((-\Delta)^s\) we improve our previous results!
Novelty: We obtain higher regularity for $u/d^s$!

The exponent $s - \epsilon$ is optimal for $f \in L^\infty$

Also, it cannot be improved if $a \in L^\infty(S^{n-1})$

Very important: $\mathcal{L}_*$ is the good class for boundary regularity!

The class $\mathcal{L}_0$ is too large for fine boundary regularity

There exist positive numbers $0 < \beta_1 < s < \beta_2$ such that

$$l_1(x_+)^{\beta_1} \equiv 0, \quad l_2(x_+)^{\beta_2} \equiv 0 \quad \text{in} \{x > 0\}$$

Solutions are not even comparable near the boundary!
Main steps of the proof of $u/d^s \in C^{s-\epsilon}(\Omega)$:

1. Bounded measurable coefficients $\implies u/d^s \in C^\gamma(\Omega)$

2. Blow up the equation at $x \in \partial \Omega$ + compactness argument.

3. Liouville theorem in half-space

Advantages of the method:

- It allows us to obtain higher regularity of $u/d^s$, also in the normal direction!
- After blow up, you do not see the geometry of the domain
- Also non translation invariant equations
- Discontinuous kernels $a \in L^\infty(S^{n-1})$
PART II:

Regularity of stable solutions to elliptic equations
Regularity of minimizers

Classical problem in the Calculus of Variations: Regularity of minimizers

Example in Geometry: Regularity of hypersurfaces in $\mathbb{R}^n$ which minimize the area functional.

- These hypersurfaces are smooth if $n \leq 7$
- In $\mathbb{R}^8$ the Simons cone minimizes area and has a singularity at $x = 0$

As we will see, the same happens for other nonlinear PDE in bounded domains.
Regularity of minimizers

\[
\begin{cases}
-\Delta u = f(u) \quad \text{in } \Omega \subset \mathbb{R}^n \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

Open problem:

\( u \) local minimizer (or stable solution) \& \( n \leq 9 \implies u \in L^\infty \)?

- In \( \mathbb{R}^{10} \), \( u(x) = \log \frac{1}{|x|^2} \) is a stable solution in \( B_1 \)
- \( f(u) = \lambda e^u \) or \( f(u) = \lambda (1 + u)^p \) \& \( n \leq 9 \) \[Crandall-Rabinowitz '75\]
- \( \Omega = B_1 \) \& \( n \leq 9 \) \[Cabré-Capella '06\]
- \( n \leq 4 \) \[n \leq 3 \text{ Nedev '00; } n \leq 4 \text{ Cabré '10}\]
The extremal solution

If $f(u) \rightsquigarrow \lambda f(u)$, then there is $\lambda^* \in (0, +\infty)$ s.t.

- For $0 < \lambda < \lambda^*$, there is a bounded solution $u_\lambda$.
- For $\lambda > \lambda^*$, there is no solution.
- For $\lambda = \lambda^*$,

$$u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$$

is a weak solution, called the extremal solution. Moreover, it is stable.

**Question:** Is the extremal solution bounded? [Brezis-Vázquez '97]
Our work

We have studied the regularity of stable solutions to

\[
\begin{cases}
-\Delta u = f(u) \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

[Comm. PDE ’13]

**Thm:** $L^\infty$ for $n \leq 7$ & $\Omega$ of double revolution

\[
\begin{cases}
(-\Delta)^s u = f(u) \quad \text{in } \Omega \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

[Calc. Var. PDE ’13]  

**Thm:** $L^\infty$ and $H^s$ bounds in general domains;

**Thm:** optimal regularity for $f(u) = \lambda e^u$ in $x_i$-symmetric domains
Sobolev inequalities with weights

When studying $-\Delta u = f(u)$, we needed

$$\int_{\Omega} \left\{ s^{-\alpha} |u_s|^2 + t^{-\beta} |u_t|^2 \right\} ds \, dt \leq C \quad \Rightarrow \quad u \in L^q(\Omega) ? \quad q(\alpha, \beta) = ?$$

After a change of variables, we want

$$\left( \int_{\tilde{\Omega}} |u|^q x_1^a x_2^b \, dx_1 \, dx_2 \right)^{1/q} \leq C \left( \int_{\tilde{\Omega}} |\nabla u|^2 x_1^a x_2^b \, dx_1 \, dx_2 \right)^{1/2}, \quad q = q(a, b)$$

Thus, we want Sobolev inequalities with weights

$$\left( \int_{\mathbb{R}^n} |u|^q w(x) \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |\nabla u|^p w(x) \, dx \right)^{1/p}, \quad w(x) = x_1^{A_1} \cdots x_n^{A_n}$$
PART III:

Isoperimetric inequalities with densities
Sobolev inequalities with weights

**Theorem (Cabré-R; J. Differential Equations’13)**

Let $A_i \geq 0$,

$$w(x) = x_1^{A_1} \cdots x_n^{A_n}, \quad D = n + A_1 + \cdots + A_n.$$ 

Let $1 \leq p < D$. Then,

$$\left( \int_{\mathbb{R}^n} |u|^q w(x)dx \right)^{1/q} \leq C_{p,A} \left( \int_{\mathbb{R}^n} |\nabla u|^p w(x)dx \right)^{1/p},$$

with $q = pD/(D - p)$.

To prove the result, we establish a new weighted isoperimetric inequality.
Isoperimetric inequalities with monomial weights

Theorem (Cabré-R; J. Differential Equations’13)

Let $A_i > 0$, $w(x)$ and $D$ as before, and

$$\Sigma = \{ x \in \mathbb{R}^n : x_1, \ldots, x_n > 0 \}.$$

Then, for any $E \subset \Sigma$,

$$\frac{P_w(E)}{w(E)^{\frac{D-1}{D}}} \geq \frac{P_w(B_1 \cap \Sigma)}{w(B_1 \cap \Sigma)^{\frac{D-1}{D}}}.$$

We denoted the weighted volume and perimeter

$$w(E) = \int_E w(x) dx \quad \quad P_w(E) = \int_{\Sigma \cap \partial E} w(x) dS.$$
This type of isoperimetric inequalities have been widely studied:

- $w(x) = e^{-|x|^2}$ [Borell; Invent. Math.’75]
- Existence and regularity of minimizers (Pratelli, Morgan, …)
- Log-convex radial densities $w(|x|)$ [Figalli-Maggi ’13], [Chambers ’14]
- With $w(x) = e^{|x|^2}$, $w(x) = |x|^\alpha$, or other particular weights
- …
Isoperimetric inequalities in cones

In general cones $\Sigma$, a well known result is the following:

**Theorem (Lions-Pacella ’90)**

Let $\Sigma$ be any open convex cone in $\mathbb{R}^n$. Then, for any $E \subset \Sigma$,

\[
\frac{|\Sigma \cap \partial E|}{|E|^{\frac{n}{n-1}}} \geq \frac{|\Sigma \cap \partial B_1|}{|B_1 \cap \Sigma|^{\frac{n}{n-1}}}.
\]

Important: Only the perimeter inside $\Sigma$ is counted
Theorem (Cabrè-R-Serra; preprint ’13)

Let \( \Sigma \) be any convex cone in \( \mathbb{R}^n \). Assume

\[ w(x) \text{ homogeneous of degree } \alpha \geq 0, \quad \& \quad w^{1/\alpha} \text{ concave in } \Sigma. \]

Then, for any \( E \subset \Sigma \),

\[ \frac{P_w(E)}{w(E)^{\frac{D-1}{D}}} \geq \frac{P_w(B_1 \cap \Sigma)}{w(B_1 \cap \Sigma)^{\frac{D-1}{D}}} . \]

Recall

\[ w(E) = \int_E w(x) dx \quad P_w(E) = \int_{\Sigma \cap \partial E} w(x) dS. \]
Comments

- Minimizers are radial, while \( w(x) \) is not!
- When \( w \equiv 1 \) we recover the result of Lions-Pacella (with new proof!).
- We can also treat **anisotropic perimeters**

\[
P_{w,H}(E) = \int_{\Sigma \cap \partial E} H(\nu) \, w(x) dS.
\]

\( w \equiv 1 \implies \) new proof of the **Wulff theorem**.

- Some examples of weights are

\[
w(x) = \text{dist}(x, \partial \Sigma)^\alpha, \quad w(x) = \sqrt{x} + \sqrt{y}, \quad w(x) = \frac{xyz}{x + y + z}, \quad ...
\]
The proof

- The proof uses the ABP technique applied to an appropriate PDE
- When \( w \equiv 1 \), the idea goes back to the work [Cabré ’00] (for the classical isoperimetric inequality)
- Here, we need to consider a linear Neumann problem in \( E \subset \Sigma \) involving the operator \( w^{-1} \text{div}(w \nabla u) \)
The end

Thank you!