Integro-differential equations:
Regularity theory and Pohozaev identities

Xavier Ros Oton
Departament Matemàtica Aplicada I, Universitat Politècnica de Catalunya
PhD Thesis
Advisor: Xavier Cabré
Structure of the thesis

- **PART I**: Integro-differential equations

\[ Lu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x + y))K(y)dy \]

- **PART II**: Regularity of stable solutions to elliptic equations

\[ -\Delta u = \lambda f(u) \quad \text{in } \Omega \subset \mathbb{R}^n \]

- **PART III**: Isoperimetric inequalities with densities

\[ \frac{|\partial \Omega|}{|\Omega|^\frac{n-1}{n}} \geq \frac{|\partial B_1|}{|B_1|^\frac{n-1}{n}} \]
PART I

1. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, [J. Math. Pures Appl. '14]

2. The Pohozaev identity for the fractional Laplacian, [ARMA '14]

3. Nonexistence results for nonlocal equations with critical and supercritical nonlinearities, [Comm. PDE '14]

PART II

5. Regularity of stable solutions up to dimension 7 in domains of double revolution, \([\textit{Comm. PDE '13}]\)

6. The extremal solution for the fractional Laplacian, \([\textit{Calc. Var. PDE '14}]\)

PART III

8. Sobolev and isoperimetric inequalities with monomial weights, 

[J. Differential Equations ’13]

PART I:

Integro-differential equations
Nonlocal equations

Linear **elliptic** integro-differential operators:

\[ Lu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x + y)) K(y) dy, \]

with \( K \geq 0 \), \( K(y) = K(-y) \), and

\[ \int_{\mathbb{R}^n} \min(1, |y|^2) K(y) dy < \infty. \]
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- Brownian motion \[\rightarrow\] 2nd order PDEs
- Lévy processes \[\rightarrow\] Integro-Differential Equations
Expected payoff

Brownian motion

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
 u = \phi & \text{on } \partial \Omega
\end{cases}
\]

\[u(x) = \mathbb{E}(\phi(X_\tau)) \quad \text{(expected payoff)}\]

\[X_t = \text{Random process, } X_0 = x\]

\[\tau = \text{first time } X_t \text{ exits } \Omega\]
Expected payoff

Brownian motion

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\]

Lévy processes

\[
\begin{align*}
Lu &= 0 & \text{in } \Omega \\
u &= \phi & \text{in } \mathbb{R}^n \setminus \Omega
\end{align*}
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| Optimal stopping time | Obstacle problem |
The fractional Laplacian

- Most canonical example of elliptic integro-differential operator:

\[
(-\Delta)^s u(x) = c_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x + y)}{|y|^{n+2s}} \, dy, \quad s \in (0, 1).
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- Notation justified by

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(-\Delta)^s u(\xi) = |\xi|^{2s} \hat{u}(\xi), \quad \rightarrow \quad (-\Delta)^s \circ (-\Delta)^t = (-\Delta)^{s+t}.
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\]

It corresponds to \textit{stable} and radially symmetric Lévy process.
Stable Lévy processes

Special class of Lévy processes: stable processes

\[ Lu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x + y)) \frac{a(y/|y|)}{|y|^{n+2s}} \, dy \]

- Very important and well studied in Probability
- These are processes with self-similarity properties \((X_t \approx t^{-1/\alpha} X_1)\)
- Central Limit Theorems \(\longleftrightarrow\) stable Lévy processes
- \(a(\theta)\) is called the *spectral measure* (defined on \(S^{n-1}\)).
Why studying nonlocal equations?

Nonlocal equations are used to model (among others):

- Prices in Finance (since the 1990's)
- Anomalous diffusions (Physics, Ecology, Biology):
  \[ u_t + Lu = f(x, u) \]
- Also, they arise naturally when long-range interactions occur:

  - Image Processing
  - Relativistic Quantum Mechanics
  - Boltzmann equation

Xavier Ros Oton (UPC, Barcelona)
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Also, they arise naturally when long-range interactions occur:

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- Relativistic Quantum Mechanics $\sqrt{-\Delta} + m$
- Boltzmann equation
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Still, these operators appear in:

- Fluid Mechanics (surface quasi-geostrophic equation)
- Conformal Geometry
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Finally, all PDEs are limits of nonlocal equations (as \( s \uparrow 1 \)).
Important works

Works in Probability 1950-2014 (Kac, Getoor, Bogdan, Bass, Chen,...)

Fully nonlinear equations: Caffarelli-Silvestre '07-10 [CPAM, Annals, ARMA]

Reaction-diffusion equations
\[ u_t + L u = f(x, u) \]

Obstacle problem, free boundaries

Nonlocal minimal surfaces, fractional perimeters

Math. Physics: (Lieb, Frank,...) [JAMS'08], [Acta Math.'13]

Fluid Mech.: Caffarelli-Vasseur [Annals'10], [JAMS'11]
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The classical Pohozaev identity

\(-\Delta u = f(u)\) in \(\Omega\)
\[\begin{align*}
    u &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}\]

\[\int_{\Omega} \left\{ n F(u) - n - 2 \frac{\partial u}{\partial \nu} \right\} d\sigma = \frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) d\sigma \]

Theorem (Pohozaev, 1965)

Follows from: For any function \(u\) with \(u = 0\) on \(\partial \Omega\),
\[\int_{\Omega} (x \cdot \nabla u) \Delta u = 2 - n \int_{\Omega} u \Delta u + \frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) d\sigma \]

And this follows from the divergence theorem.
The classical Pohozaev identity

\[
\begin{aligned}
-\Delta u &= f(u) \quad \text{in } \Omega \\
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\]

**Theorem (Pohozaev, 1965)**

\[
\int_{\Omega} \left\{ n F(u) - \frac{n-2}{2} u f(u) \right\} = \frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (\mathbf{x} \cdot \nu) d\sigma
\]

Follows from: For any function \( u \) with \( u = 0 \) on \( \partial \Omega \),

\[
\int_{\Omega} (\mathbf{x} \cdot \nabla u) \Delta u = \frac{2-n}{2} \int_{\Omega} u \Delta u + \frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (\mathbf{x} \cdot \nu) d\sigma
\]

And this follows from the divergence theorem.
Applications of the identity:

- Nonexistence of solutions: critical exponent $-\Delta u = u^{\frac{n+2}{n-2}}$
- Unique continuation “from the boundary”
- Monotonicity formulas
- Concentration-compactness phenomena
- Radial symmetry
- Stable solutions: uniqueness results, $H^1$ interior regularity
- Other: Geometry, control theory, wave equation, harmonic maps, etc.
Pohozaev identities for \((-\Delta)^s\)

Assume \(|\nabla (u \cdot -\Delta)^s u| \leq C\) in \(\Omega\)

\(u = 0\) in \(\mathbb{R}^n \setminus \Omega\)

(Plus some interior regularity on \(u\))

Theorem (R-Serra'12; ARMA)

If \(\Omega\) is \(C^{1,1}\),

\[
\int_{\Omega} (x \cdot \nabla u) (-\Delta)^s u = 2s - n^2 \int_{\Omega} u (-\Delta)^s u - \Gamma(1 + s)^2 \int_{\partial \Omega} (u d_s(x))^2 (x \cdot \nu)
\]

Here, \(\Gamma\) is the gamma function.
Pohozaev identities for \((-\Delta)^s\)

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**Theorem (R-Serra’12; ARMA)**

If \(\Omega\) is \(C^{1,1}\),

\[
\int_{\Omega} (x \cdot \nabla u) (-\Delta)^s u = \frac{2s - n}{2} \int_{\Omega} u (-\Delta)^s u - \frac{\Gamma(1 + s)^2}{2} \int_{\partial \Omega} \left( \frac{u}{d^s(x)} \right)^2 (x \cdot \nu)
\]

Here, \(\Gamma\) is the gamma function.
Remark

\[
\frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) \quad \rightsquigarrow \quad \frac{\Gamma (1 + s)^2}{2} \int_{\partial \Omega} \left( \frac{u}{d^s} \right)^2 (x \cdot \nu)
\]

\[
\left. \frac{u}{d^s} \right|_{\partial \Omega} \quad \text{plays the role that} \quad \frac{\partial u}{\partial \nu} \quad \text{plays in 2nd order PDEs}
\]
Pohozaev identities for $(-\Delta)^s$

Changing the origin in our identity, we find

\[ \int_{\Omega} u_{x_i} (-\Delta)^s u = \frac{\Gamma(1 + s)^2}{2} \int_{\partial\Omega} \left( \frac{u}{d^s} \right)^2 \nu_i \]

Thus,
Pohozaev identities for \((-\Delta)^s\)

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\]

Thus,

**Corollary**

*Under the same hypotheses as before*

\[
\int_{\Omega} (-\Delta)^s u \, \nu_{x_i} = -\int_{\Omega} u_{x_i} (-\Delta)^s \nu + \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{d^s} \frac{\nu}{d^s} \nu_i
\]

Note the contrast with the nonlocal flux in the formula \(\int_{\Omega} (-\Delta)^s w = \int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} \ldots\)
Ideas of the proof

1. \( u_\lambda(x) = u(\lambda x) \), \( \lambda > 1 \), \( \Rightarrow \)

\[
\int_\Omega (x \cdot \nabla u)(-\Delta)^s u = \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_\Omega u_\lambda(-\Delta)^s u
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2. \( \Omega \) star-shaped \( \Rightarrow \)

\[
\int_{\Omega} (x \cdot \nabla u) (-\Delta)^s u = \frac{2s - n}{2} \int_{\Omega} u (-\Delta)^s u + \frac{1}{2} \left. \frac{d}{d\lambda} \right|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda},
\]

\( w = (-\Delta)^{s/2} u \)
Ideas of the proof

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where $w = (-\Delta)^{s/2} u$

3. Analyze very precisely the singularity of $(-\Delta)^{s/2} u$ along $\partial\Omega$, and compute.
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\[ w = (-\Delta)^{\frac{s}{2}} u \]

3. Analyze very precisely the singularity of \((-\Delta)^{\frac{s}{2}} u\) along \( \partial \Omega \), and compute.

4. Deduce the result for general \( C^{1,1} \) domains.
Fractional Laplacian: Two explicit solutions

1. \( u(x) = (x_+)^s \) satisfies \( (-\Delta)^s u = 0 \) in \( (0, +\infty) \).
Fractional Laplacian: Two explicit solutions

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2. Explicit solution by [Getoor, 1961]:

\[
\begin{align*}
(-\Delta)^s u &= 1 \quad \text{in } B_1 \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus B_1
\end{align*}
\]

\[
\implies u(x) = c(1 - |x|^2)^s
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$\implies$ $u(x) = c(1 - |x|^2)^s$

- They are $C^\infty$ inside $\Omega$, but $C^s(\cl{\Omega})$ and not better!
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$$

- They are $C^\infty$ inside $\Omega$, but $C^s(\overline{\Omega})$ and not better!
- In both cases, they are comparable to $d^s$, where $d(x) = \text{dist}(x, \mathbb{R}^n \setminus \Omega)$. 
Boundary regularity: First results

\[
\begin{cases}
(-\Delta)^s u = g \quad \text{in } \Omega \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

Then, \( \|u\|_{C^s(\overline{\Omega})} \leq C\|g\|_{L^\infty} \). Moreover,
Boundary regularity: First results

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Theorem (R-Serra’12; J. Math. Pures Appl.)

\( \Omega \) bounded and \( C^{1,1} \) domain. Then,

- \( \|u/d^s\|_{C^{\gamma}(\overline{\Omega})} \leq C\|g\|_{L^\infty} \) for some small \( \gamma > 0 \),

where \( d \) is the distance to \( \partial \Omega \).
Boundary regularity: First results

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where \( d \) is the distance to \( \partial \Omega \).

Proof: Can not do odd reflection! (boundary behavior different from interior!)
Boundary for integro-differential operators?

\[\begin{cases}
(-\Delta)^s u = g(x) & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}\quad \implies \quad u/d^s \in C^\gamma(\overline{\Omega}).\]
We answer an open question: What about boundary regularity for more general operators of “order” 2s?

\[ Lu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x + y)) K(y) dy \]

Is it true that \( u/d^s \) is Hölder continuous? At least bounded?
Boundary for integro-differential operators?

\[
\begin{cases}
(–\Delta)^s u = g(x) \quad \text{in } \Omega \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\Rightarrow u/d^s \in C^\gamma(\overline{\Omega}).
\]

We answer an open question: What about boundary regularity for more general operators of “order” 2s?

\[Lu(x) = PV \int_{\mathbb{R}^n} (u(x) – u(x + y))K(y) dy\]

Is it true that \(u/d^s\) is Hölder continuous? At least bounded?

- We answer this for linear and also for fully nonlinear equations
Fully nonlinear integro-differential equations

Let us consider solutions to

\[
\begin{cases}
lu = f & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

where \( I \) is a fully nonlinear operator like

\[
lu(x) = \sup_{\alpha} L_\alpha u(x)
\]

(controlled diffusion)
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where \( I \) is a fully nonlinear operator like

\[
l u (x) = \sup_{\alpha} L_{\alpha} u(x) \quad \text{(controlled diffusion)}
\]

Here, all \( L_{\alpha} \in \mathcal{L} \) for some class of linear operators \( \mathcal{L} \). 

- The class \( \mathcal{L} \) is called the ellipticity class.
Fully nonlinear integro-differential equations

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(When \( L_{\alpha} \) are 2nd order operators, we have \( F(D^2 u) = f(x) \) in \( \Omega \))
Interior regularity:

- Was developed by Caffarelli and Silvestre in 2007-2010 (CPAM, Annals, ARMA)
- They established: Krylov-Safonov, Evans-Krylov, perturbative theory, etc.
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$L_* = \text{Subclass of } L_0, \text{ corresponding to stable Lévy processes}$
Boundary regularity for fully nonlinear equations

Let \( l(u, x) \) be a fully nonlinear operator elliptic w.r.t. \( \mathcal{L}_* \), and

\[
\begin{cases}
  l(u, x) = f(x) & \text{in } \Omega \\
  u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

Theorem (R-Serra; preprint'14)

If \( \Omega \) is \( C^{1,1} \), then any viscosity solution satisfies

\[
\|u/ds\|_{C^{s,\epsilon}(\Omega)} \leq C \|f\|_{L^\infty(\Omega)}
\]

for all \( \epsilon > 0 \)

Even for \((-\Delta)^s\) we improve our previous results!

Xavier Ros Oton (UPC, Barcelona)

PhD Thesis

Barcelona, June 2014 26 / 43
Boundary regularity for fully nonlinear equations

Let $I(u, x)$ be a fully nonlinear operator elliptic w.r.t. $\mathcal{L}_*$, and

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Theorem (R-Serra; preprint’14)

If $\Omega$ is $C^{1,1}$, then any viscosity solution satisfies

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Very important: $L_*$ is the good class for boundary regularity!

The class $L_0$ is too large for fine boundary regularity

There exist positive numbers $0 < \beta_1 < s < \beta_2$ such that

$$l_1(x_+)^{\beta_1} \equiv 0, \quad l_2(x_+)^{\beta_2} \equiv 0 \quad \text{in} \{x > 0\}$$

Solutions are not even comparable near the boundary!
Main steps of the proof of $u/d^s \in C^{s-\epsilon}(\Omega)$:

1. Bounded measurable coefficients $\implies u/d^s \in C^\gamma(\Omega)$
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- Discontinuous kernels $a \in L^\infty(S^{n-1})$
PART II:

Regularity of stable solutions to elliptic equations
Regularity of minimizers

Classical problem in the Calculus of Variations: Regularity of minimizers
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Classical problem in the Calculus of Variations: Regularity of minimizers

Example in Geometry: Regularity of hypersurfaces in $\mathbb{R}^n$ which minimize the area functional.
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As we will see, the same happens for other nonlinear PDE in bounded domains.
Regularity of minimizers

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\begin{cases}
-\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^n \\
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\end{cases}
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Open problem:

\[u \text{ local minimizer (or stable solution) } \& \ n \leq 9 \implies u \in L^\infty?\]
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- \(\Omega = B_1\) \& \(n \leq 9\) [Cabré-Capella ’06]
- \(n \leq 4\) [\(n \leq 3\) Nedev ’00; \(n \leq 4\) Cabré ’10]
The extremal solution

If $f(u) \Rightarrow \lambda f(u)$, then there is $\lambda^* \in (0, +\infty)$ s.t.

For $0 < \lambda < \lambda^*$, there is a bounded solution $u_\lambda$.

For $\lambda > \lambda^*$, there is no solution.

For $\lambda = \lambda^*$, $u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$ is a weak solution, called the extremal solution. Moreover, it is stable.

Question: Is the extremal solution bounded? [Brezis-Vázquez '97]

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Our work

We have studied the regularity of stable solutions to

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[Comm. PDE ’13]

Thm: $L^\infty$ for $n \leq 7$ & $\Omega$ of double revolution
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\[\text{Thm: } L^\infty \text{ for } n \leq 7 \& \Omega \text{ of double revolution} \]

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\]

\[\text{Thm: } L^\infty \text{ and } H^s \text{ bounds in general domains;}\]

\[\text{Thm: optimal regularity for } f(u) = \lambda e^u \text{ in } x_i\text{-symmetric domains}\]
When studying $-\Delta u = f(u)$, we needed

$$\int_{\Omega} \left\{ s^{-\alpha} |u_s|^2 + t^{-\beta} |u_t|^2 \right\} ds dt \leq C \quad \Rightarrow \quad u \in L^q(\Omega) \quad ? \quad q(\alpha, \beta) = ?$$
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After a change of variables, we want

$$\left( \int_{\tilde{\Omega}} |u|^q x_1^a x_2^b \, dx_1 \, dx_2 \right)^{1/q} \leq C \left( \int_{\tilde{\Omega}} |\nabla u|^p x_1^a x_2^b \, dx_1 \, dx_2 \right)^{1/2}, \quad q = q(a, b)$$
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Thus, we want Sobolev inequalities with weights

\[ \left( \int_{\mathbb{R}^n} |u|^q w(x) dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |\nabla u|^p w(x) dx \right)^{1/p}, \quad w(x) = x_1^{A_1} \cdots x_n^{A_n} \]
PART III:

Isoperimetric inequalities with densities
Theorem (Cabré-R; J. Differential Equations’13)

Let $A_i \geq 0$,

$$w(x) = x_1^{A_1} \cdots x_n^{A_n}, \quad D = n + A_1 + \cdots + A_n.$$  

Let $1 \leq p < D$. Then,

$$\left( \int_{\mathbb{R}^n} |u|^q w(x) \, dx \right)^{1/q} \leq C_{p,A} \left( \int_{\mathbb{R}^n} |\nabla u|^p w(x) \, dx \right)^{1/p},$$

with $q = pD/(D - p)$.

To prove the result, we establish a new weighted isoperimetric inequality.
Isoperimetric inequalities with monomial weights

Theorem (Cabré-R; J. Differential Equations’13)

Let $A_i > 0$, $w(x)$ and $D$ as before, and

$$\Sigma = \{x \in \mathbb{R}^n : x_1, ..., x_n > 0\}.$$ 

Then, for any $E \subset \Sigma$,

$$\frac{P_w(E)}{w(E)^{\frac{D-1}{D}}} \geq \frac{P_w(B_1 \cap \Sigma)}{w(B_1 \cap \Sigma)^{\frac{D-1}{D}}}.$$

We denoted the weighted volume and perimeter

$$w(E) = \int_E w(x)dx \quad \quad P_w(E) = \int_{\Sigma \cap \partial E} w(x)dS.$$
This type of isoperimetric inequalities have been widely studied:

- $w(x) = e^{-|x|^2}$ [Borell; Invent. Math.’75]
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- \( w(x) = e^{-|x|^2} \) [Borell; Invent. Math.'75]
- Existence and regularity of minimizers (Pratelli, Morgan,...)
- Log-convex radial densities \( w(|x|) \) [Figalli-Maggi '13], [Chambers '14]
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- $w(x) = e^{-|x|^2}$ [Borell; Invent. Math.’75]
- Existence and regularity of minimizers (Pratelli, Morgan,...)
- log-convex radial densities $w(|x|)$ [Figalli-Maggi ’13], [Chambers ’14]
- with $w(x) = e^{|x|^2}$, $w(x) = |x|^\alpha$, or other particular weights
- ...
Isoperimetric inequalities in cones

In general cones \( \Sigma \), a well known result is the following:

**Theorem (Lions-Pacella ’90)**

Let \( \Sigma \) be any open convex cone in \( \mathbb{R}^n \). Then, for any \( E \subset \Sigma \),

\[
\frac{|\Sigma \cap \partial E|}{|E|^{\frac{n}{n-1}}} \geq \frac{|\Sigma \cap \partial B_1|}{|B_1 \cap \Sigma|^{\frac{n}{n-1}}}.
\]

Important: Only the perimeter inside \( \Sigma \) is counted.
New isoperimetric inequalities weights

Theorem (Cabré-R-Serra; preprint '13)

Let $\Sigma$ be any convex cone in $\mathbb{R}^n$. Assume

$$w(x) \text{ homogeneous of degree } \alpha \geq 0, \quad \& \quad w^{1/\alpha} \text{ concave in } \Sigma.$$ 

Then, for any $E \subset \Sigma$,

$$\frac{P_w(E)}{w(E)^{D-1/D}} \geq \frac{P_w(B_1 \cap \Sigma)}{w(B_1 \cap \Sigma)^{D-1/D}}.$$ 

Recall

$$w(E) = \int_E w(x) dx \quad \quad P_w(E) = \int_{\Sigma \cap \partial E} w(x) dS.$$
• Minimizers are radial, while $w(x)$ is not!
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When \( w \equiv 1 \) we recover the result of Lions-Pacella (with new proof!).
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We can also treat anisotropic perimeters

$$P_{w,H}(E) = \int_{\Sigma \cap \partial E} H(\nu) w(x) dS.$$ 

$w \equiv 1 \implies$ new proof of the Wulff theorem.
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P_{w,H}(E) = \int_{\Sigma \cap \partial E} H(\nu) \, w(x) \, dS.
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\( w \equiv 1 \implies \) new proof of the **Wulff theorem**.

Some examples of weights are

\[
w(x) = \text{dist}(x, \partial \Sigma)^\alpha, \quad w(x) = \sqrt{x} + \sqrt{y}, \quad w(x) = \frac{xyz}{x + y + z}, \quad ...
\]
The proof

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- When $w \equiv 1$, the idea goes back to the work [Cabré ’00] (for the classical isoperimetric inequality)
- Here, we need to consider a linear Neumann problem in $E \subset \Sigma$ involving the operator $w^{-1}\text{div}(w \nabla u)$
The end

Thank you!