Annals of Mathematics, 114 (1981), 193–228

An ordinal L^p-index for Banach spaces, with application to complemented subspaces of L^p

By J. BOURGAIN, H. P. ROSENTHAL,[1] G. SCHECHTMAN[2]

One of the central problems in the Banach space theory of the L^p-spaces is to classify their complemented subspaces up to isomorphism (i.e., linear homeomorphism). Let us fix $1 < p < \infty$, $p \neq 2$. There are five "simple" examples, L^p, l^p, l^2, $l^2 \oplus l^p$, and $(l^2 \oplus l^2 \oplus \ldots)_p$. Although these were the only infinite-dimensional ones known for some time, further impetus to their study was given by the discoveries of Lindenstrauss and Pełczyński [15] and Lindenstrauss and Rosenthal [16]. These discoveries showed that a separable infinite-dimensional Banach space is isomorphic to a complemented subspace of L^p if and only if it is isomorphic to l^2 or is an "$\mathcal{L}_p$-space", that is, equal to the closure of an increasing union of finite-dimensional spaces uniformly close to l_n^p's. By making crucial use of statistical independence, the second author produced several more examples in [19], and the third author built infinitely many non-isomorphic examples in [23]. These discoveries left unanswered: Does there exist a λ_p and infinitely many non-isomorphic λ_p-complemented subspaces of L^p (equivalently, are there infinitely many separable $\mathcal{L}_{p,\lambda}$-spaces for some λ depending on p)? We answer these questions by obtaining uncountably many non-isomorphic complemented subspaces of L^p.* Before our work, it was suspected that every $\mathcal{L}_p$-space non-isomorphic to L^p embedded in $(l^2 \oplus l^2 \oplus \ldots)_p$ (for $2 < p < \infty$) (see Problem 1 of [23]). Indeed, all the known examples had this property. However our results show that there is no universal $\mathcal{L}_p$-space besides L^p. To obtain these results, we use rather deep properties of martingales together with a new ordinal index, called the local L^p-index, which assigns "large" countable ordinals to any

003-486X/81/0114-2/0193/036$01.80/1

*It is not known if there are c non-isomorphic complemented subspaces of L^p, $1 < p < \infty$, $p \neq 2$, c the cardinality of the continuum. For $p = \infty$ the question becomes, are there c nonisomorphic complemented subspaces of $c([0,1])$? It is conjectured in this case that an affirmative answer implies the continuum hypothesis.

[1]Supported in part by NSF-MCS-7827056
[2]Supported in part by NSF-MCS-7606565 and 7903042.

separable Banach space not containing L^p-isomorphically. We pass now to a more detailed summary of our work.

Our main result concerning the classification of the complemented subspaces of L^p is as follows:

THEOREM A. *Let $1 < p < \infty$, $p \neq 2$, and let ω_1 denote the first uncountable ordinal. There exists a family $(X_\alpha^p)_{\alpha<\omega_1}$ of complemented subspaces of L^p so that for all $\alpha < \beta < \omega_1$, X_α^p is isometric to a subspace of X_β^p but X_β^p is not isomorphic to a subspace of X_α^p. Moreover if B is a separable Banach space such that X_α^p is isomorphic to a subspace of B for all α, then L^p is isomorphic to a subspace of B.*

Since at most one of the spaces X_α^p can be isomorphic to Hilbert space, we obtain that there exist uncountably many non-isomorphic $\mathcal{L}_p$-spaces, $1 < p < \infty$, $p \neq 2$, thus answering a question raised in [23]. (It has recently been proved that there are uncountably many non-isomorphic separable $\mathcal{L}_1$-spaces. See [12].) It of course follows immediately that there is a λ (depending on p) so that there are uncountably many non-isomorphic $\mathcal{L}_{p,\lambda}$ spaces; as noted above, the existence of infinitely many such had remained an open question until now.

Given Banach spaces X and Y, we use the notation $X \hookrightarrow Y$ to mean X is isomorphic (linearly homeomorphic) to a subspace of Y; $X \overset{c}{\hookrightarrow} Y$ means X is isomorphic to a complemented subspace of Y. Given a class K of Banach spaces and a Banach space B, we say that B is universal for K if $E \hookrightarrow B$ for all $E \in K$.

Our main result then yields the following consequence:

COROLLARY. *Let $1 < p < \infty$, $p \neq 2$, and let $\mathcal{W}_p$ denote the class of all complemented subspaces X of L^p such that $L^p \not\hookrightarrow X$. Let B be a separable Banach space universal for $\mathcal{W}_p$. Then $L^p \hookrightarrow B$.*

In our proof of the Main Theorem, we make essential use of the following result established in [13]:

$$(\Delta) \textit{ If } \quad X \subset L^p \quad \textit{and} \quad L^p \hookrightarrow X, \quad \textit{then} \quad L^p \overset{c}{\hookrightarrow} X.$$

It follows, incidentally, that if $X \overset{c}{\hookrightarrow} L^p$, then X is isomorphic to L^p if (and only if) $L^p \hookrightarrow X$. Hence the corollary may be rephrased: *if B separable is universal for the class of all separable $\mathcal{L}_p$-spaces non-isomorphic to L^p, then $L^p \hookrightarrow B$* ($1 < p < \infty$, $p \neq 2$).

It is a long standing conjecture that every infinite-dimensional complemented subspace of L^1 is isomorphic to l^1 or L^1. Thus the analogue of our main result is thought to be false for $p = 1$ (although this is an open question). If we drop the word "complemented", then the analogue of our main result and its attendant corollary prove true for $p = 1$; in fact, we obtain the following improvement:

PROPOSITION. *Let $\mathcal{C}$ denote the class of all subspaces of L^1 satisfying the Radon-Nikodym property and let B be universal for $\mathcal{C}$ with B separable. Then* $L^1 \hookrightarrow B$.

In previous (unpublished) work, M. Talagrand had obtained that the class of all separable Banach spaces with the RNP has no universal element.

To obtain our results, we introduce (in Section 2) an ordinal index for separable Banach spaces, called the *local L^p-index*. Ordinal indices with similar properties were introduced by the first author in [2] for l^1-structures and in [3] for quite general structures. (For a discussion of the local L^∞-index and its connection with the classical theory of analytical sets, see [21]. Also, see [22] for a summary of the proof of the Main Theorem without the complementation assertion (unknown at the time [22] was written).)

The properties of this index are as follows (ω_1 denotes the first uncountable ordinal):

THEOREM 2.1. *For each $1 \le p \le \infty$ and separable Banach space B, there exists an ordinal number $h_p(B) \le \omega_1$, the local L^p-index of B, so that*

(a) *$h_p(B) < \omega_1$ if and only if $L^p \not\hookrightarrow B$ and $p < \infty$, or $C([0,1]) \not\hookrightarrow B$ and $p = \infty$; and*

(b) *if X is a Banach space such that $X \hookrightarrow B$, then $h_p(X) \le h_p(B)$.*

We construct the family of Theorem A by alternately taking disjoint and independent sums of subspaces of L^p. Precisely, let $1 \le p < \infty$ and let R_0^p be the one-dimensional space of constant functions. If R_α^p has been defined, we let $R_{\alpha+1}^p$ equal the L^p-direct sum in L^p of R_α^p with itself. If α is a limit ordinal and R_β^p has been defined for all $\beta < \alpha$, we let R_α^p equal the independent L^p-sum in L^p of the R_β^p's for $\beta < \alpha$. It is important that the R_α^p's are presented as specific spaces of random variables; the precise definitions of disjoint and independent sums in L^p may be found in the second part of Section 2.

Incidentally, it follows easily that for $\alpha < \beta$, R_α^p isometrically embeds in R_β^p. In fact, the natural embedding is implemented by a projection of norm one (for $p = 1$ as well).

Theorem A then follows easily from the following result:

THEOREM B. *Let $1 \le p < \infty$, $p \ne 2$ and $\alpha < \omega_1$. Then*

(1) $L^p \not\hookrightarrow R_\alpha^p$,

(2) $h_p(R_\alpha^p) \ge \alpha + 1$ *and*

(3) *R_α^p is complemented in L^p if $p \ne 1$.*

Proof that B $\Rightarrow$ A. We simply construct an increasing function $\tau\colon \omega_1 \to \omega_1$ so that $X_\alpha^p = R_{\tau(\alpha)}^p$ for all $\alpha < \omega_1$. Let $\tau(0) = \omega$. (Thus X_0^p is the first R_α^p which is infinite dimensional.) Suppose $\beta > 0$ is a countable ordinal and $\tau(\alpha)$ has been defined for all $\alpha < \beta$. Now Theorem 2.1 and (1) yield that $h_p(R^p_\gamma) < \omega_1$ for all $\gamma < \omega_1$. Let $\tau(\beta) = \sup\{h_p(R^p_{\tau(\alpha)}): \alpha < \beta\}$. By (2) of Theorem B, $h_p(R^p_{\tau(\beta)}) = h_p(X^p_\beta) \geq \tau(\beta) + 1$. Since $\alpha < \beta$ implies $h_p(X^p_\alpha) \leq \tau(\beta) < h_p(X^p_\beta)$, it follows from Theorem 2.1(a) that X^p_β is not isomorphic to a subspace of X^p_α. Finally, suppose B is as in Theorem A. Then by Theorem 2.1 and part (2) of Theorem B, for all $\alpha < \omega_1$, $\alpha + 1 \leq h_p(X^p_\alpha) \leq h_p(B)$. Hence $h_p(B) = \omega_1$, so by Theorem 2.1, $L^p \hookrightarrow B$, proving Theorem A.

It is easily seen that the R^1_α's all have the RNP; hence the above argument also proves the proposition.

The assertions (1)–(3) of Theorem B are essentially established in Sections 1–3 respectively. We pass to a brief summary of how this is done.

Section 1 is devoted exclusively to the proof of the following result:

THEOREM 1.1. *Let $1 < p < \infty$, Y be a Banach space with an unconditional Schauder decomposition (Y_i), and suppose $L^p \overset{c}{\hookrightarrow} Y$. Then either $L^p \overset{c}{\hookrightarrow} Y_i$ for some i, or there exists a block basic sequence of the Y_i's equivalent to the Haar-basis in L^p, with closed linear span complemented in Y^p.*

Theorem B(1) for $p > 1$ now follows easily from the above result, (Δ), and the fact that no independent sequence of random variables is equivalent to the Haar basis in L^p (for $p \neq 2$). The details are given in Section 2. Of course, B(1) for $p = 1$ also follows immediately from the fact that the R^p_α's all have the RNP, established in Section 2. We do not know if Theorem 1.1 holds if the words "unconditional" or "complemented" are deleted from its statement. The techniques of Enflo and Starbird [9] (see also Kalton [14]) may be used to show that 1.1 does hold for $p = 1$ (in which case only the first alternative occurs).

Section 2 is devoted to the definition and properties of the local L^p-index, the proof of Theorem B(2), and the demonstration of a few other properties of the R^p_α's. (For example, it is proved that R^p_α has an unconditional basis for all $1 < p < \infty$ and $\alpha < \omega_1$.)

In Section 3, we obtain that the R^p_α's are complemented in L^p for $1 < p < \infty$. To accomplish this, we require a fundamentally different description of these spaces.

Let T be a countable partially ordered set such that the set of predecessors of any element of T is finite and linearly ordered; we call such a T a *tree*. Call a subset Γ of T a branch if it contains all the predecessors of all its elements. Now let $\{0,1\}^T$ be endowed with the product measure of the "fair" measure on the two point set $\{0,1\}$, and let X^p_T denote the closed linear span in $L^p\{0,1\}^T$ over all branches Γ of those functions which depend only on the coordinates in Γ.

Thus we show in Section 3 (Theorem 3.8) that for any tree, T, X_T^p is complemented in $L^p(\{0,1\}^T)$, $1 < p < \infty$, and verify (Lemma 3.9) that for all α there is a tree T_α so that R_α^p may be identified with $X_{T_\alpha}^p$ for all $1 \leq p < \infty$.

The complementation result makes crucial use of some martingale inequalities due to Stein, Burkholder, Davis and Gundy. We also note at the end of Section 3 that each R_α^p may be identified with the closed linear span in L^p of a certain set of Walsh functions; that is, with a translation invariant subspace of $L^p(\{0,1\}^N)$. Several open questions are posed throughout; in particular, at the end of Section 3.

Much of this research was conducted while the authors held visiting positions in France—the first and second at Université de Paris VI and the third at Ecole Polytechnique, Palaiseau. We would like to thank our French colleagues for their warm hospitality and support. In particular, we would like to thank G. Pisier for stimulating conversations concerning the work presented here.

1. Complemented embeddings of L^p into spaces with unconditional Schauder decompositions

The main result of this section is as follows:

THEOREM 1.1. *Let $1 < p < \infty$ and suppose L^p is isomorphic to a complemented subspace of a Banach space Y with an unconditional Schauder decomposition (Y_i). Then one of the following holds:*

(1) *There is an i so that L^p is isomorphic to a complemented subspace of Y_i;*

(2) *A block basic sequence of the Y_i's is equivalent to the Haar basis of L^p and has closed linear span complemented in Y.*

(We recall that (Y_i) is an unconditional Schauder decomposition of Y if each Y_j is a closed linear subspace of Y, and if for all $y \in Y$, there exists a unique sequence (y_j) with $y_j \in Y_j$ for all j and Σy_j converging unconditionally to y. A sequence (b_i) in Y is called a block basic sequence of the Y_i's if there exist $y_j \in Y_j$ and integers $n_1 < n_2 < \cdots$ with $b_i = \sum_{j=n_i}^{n_{i+1}-1} y_j$ for all i.)

The proof is accomplished by using many standard results about L^p and general unconditional Schauder decompositions. In particular, we make essential use of the results and techniques of Alspach, Enflo and Odell [1]. We first assemble these standard results. For the convenience of the reader, we have labeled those used directly in the proof of Theorem 1.1 as scholia; the others are called lemmas.

We first need facts about unconditional bases and decompositions. Let N denote the set of positive integers. Given a Banach space B with an unconditional basis (b_i) and (x_i) a sequence of non-zero elements in B, *say that (x_i) is disjoint if there exist disjoint subsets $M_1, M_2, \ldots$ of N with $x_i \in [b_j]_{j \in M_i}$ for all i. Say* *that* (x_i) *is essentially disjoint if there exists a disjoint sequence* (y_i) *such that* $\Sigma\|x_i - y_i\|/\|x_i\| < \infty$. Of course, if (x_i) is essentially disjoint, then (x_i) is essentially a block basis of a permutation of (b_i). Also, (x_i) is an unconditional basic sequence. (Throughout this paper, if $\{b_i\}_{i \in J}$ is an indexed family of elements of a Banach space B, $[b_i]_{i \in J}$ denotes the closed linear span of $\{b_i\}_{i \in J}$ in B.)

We next slightly rephrase the useful Lemma 1.1 of [1] (which, as noted in [1], follows easily from the ideas of [7]).

LEMMA 1.2. *Let* (b_n) *be an unconditional basis for* B *with biorthogonal functionals* (b_n^*), T: $B \to B$ *an operator*, $\varepsilon > 0$, *and* (b_{n_i}) *a subsequence of* (b_n) *so that* (Tb_{n_i}) *is essentially disjoint and* $|b_{n_i}^*(Tb_{n_i})| \geq \varepsilon$ *for all* i. *Then* (Tb_{n_i}) *is equivalent to* (b_{n_i}) *and* $[Tb_{n_i}]$ *is complemented in* B.

Our next result follows immediately from the proof of the remarkable diagonalization theorem of Tong [26]; (see also Proposition 1.c.8 of [17]). If (X_i) is an unconditional Schauder decomposition, say that P_i is the natural projection onto X_i if $P_i x = x_i$ provided $x = \Sigma x_j$ with $x_j \in X_j$ for all j. We shall refer to (P_i) as the projections corresponding to the decomposition.

LEMMA 1.3. *Let* X *and* Y *be Banach spaces with unconditional Schauder decompositions* (X_i) *and* (Y_i) *respectively; and let* (P_i) *(resp.* Q_i*) be the natural projection from* X *(resp.* Y*) onto* X_i *(resp.* Y_i*). Then if* T: $X \to Y$ *is a bounded linear operator, so is* $\Sigma Q_i T P_i$. *(In other words, there is a* $K < \infty$ *so that for all* $x \in X$, $\Sigma Q_i T P_i x$ *converges and* $\|\Sigma Q_i T P_i x\| \leq K\|x\|$.*)*

Our next result is used directly in the proof of case 2 of the Main Theorem. (Throughout this paper, "projection" means "bounded linear projection", "operator" means "bounded linear operator".)

SCHOLIUM 1.4. *Let* Y *have an unconditional Schauder decomposition with corresponding projections* (Q_i) *(as in the previous result), and let* X *be a complemented subspace of* Y *with an unconditional basis* (x_i) *with biorthogonal functionals* (x_i^*). *Suppose there exist* $\varepsilon > 0$, *a projection* U: $Y \to X$ *and disjoint subsets* $M_1, M_2, \ldots$ *of* N *with the following properties*:

(a) $(UQ_i x_l)_{l \in M_i, i \in N}$ *is essentially disjoint and*

(b) $|x_l^*(UQ_i x_l)| \geq \varepsilon$ *for all* $l \in M_i$, $i \in N$.

Then $(Q_i x_l)_{l \in M_i, i \in N}$ *is equivalent to* $(x_l)_{l \in M_i, i \in N}$ *and* $[Q_i x_l]_{l \in M_i, i \in N}$ *is complemented in* Y.

Proof. Let $M = \cup_{i=1}^{\infty} M_i$ and $L = N \sim M$. Let $X_i = [x_l]_{l \in M_i}$ for $i > 1$ and $X_1 = [x_l]_{l \in M_1 \cup L}$. Then of course (X_i) is an unconditional Schauder decomposition for X; let (P_i) be the corresponding projections. Also, let T be the natural projection from X onto $[x_l]_{l \in M}$. Now if we regard T as an operator from X into

Y, $V = \Sigma Q_i T P_i$ is also an operator, by the preceding lemma. Fixing i and $l \in M_i$, we have

$$V(x_l) = Q_i T P_i(x_l) = Q_i(x_l). \tag{1.1}$$

Hence by (b),

$$|x_l^* U V(x_l)| = |x_l^* U Q_i(x_l)| \geq \varepsilon. \tag{1.2}$$

Moreover, $(UV(x_l))_{l \in M}$ is almost disjoint by (a). Thus Lemma 1.2 applies and $(UV(x_l))_{l \in M}$ is equivalent to $(x_l)_{l \in M}$ and $[UV(x_l)]_{l \in M}$ is complemented in X. It now follows directly that $(V(x_l))_{l \in M}$ is equivalent to $(x_l)_{l \in M}$ with $[V(x_l)]$ complemented in Y, which proves the theorem by virtue of (1.1). (To see the final assertion, $(V(x_l))$ is dominated by (x_l) but dominates $(UV(x_l))$, hence $(V(x_l))$ is equivalent to (x_l)). Let P be a projection from X onto $[UV(x_l)]_{l \in M}$ and let S: $[UV(x_l)]_{l \in M} \to [x_l]_{l \in M}$ be the isomorphism with $SUV(x_l) = x_l$ for all $l \in M$. Then $Q = VSPU$ is a projection from Y onto $[Vx_l]_{l \in M}$, as is seen by considering the commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\;U\;} & X & \xrightarrow{\;P\;} & [UV(x_l)]_{l \in M} \\ V \uparrow & & & & \downarrow S \\ [x_l]_{l \in M} & & \xrightarrow{\;\;\;I\;\;\;} & & [x_l]_{l \in M} \end{array}$$

We next recall the fundamental result of Gamlen and Gaudet [10]; *throughout this paper*, (h_i) *denotes the Haar-basis, normalized in* L^∞.

LEMMA 1.5. *Let* $1 < p < \infty$ *and* $I \subset N$ *such that if* $E = \{t \in [0,1]$: t *belongs to infinitely many* h_i*'s with* $i \in I\}$, *then* E *is of positive Lebesgue measure. Then* $[h_i]_{i \in I}$ *is isomorphic to* L^p.

Now fix p, $1 < p < \infty$. Following [1], we recall that L^p is isomorphic to

$$L^p(l^2) = \left\{ (f_i) \colon f_i \in L^p \quad \text{and} \quad \|(f_i)\| = \left(\int \Big(\sum |f_i|^2 \Big)^{p/2} \right)^{1/p} < \infty \right\}.$$

Fixing i and letting (h_{ij}) be the element of $L^p(l_2)$ whose j-th coordinate equals h_i, all other coordinates 0, we see that $(h_{ij})_{i,j}$ is an unconditional basis for $L^p(l^2)$, thanks to the fact that (h_i) is an unconditional basis for L^p. Now any unconditional basic sequence (x_i) in L^p is equivalent to the diagonal sequence $x_{ij} = x_i$ if $j = i$; $x_{ij} = 0$ otherwise, in $L^p(l^2)$; hence as observed in [1], we have the following fact:

SCHOLIUM 1.6. *There is a constant* K_p *depending only on p so that for any function* j: $N \to N$, $(h_{ij(i)})_{i=1}^\infty$ *in* $L^p(l^2)$ *is* K_p*-equivalent to* (h_i) *in* L^p.

We are now prepared for the following consequence of the proof of Alspach, Enflo and Odell that L^p is primary [1]. Let (h_{ij}^*) denote the biorthogonal functionals to (h_{ij}) as defined above.

SCHOLIUM 1.7. *Let $1 < p < \infty$ and $T\colon L^p(l^2) \to L^p(l^2)$ be a given operator. Suppose there is a $c > 0$ so that when $I = \{i\colon |h_{ij}^* T h_{ij}| \geq c$ for infinitely many $j\}$, then E has positive Lebesgue measure, where*

$$E = \{t \in [0,1]\colon t \text{ belongs to infinitely many } h_i\text{'s with } i \in I\}.$$

Then there is a subspace Y of $L^p(l^2)$ with Y isomorphic to L^p, $T|Y$ an isomorphism, and TY complemented in $L^p(l^2)$.

Proof. We shall show that Y may be chosen of the form $Y = [h_{ij(i)}]_{i \in I}$ for some $j\colon I \to N$.

Fix $i \in I$. By the definition of I, there is a sequence $j_1 < j_2 < \cdots$ with $\|Th_{ij_k}\| \geq c > 0$ for all k; of course $(Th_{ij_k})_{k=1}^{\infty}$ is weakly null. It then follows by the standard gliding hump argument and the definition of I that there exists a function $j\colon I \to N$ so that $(Th_{ij(i)})_{i \in I}$ is essentially disjoint with respect to $(h_{ik})_{i,k=1}^{\infty}$ and $|h_{ij(i)}^* Th_{ij(i)}| \geq c$ for all $i \in I$. Then by Lemma 1.2, $[Th_{ij(i)}]_{i \in I}$ is complemented in $L^p(l^2)$ and $(Th_{ij(i)})_{i \in I}$ is equivalent to $(h_{ij(i)})_{i \in I}$, which is equivalent to $(h_i)_{i \in I}$ by Lemma 1.6. In turn, $[h_i]_{i \in I}$ is isomorphic to L^p by the result of Gamlen-Gaudet, Lemma 1.5. This completes the proof.

COROLLARY 1.8. *Let $1 < p < \infty$ and $T\colon L^p \to L^p$ be a given operator. Then for $S = T$ or $I - T$, there exists a subspace Y of L^p with Y isomorphic to L^p, $S|Y$ an isomorphism, and $S(Y)$ complemented in L^p.*

Proof. Since L^p is isomorphic to $L^p(l^2)$, it suffices to prove 1.8 with L^p replaced by $L^p(l^2)$ in its statement. Let $I_1 = \{i\colon |h_{ij}^* Th_{ij}| \geq \frac{1}{2}$ for infinitely many $j\}$ and $I_2 = \{i\colon |h_{ij}^*(I - T)h_{ij}| \geq \frac{1}{2}$ for infinitely many $j\}$. Then $N = I_1 \cup I_2$; hence for $j = 1$ or 2, E_j has positive Lebesgue measure, where $E_j = \{t\colon t$ belongs to infinitely many h_i's for $i \in I_j\}$. The result now follows from the preceding theorem.

Remark: 1.8 was first established by Enflo. The work of Enflo-Starbird [9] shows that it holds for $p = 1$ (see also [14]).

THEOREM 1.9. *Let $1 < p < \infty$, and X and Y be given Banach spaces. If L^p is isomorphic to a complemented subspace of $X \oplus Y$, then L^p is isomorphic to a complemented subspace of X or to a complemented subspace of Y.*

Proof. Let P (resp. Q) denote the natural projection from $X \oplus Y$ onto X (resp. Y). Hence $P + Q = I$. Let Z be a complemented subspace of

$X \oplus Y$ isomorphic to L^p and let U: $X \oplus Y \to Z$ be a projection. Since $UP|Z + UQ|Z = I|Z$, the preceding result shows that there is a subspace W of Z with W isomorphic to L^p, $T|W$ an isomorphism, and TW complemented in Z, where $T = UP|Z$ or $T = UQ|Z$. Suppose the former: Let S be a projection from Z onto TW and $R = (T|W)^{-1}$. Then $I|W = RSUP|W$; hence since the identity on W may be factored through X, W is isomorphic to a complemented subspace of X.

Remark: Of course this result also holds for $p = 1$, by virtue of the preceding remarks. Also, it thus follows trivially by induction that if $X_1, \ldots, X_n$ are given Banach spaces with L^p isomorphic to a complemented subspace of $X_1 \oplus \cdots \oplus X_n$, then L^p is isomorphic to a complemented subspace of X_i for some i.

We need two more preliminary results dealing with sequences equivalent to the Haar basis. We recall the explicit definition of the latter, normalized in L^∞: $h_1 \equiv 1$ and for $n = 2^k + j$ with $0 \leq k$ and $1 \leq j \leq 2^k$,

$$h_n = \chi_{\left[\frac{j-1}{2^k}, \frac{2j-1}{2^{k+1}}\right)} - \chi_{\left[\frac{2j-1}{2^{k+1}}, \frac{j}{2^k}\right)}.$$

The next result is essentially Lemma 4 of [10]. (We employ the notation $[f = a]$ for $\{t\colon f(x) = a\}$; μ denotes Lebesgue measure. For a measurable function f, $\operatorname{supp} f = [f \neq 0]$.)

LEMMA 1.10. *Let (x_i) be a sequence of measurable functions on $[0,1]$ with x_1 $\{0,1\}$-valued and (x_i) $\{1,0,-1\}$-valued for $i > 1$. Suppose there exist positive constants a and b so that, for all positive l, with k the unique integer, $1 \leq k \leq l$, and α the unique choice of $+1$ or -1 so that* $\operatorname{supp} h_{l+1} = [h_k = \alpha]$, *then*

(a) $[x_k = \alpha] = \operatorname{supp} x_{l+1}$ *(up to a set of measure zero) and*

(b) $a/2\int |h_k| \leq \mu([x_{l+1} = \beta]) \leq b/2\int |h_k|$ *for* $\beta = \pm 1$.

Then for all p, $1 \leq p < \infty$, (x_n) is equivalent to (h_n) in L^p, $[x_n]$ is isometric to L^p and hence is the range of a norm-one projection defined on L^p.

Remark: In the above statement, $k = [(l + 1)/2]$ and $\alpha = (-1)^{l+1}$. Also, if $a = b = 1$, (x_n) is isometrically equivalent to (h_n) in the L^p-norm.

The hypotheses of our final preliminary result yield sequences equal to a small perturbation of the x_i's of the preceding result, hence these sequences are again equivalent to the Haar basis.

SCHOLIUM 1.11. *Let (z_i) be a sequence of measurable functions on $[0,1]$ with z_1 $\{1,0\}$-valued non-zero in L^1 and (z_i) $\{1,0,-1\}$-valued with $\int z_i = 0$ for all $i > 1$. Suppose that for all positive l, letting k be the unique integer, $1 \leq k \leq l$, and α the unique choice of $+1$ or -1 so that* $\operatorname{supp} h_{l+1} = [h_k = \alpha]$, *then*

$$\operatorname{supp} z_{l+1} \subset [z_k = \alpha]$$

and $\mu([z_k = \alpha] \sim \operatorname{supp} z_{l+1}) \leq \varepsilon_l \int |z_1|$ *(where* $\varepsilon_j = 1/2^{j^2}$ *for all* j*). Then for all* p, $1 \leq p < \infty$, (z_n) *is equivalent to* (h_n) *in the* L^p*-norm and* $[z_n]$ *is complemented in* L^p.

Proof. Buried in the indexing of the Haar system by N is the fact that the supports form a dyadic tree of sets. We introduce the perhaps more natural dyadic indexing as follows: Let D_k denote the set of all k-tuples of 0's and 1's and let $\mathfrak{D} = \bigcup_{k=0}^{\infty} D_k$. (Thus $\mathfrak{D}$ is the set of all finite sequences of 0's and 1's.) For $n > 1$, let $t = (t_1 \cdots t_k)$ be the unique element of $\mathfrak{D}$ such that

$$n - 1 = 2^k + \sum_{i=1}^{k} t_i 2^{k-i}. \tag{1.3}$$

(Here $k = 0$ is possible; then "t" denotes the empty sequence $\varnothing$.) Now for $\varepsilon = 0$ or 1, set $E_{t\varepsilon} = [z_n = (-1)^{\varepsilon}]$. Thus $z_n = 1$ on E_{t0}, $z_n = -1$ on E_{t1} and $z_n = 0$ elsewhere. Also, set $E_{\varnothing} = [z_1 = 1]$. Let $\varepsilon(t) = \varepsilon_{n-1}$ and $b = \int |z_1|$. Our hypotheses are then equivalent to the following: For all $t \in \mathfrak{D}$,

$$\mu(E_{t0}) = \mu(E_{t1}), \tag{1.4}$$

$$E_t \supset E_{t0} \cup E_{t1} \tag{1.5}$$

and

$$\mu(E_t \sim (E_{t0} \cup E_{t1})) < b\varepsilon(t). \tag{1.6}$$

It then follows easily that for $t \in D_k$, $k \geq 1$,

$$\begin{gathered} \mu(E_t) \leq \frac{b}{2^k} \text{ and} \\ \mu(E_t) \geq \frac{b}{2^k}\left(1 - \sum_{j=0}^{k-1} \varepsilon(t_1 \cdots t_j) 2^j\right). \end{gathered} \tag{1.7}$$

Now define $F_t = \bigcap_{l=1}^{\infty} \bigcup_{v \in D_l} E_{tv}$. Then fixing $t \in D_k$, letting n be as in (1.3), we have

$$\mu(E_t \sim F_t) \leq b \sum_{l=1}^{\infty} \sum_{v \in D_l} \varepsilon(tv) \leq b \sum_{l \geq n} \varepsilon_l \leq \frac{b}{2^{(n-1)^2}} \leq \frac{b}{2^{k^2}}. \tag{1.8}$$

Since $\varepsilon(t_1 \cdots t_j) \leq \dfrac{1}{2^{(j+1)^2}}$, we easily obtain from (1.7) and (1.8) that there is a constant $a > 0$ so that

$$\frac{a}{2^k} \leq \mu(F_t) \quad \text{for all} \quad t \in D_k, \quad \text{for all} \quad k. \tag{1.9}$$

Again let n and $(t_1 \cdots t_k)$ satisfy (1.3) and let x_n be defined by

(1.10) $x_n = 1$ on F_{t0}, $x_n = -1$ on F_{t1} and $x_n = 0$ elsewhere.

It follows easily that (x_n) satisfies the hypotheses of the preceding lemma. Finally, we obtain from (1.8) that there is a constant c so that for all p, $1 \le p < \infty$,

$$\frac{\|x_n - z_n\|_p^p}{\|x_n\|_p^p} = \frac{\|x_n - z_n\|_1}{\|x_n\|_1} \le \frac{c}{2^{k^2-k}}. \tag{1.11}$$

Hence,

$$\sum_{n=3}^{\infty} \frac{\|x_n - z_n\|_p}{\|x_n\|_p} \le \sum_{k=2}^{\infty} \frac{c}{2^{k^2-2k}},$$

which proves the result in view of the preceding lemma and standard perturbation arguments.

We are at last prepared for the proof of our main theorem. Let us first outline the procedure. We assume that $L^p(l^2)$ is a complemented subspace of Y; let U: $Y \to L^p(l^2)$ be a projection. Let (Y_i) be an unconditional decomposition of Y. *Suppose that there is no i with L^p isomorphic to a complemented subspace of* Y_i. We shall then construct a "blocking" of the decomposition (Y_i) with corresponding projections (Q_i), finite disjoint subsets $M_1, M_2, \ldots$, of N, and a map j: $\bigcup_{i=1}^{\infty} M_i \to N$ so that

(i) $(Q_k h_{ij(i)})_{i \in M_k, k \in N}$ is equivalent to $(h_i)_{i \in M_k, k \in N}$ with $[Q_k h_{ij(i)}]_{i \in M_k, k \in N}$ complemented in $L^p(l^2)$ and

(ii) (z_k) is equivalent to the Haar basis and $[z_k]$ is complemented in L^p, where $z_k = \sum_{i \in M_k} h_i$ for all k.

(The h_{ij}'s are as defined preceding Scholium 1.6.) This is enough to prove the theorem, for we simply let $b_k = \sum_{i \in M_k} Q_k h_{ij(i)}$ for all k; then (b_k) is the desired block basic sequence equivalent to the Haar basis with $[b_k]$ complemented.

We pass now to the details. Let P_i be the natural projection from Y onto Y_i. More generally, for F a subset of N, we let $P_F = \sum_{i \in F} P_i$. Also, we let $R_n = I - \sum_{i=1}^{n} P_i$ $(= P_{(n,\infty)})$. We first draw a consequence from our assumption that no Y_i contains a complemented isomorph of L^p.

SUBLEMMA 1. *For each n, let*

$$I = \left\{ i \in N: h_{ij}^* U R_n h_{ij} > \tfrac{1}{2} \quad \textit{for infinitely many integers } j \right\}.$$

Let $E_I = \{t$: t belongs to the support of h_i for infinitely many $i \in I\}$. Then $\mu(E_I) = 1$ (where μ denotes Lebesgue measure).

Indeed, let $L = \{i \in N\colon h_{ij}^* UP_{[1,n]} h_{ij} \geq \frac{1}{2}$ for infinitely many integers $j\}$; then $I \cup L = N$.

Hence $E_I \cup E_L = [0,1]$. So if $\mu(E_I) < 1$, $\mu(E_L) > 0$. But then $T = UP_{[1,n]}$ satisfies the hypotheses of Scholium 1.7. Hence there is a subspace Z of $L^p(l^2)$, with Z isomorphic to L^p and TZ complemented in $L^p(l^2)$. It follows easily that then $P_{[1,n]}|Z$ is an isomorphism with $P_{[1,n]}Z$ complemented; that is, L^p embeds as a complemented subspace of $Y_1 \oplus \cdots \oplus Y_n$. Hence by Scholium 1.9, L^p embeds as a complemented subspace of Y_i for some i.

We next need a simple but crucial observation.

Sublemma 2. *Let $I \subset N$, E_I be as in Sublemma 1 with $\mu(E_I) = 1$, and $S \subset [0,1]$ with S a finite union of disjoint left-closed dyadic intervals. Then there exists a $J \subset I$ so that $\operatorname{supp} h_i \cap \operatorname{supp} h_l = \varnothing$ for all $i \neq l$, $i, l \in J$, with $S \supset \bigcup_{j \in J} \operatorname{supp} h_j$ and $S \sim \bigcup_{j \in J} \operatorname{supp} h_j$ of measure zero.*

Proof. It suffices to prove the result for S equal to a left-closed dyadic interval. Now any two Haar functions either have disjoint supports or the support of one is contained in that of the other. Moreover, for all but finitely many $i \in I$, $\operatorname{supp} h_i \subset S$ or $\operatorname{supp} h_i \cap S = \varnothing$. Hence S differs from $\bigcup\{\operatorname{supp} h_j\colon \operatorname{supp} h_j \subset S$, $j \in I\}$ by a measure-zero set. Now simply let $J = \{j \in I\colon \operatorname{supp} h_j \subset S$ and there is no $l \in I$ with $\operatorname{supp} h_j \subsetneqq \operatorname{supp} h_l \subset S\}$.

We now choose $M_1, M_2, \ldots$ disjoint finite subsets of N, a map $j\colon \bigcup_{i=1}^{\infty} M_i \to N$, and $1 = m_0 < m_1 < m_2, \ldots$ with the following properties:

A. For each k, the h_i's for $i \in M_k$ are disjointly supported. Set $z_k = \sum_{i \in M_k} h_i$. Then (z_k) satisfies the hypotheses of Scholium 1.11.

B. Let $Q_k = P_{[m_{k-1}, m_k)}$ for all k. Then $(UQ_k h_{ij(i)})_{i \in M_k, k \in N}$ is essentially disjoint and $h_{ij(i)}^* UQ_k h_{ij(i)} > \frac{1}{2}$ for all $i \in M_k$, $k \in N$.

Having accomplished this, we set $b_k = \sum_{i \in M_k} Q_k h_{ij(i)}$ for all k. Then by B, (b_k) is a block basic sequence of the Y_i's.

By Scholium 1.4,

$$(1.12) \qquad (Q_k h_{ij(i)})_{i \in M_k, k \in N} \sim (h_{ij(i)})_{i \in M_k, k \in N} \sim (h_i)_{i \in M_k, k \in N}$$

where "$\sim$" denotes equivalence of basic sequences; the last equivalence follows from Scholium 1.6, i.e., the unconditionality of the Haar basis. Hence by the definitions of (b_k) and (z_k), (b_k) is equivalent to (z_k) which is equivalent to (h_k), the Haar basis, by Scholium 1.11. Also, since $[z_k]$ is complemented in L^p by 1.11, $[b_k]$ is complemented in $[Q_k h_{ij(i)})_{i \in M_k, k \in N}$ by (1.12). Again by Scholium 1.4, $[Q_k h_{ij(i)}]_{i \in M_k, k \in N}$ is complemented in Y, hence also $[b_k]$ is complemented in Y.

It remains now to choose the M_i's, m_i's and map j. To insure B, we shall also choose a sequence $(f_i)_{i \in M_k, k \in N}$ of disjointly finitely supported elements of $L^p(l^2)$

(disjointly supported with respect to the basis (h_{ij})) so that

$$\sum_{i\in M_k} \frac{\|UQ_k h_{ij(i)} - f_i\|}{\|UQ_k h_{ij(i)}\|} < \frac{1}{2^k} \quad \text{for all } k. \tag{1.13}$$

To start, we let $M_1 = \{1\}$ and $j(1) = 1$. Thus $z_1 = 1$; we also set $f_1 = h_{11}$. Then $h_{11} = Uh_{11} = \lim_{n\to\infty} UP_{[1,n]}h_{11}$. So it is obvious that we can choose $m_1 > 1$ such that $\|UP_{[1,m_1]}h_{11} - h_{11}\| < \frac{1}{2}$; hence $h_{11}^* UP_{[1,m_1]}h_{11} > \frac{1}{2}$. Thus, the first step is essentially trivial.

Now suppose $l \geq 1$, $M_1, \ldots, M_l$, $m_1 < \cdots < m_l$, j: $\bigcup_{i=1}^l M_i \to N$ and $(f_i)_{i\in M_k, 1\leq k\leq l}$ have been chosen. We set $z_i = \sum_{j\in M_i} h_j$ for all $i, 1 \leq i \leq l$.

Let $1 \leq k \leq l$ be the unique integer and α the unique choice of ± 1 so that $\operatorname{supp} h_{l+1} = [h_k = \alpha]$. Let $S = [z_k = \alpha]$. Set $n = m_l$ and let I be as in Sublemma 1. Since S is a finite union of disjoint left-closed dyadic intervals, by Sublemma 2 we may choose a finite set $M_{l+1} \subset I$, disjoint from $\bigcup_{i=1}^l M_i$, so that the h_i's for $i \in M_{l+1}$ are disjointly supported with $\operatorname{supp} h_i \subset S$ for $i \in M_{l+1}$ and

$$\mu\left(S \sim \bigcup_{i\in M_{l+1}} \operatorname{supp} h_i\right) \leq \varepsilon_l \tag{1.14}$$

(where $\varepsilon_j = 1/2^{j^2}$ for all j). At this point, we have that $z_{l+1} = \sum_{i\in M_{l+1}} h_i$ satisfies the conditions of Scholium 1.11.

By the definition of I, for each $i \in M_{l+1}$ there is an infinite set J_i with

$$h_{ij}^* UR_n h_{ij} > \tfrac{1}{2} \quad \text{for all} \quad j \in J_i.$$

Now $(UR_n h_{ij})_{j=1}^\infty$ is a weakly null sequence; hence it follows that we may choose j: $M_{l+1} \to N$ and disjointly finitely supported elements $(f_i)_{i\in M_{l+1}}$, with supports (relative to the h_{ij}'s) disjoint from those of $\{f_i\colon i \in \bigcup_{i=1}^l M_i\}$, so that

$$\sum_{i\in M_{l+1}} \frac{\|UR_n h_{ij(i)} - f_i\|}{\|UR_n h_{ij(i)}\|} < \frac{1}{2^{l+1}}.$$

At last, since $R_n g = \lim_{k\to\infty} P_{[m_l,k)} g$ for any $g \in L^p(l^2)$, we may choose an $m_{l+1} > m_l$ so that setting $Q_{l+1} = P_{[m_l, m_{l+1})}$, (1.13) holds for $k = l + 1$ and also

$$h_{ij(i)}^* UQ_k h_{ij(i)} > \tfrac{1}{2} \quad \text{for all} \quad i \in M_k.$$

This completes the construction of the M_i's, m_i's and map j. Since (1.12) holds, A and B hold; thus the proof is complete.

2. The local L^p-index

Our object in this section is to construct the local L^p-index and verify its properties, then apply it to the R^p_α's defined in the introduction. The basic theorem is 2.1 of the introduction, which we recall here.

THEOREM 2.1. *For each $1 \le p \le \infty$ and separable Banach space B, there exists an ordinal number $h_p(B) \le \omega_1$, the local L^p-index of B, so that*

(a) *$h_p(B) < \omega_1$ if and only if $L^p \not\hookrightarrow B$ and $p < \infty$ or $C([0,1]) \not\hookrightarrow B$ and $p = \infty$, and*

(b) *If X is a Banach space such that $X \hookrightarrow B$, then $h_p(X) \le h_p(B)$.*

The formal definition of the index requires some preliminary formulations (Proposition 2.2 and Definition 1). The index is given in Definition 2 and the "boundedness principle" Theorem 2.1(a) is established in Proposition 2.3, by use of an evident but crucial permanence property of well-founded relations (Lemma 2.4). Theorem 2.1(b) is then quickly obtained, after which we give a general concatenation lemma (Lemma 2.5) which shows that if $h_p(B) > \alpha$ then $h_p(B \oplus B)_p > \alpha + 1$. We then resume our discussion of the R^p_α's, giving the formal definitions of independent and disjoint sums in L^p, and of the R^p_α's themselves in Definition 3. We show in Theorem 2.6 that $h_p(R^p_\alpha) \ge \alpha + 1$ and $L^p \not\hookrightarrow R^p_\alpha$ in Proposition 2.7, thus completing parts (1) and (2) of Theorem B of the introduction. Finally, we establish in Proposition 2.8 that the R^p_α's have unconditional bases for all $1 < p < \infty, \alpha < \omega_1$. This is false for $p = 1$; see the remarks at the end of this section.

Before formally defining the index and establishing its properties, we begin with some intuitive comments. We may think of $L^p[0,1]$ as given by an increasing sequence (E_n) of spaces with E_n isometric to $l^p_{2^n}$ for all n, where E_{n+1} is obtained from E_n by "splitting" each element of the natural basis for E_n in two. Thus, we let

$$E_0 = [1],$$

$$E_1 = \left[2^{1/p}\chi_{[0,\frac{1}{2}]}, 2^{1/p}\chi_{[\frac{1}{2},1]}\right],$$

$$E_2 = \left[2^{2/p}\chi_{[0,\frac{1}{4}]}, 2^{2/p}\chi_{[\frac{1}{4},\frac{1}{2}]}, 2^{2/p}\chi_{[\frac{1}{2},\frac{3}{4}]}, 2^{2/p}\chi_{[\frac{3}{4},1]}\right], \quad \text{etc.}$$

Now a Banach space B contains an isomorph of L^p provided it contains an increasing sequence (F_n) of finite dimensional spaces which "look like" the E_n's. We may interpret the natural basis for E_n as an element e_n of $(L^p)^{D_n}$ (i.e., a function from D_n to L^p) rather than as a 2^n-tuple of vectors, where D_n denotes the set of all n-tuples of 0's and 1's. Suppose $F_n = [u_n(x): x \in D_n]$ with $u_n \in B^{D_n}$

for all n. Then (F_n) looks like (E_n) provided

$$u_n(x) = \frac{u_{n+1}(x,0) + u_{n+1}(x,1)}{2^{1/p}} \quad \text{for all } x \in D_n, \quad \text{all } n,$$

and $\{u_n(x)\colon x \in D_n\}$ is uniformly equivalent to the $l^p_{2^n}$ basis. Then with $\delta > 0$ given, we can introduce a partial order on a subset of $\bigcup_{n=0}^{\infty} B^{D_n}$ so that B contains a $1/\delta$-isomorph of L^p provided the partially ordered set has an infinite linearly ordered subset.

We now introduce the needed formal definitions and notation. *Let* $\mathfrak{D}$ *denote the set of all finite sequences of* 0's *and* 1's. That is, $\mathfrak{D} = \bigcup_{n=0}^{\infty} D_n$ where $D_n = \{(t_1,\dots,t_n)\colon t_i = 0 \text{ or } 1 \text{ for all } i\}$. Let B be a separable Banach space; if $u \in B^{\mathfrak{D}}$, let $|u| = k$ if $u \in D_k$. (We shall refer to $|u|$ as the rank of u.)

Since D_0 is the set consisting of the empty sequence, B^{D_0} can be identified with B itself. Now fix $p, 1 \le p \le \infty$. For $u, v \in B^{\mathfrak{D}}$, we set $u < v$ provided $|u| < |v|$ and

$$(2.1)\quad u(x) = 2^{-k/p} \sum_{\tau \in D_k} v(x,\tau) \quad \text{for all} \quad x \in D_{|u|}, \quad \text{where} \quad k = |v| - |u|.$$

It is evident that $<$ is indeed a partial order on $B^{\mathfrak{D}}$. Now fix $\delta, 0 < \delta \le 1$, and let $\bar{B}^{\delta}$ denote the set of all $u \in B^{\mathfrak{D}}$ so that

$$(2.2)\qquad \delta\Big(\sum_{x\in D_n} |c(x)|^p\Big)^{1/p} \le \Big\|\sum_{x\in D_n} c(x)u(x)\Big\| \le \Big(\sum_{x\in D_n} |c(x)|^p\Big)^{1/p}$$

for all $c \in \mathbf{R}^{D_n}$, where $|u| = n$. (If $p = \infty$, we replace $(\Sigma_{x \in D_n}|c(x)|^p)^{1/p}$ by $\max\{|c(x)|\colon x \in D_n\}$.

For simplicity of notation, we set $\bar{B}^1 = \bar{B}$; thus the rank-n elements of $\bar{B}^1$ simply correspond to the 2^n-tuples of B that are isometrically equivalent to the usual basis for $l^p_{2^n}$. The reader may now readily establish the following result: (The case $p = \infty$ is obtained by working with $C(\Delta)$, Δ the Cantor set, rather than $C([0,1])$.)

PROPOSITION 2.2. *$L^p \hookrightarrow B$ (resp. $C([0,1]) \hookrightarrow B$ if $p = \infty$) if (and only if) there exist $0 < \delta \le 1$ and elements $u_1, u_2, \dots$ in $\bar{B}^{\delta}$ with $u_n < u_{n+1}$ for all n.*

An equivalent formulation: $L^p \not\hookrightarrow B$ if and only if every non-empty subset of $\bar{B}^{\delta}$ has a maximal element with respect to $<$. In the language of logicians, "$<$" is a well-founded relation on $\bar{B}^{\delta}$. We now follow a time-honored procedure (in logic!) to determine the "depth" of $<$; we successively erase the maximal elements until arriving at the empty set.

Definition 1. *Set* $H_0^\delta(B) = \bar{B}^\delta$. *Suppose* β *is an ordinal* > 0 *and* $H_\alpha^\delta(B)$ *has been defined for all* $\alpha < \beta$. *If* $\beta = \alpha + 1$, *let*

$$H_\beta^\delta(B) = \{u \in H_\alpha^\delta(B): \text{ there is a } v \in H_\alpha^\delta(B) \text{ with } u < v\}.$$

If β *is a limit ordinal, let* $H_\beta^\delta(B) = \bigcap_{\alpha<\beta} H_\alpha^\delta(B)$.

We note in passing that the classes $H_\alpha^\delta(B)$ are all "subtrees" of $\bar{B}^\delta$. That is, if $v \in H_\alpha^\delta(B)$, $u \in \bar{B}^\delta$ and $u < v$, then $u \in H_\alpha^\delta(B)$.

Since the H_α^δ's decrease by definition, they must become stationary after some point, that is, $H_\gamma^\delta(B) = H_{\gamma+1}^\delta(B)$ for some γ.

Definition 2. *Let* α *denote the least ordinal* γ *such that* $H_\gamma^\delta(B) = H_{\gamma+1}^\delta(B)$. *If* $H_\alpha^\delta(B) = \varnothing$, *set* $h_p(\delta, B) = \alpha$. *If* $H_\alpha^\delta(B) \neq \varnothing$, *set* $h_p(\delta, B) = \omega_1$. *Finally, set* $h_p(B) = \sup_{\delta>0} h_p(\delta, B)$.

As mentioned in the introduction, we call $h_p(B)$ *the local* L^p-index of the Banach space B.

Suppose $L^p \not\hookrightarrow B$. Then Proposition 2.1 yields that $H_\alpha^\delta = \varnothing$ where $\alpha = h_p(\delta, B)$. Evidently if $\eta < \delta$, then $H_\gamma^\eta(B) \supset H_\gamma^\delta(B)$ for any γ, hence $h_p(\eta, B) \geq h_p(\delta, B)$. Thus $h_p(B) = \lim_{\delta\to 0} h_p(\delta, B)$. It is now evident that to establish Theorem 2.1(a), we need only prove the following:

Proposition 2.3. *For all separable* B *and* $0 < \delta \leq 1$, $h_p(\delta, B) < \omega_1$ *provided* $L^p \not\hookrightarrow B$ (*resp.* $C[(0,1)] \not\hookrightarrow B$ *if* $p = \infty$).

Although we are mainly interested in isomorphic invariants, it is worth noting that L^p is isometric to a subspace of B if and only if $h_p(1, B) = \omega_1$ (resp. B is isometrically universal if and only if $h_\infty(1, B) = \omega_1$).

A general boundedness principle (see [8] and the discussion in [3]) asserts that every well-founded analytic relation has index bounded by a countable ordinal. Proposition 2.2 means that $<$ is a well-founded relation on $\bar{B}^\delta$, and it is easily seen that $<$ is analytic. Rather than appealing to a general principle, we prefer to give a direct proof based on simple though fundamental ideas concerning well-founded relations. *A relation* R *on a set* X *is said to be well-founded provided there do not exist* $x_1, x_2, \ldots$ *in* X *with* $x_n R x_{n+1}$ *for all* n. We define classes $H_\alpha(R)$ by

$$H_0(R) = X;$$

$$H_{\alpha+1}(R) = \{x \in H_\alpha(R): \text{ there exists } y \in H_\alpha(R) \text{ with } xRy\}$$

and $$H_\alpha(R) = \bigcap_{\beta<\alpha} H_\beta(R)$$

if α is a limit ordinal. If R is well-founded, there exists a least ordinal α, denoted by $h(R)$, with $H_\alpha(R) = \varnothing$.

The reader may now easily establish the following crucial permanence property:

LEMMA 2.4. *Let R and R' be well-founded relations on X and X' respectively and let τ: $X \to X'$ be an order-preserving map. That is, if xRy, then $(\tau x)R'(\tau y)$. Then $h(R) \leq h(R')$. In fact, for all ordinals α, $\tau(H_\alpha(R)) \subset H_\alpha(R')$.*

Evidently every countable well-founded relation R has bounded index $h(R)$; i.e. $h(R) < \omega_1$. Thus, if we assume $L^p \not\hookrightarrow B$, to establish the boundedness of $h_p(\delta, B)$ it suffices to exhibit an order-preserving map τ between $\bar{B}^\delta$ and a countable set $\bar{B}_0^\delta$ endowed with a well-founded relation R. Let B_0 be a countable dense subset of B and $\bar{B}_0^\delta$ denote the set of all $u \in B_0^{\mathfrak{D}}$ so that

$$(2.3) \qquad \frac{\delta}{2}\Big(\sum_{x\in D_n} |c(x)|^p\Big)^{1/p} \leq \Big\| \sum_{x\in D_n} c(x)u(x)\Big\| \leq 2\Big(\sum_{x\in D_n} |c(x)|^p\Big)^{1/p}$$

for all $c \in \mathbf{R}^{D_n}$, where $|u| = n$. Let $\eta_k = \delta 4^{-(k+1)}$ for all k, and define R on $\bar{B}_0^\delta$ by uRv provided $|u| = n, |v| = n + k$ with $k \geq 1$ and

$$(2.4) \qquad \Big\| u(x) - 2^{-k/p} \sum_{y\in D_k} v(x,y)\Big\| \leq \eta_n \quad \text{for all} \quad x \in D_n.$$

Let us check that R is well-founded. Suppose the contrary; let $u_1, u_2, \dots$ be in $\bar{B}_0^\delta$ with $u_n R u_{n+1}$ for all n. Let $k < l < m$, let $r = |u_k|$, $s = |u_l|$ and $t = |u_m|$. Let $a = s - r$ and $b = t - s$. By (2.4), we have

$$(2.5) \qquad \Big\| u_l(x,y) - 2^{-b/p} \sum_{z\in D_b} u_m(x,y,z)\Big\| \leq \eta_s$$

for all $x \in D_r$, $y \in D_a$. Then

$$(2.6) \qquad \Big\| 2^{-a/p} \sum_{y\in D_a} u_l(x,y) - 2^{-(a+b)/p} \sum_{z\in D_{a+b}} u_m(x,z)\Big\| \leq 2^{a-a/p}\eta_s \leq 2^s\eta_s \quad \text{for all} \quad x \in D_r.$$

Since $2^n\eta_n \to 0$ as $n \to \infty$, it follows from (2.6) and the completeness of B that we may define an element $\bar{u}_k \in B^{\mathfrak{D}}$ by $\bar{u}_k(x) = \lim_{l\to\infty} 2^{-a/p}\Sigma_{y\in D_a} u_l(x,y)$ for all $x \in D_r$. Then if we fix $l > k$,

$$\begin{aligned}\bar{u}_k(x) &= \lim_{m\to\infty} 2^{-a/p}2^{-b/p} \sum_{y\in D_a,\, z\in D_b} u_m(x,y,z) \\ &= 2^{-a/p} \sum_{y\in D_a} \lim_{m\to\infty} 2^{-b/p} \sum_{z\in D_b} u_m(x,y,z) = 2^{-a/p} \sum_{y\in D_a} \bar{u}_k(x,y);\end{aligned}$$

that is, $\bar{u}_k < \bar{u}_l$. Finally, (2.4) yields that

$$\|\bar{u}_k(x) - u_k(x)\| \le \eta_r \quad \text{for all} \quad x \in D_r.$$

It follows that $(u_k(x))_{x \in D_r}$ is uniformly equivalent to the usual $l^p_{2^r}$ basis, whence $L^p \hookrightarrow B$ (by Proposition 2.2). (In fact $\bar{u}_k/3 < \bar{u}_{k+1}/3$ and $\bar{u}_k/3 \in \bar{B}^{\delta/12}$ for all k.) Having established that R is well-founded, it remains to define the order preserving map τ. Set $\varepsilon_k = \delta \cdot 8^{-(k+1)}$ for all k. For each k, $x \in D_k$ and $u \in \bar{B}^\delta$ with $|u| = k$, choose $v(x) \in B_0$ with $\|u(x) - v(x)\| \le \varepsilon_k$. Then for all $c \in \mathbf{R}^{D_k}$,

$$[\delta - 2^k\varepsilon_k]\left(\sum |c(x)|^p\right)^{1/p} \left\| \sum_{x \in D_k} c(x)v(x) \right\| \le (1 + 2^k\varepsilon_k)\left(\sum |c(x)|^p\right)^{1/p}.$$

Since $2^k\varepsilon_k \le \delta/2 \le 1$, we have that $v \in \bar{B}_0^\delta$. Now set $\tau u = v$. We need only verify that τ is an order-preserving map. Let $|u| = k$, $|v| = k + l$ and suppose $u < v$. Then

$$\begin{aligned}\left\|\tau u(x) - 2^{-l/p} \sum_{y \in D_l} \tau v(x, y)\right\| & \\ &\le \|\tau u(x) - u(x)\| + 2^{-l/p} \sum_{y \in D_l} \|\tau v(x, y) - v(x, y)\| \\ &\le \varepsilon_k + 2^l \varepsilon_{k+l} \le \eta_k \quad \text{for all} \quad x \in D_k;\end{aligned}$$

thus the proof of Proposition 2.3 is complete.

We may now easily complete the proof of Theorem 2.1. Let us suppose that $X \hookrightarrow B$ and $L^p \not\hookrightarrow B$. We may choose an $\eta > 0$ and a linear map $T: X \to B$ so that

$$\eta\|x\| \le \|Tx\| \le \|x\| \quad \text{for all} \quad x \in X. \tag{2.7}$$

Now define $\tau: X^{\mathfrak{D}} \to B^{\mathfrak{D}}$ by $(\tau u)(t) = T(u(t))$ for all $u \in X^{\mathfrak{D}}, t \in D_{|u|}$. The linearity of T then implies that τ is order preserving. Finally, fix $0 < \delta \le 1$ and suppose $u \in \bar{X}^\delta$. Then by (2.7), for all $c \in \mathbf{R}^{D_{|u|}}$,

$$\begin{aligned}\eta\delta\left(\sum |c(t)|^p\right)^{1/p} \le \eta\left\|\sum c(t)u(t)\right\| \le \left\| \sum_{t \in D_{|u|}} c(t)\tau u(t)\right\| &= \left\|T\sum c(t)u(t)\right\| \\ &\le \left\|\sum c(t)u(t)\right\| \\ &\le \left(\sum |c(t)|^p\right)^{1/p}.\end{aligned}$$

That is, $\tau u \in \bar{B}^{\eta\delta}$. Hence by Lemma 2.4,

$$\begin{aligned}h_p(\delta, X) \le h_p(\eta\delta, B), \quad \text{whence} \quad h_p(X) &= \lim_{\delta \to 0} h_p(\delta, X) \\ &\le \lim_{\delta \to 0} h_p(\eta\delta, B) = h_p(B).\end{aligned}$$

This completes the proof of Theorem 2.1.

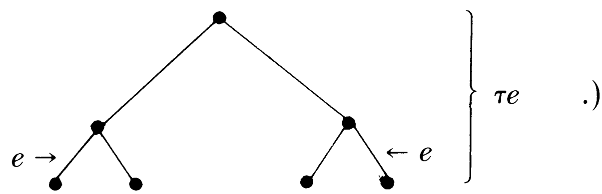
Before passing to the application of the local L^p-index to the R_α^p's given in the introduction, we need a general concatenation lemma. The lemma implies that if $h_p(B) > \alpha$, then $h_p(B \oplus B)_p > \alpha + 1$.

LEMMA 2.5. *Let B be a separable Banach space, $0 < \delta \leq 1$ and $\alpha < \omega_1$. Let $e \in H_\alpha^\delta(B)$. Let $\bar{e}$ be the element of $(B \oplus B)_p^{\mathfrak{D}}$ defined by $\bar{e}(t) = 2^{-1/p}e(t) \oplus e(t)$ for all $t \in D_{|e|}$. Then $\bar{e} \in H_{\alpha+1}^\delta((B \oplus B)_p)$.*

Proof. Let τe be the element of $(B \oplus B)_p^{\mathfrak{D}}$ defined by

$$\tau e(0,t) = e(t) \oplus 0 \quad \text{and} \quad \tau e(1,t) = 0 \oplus e(t) \quad \text{for all} \quad t \in D_{|e|}.$$

Then we have that $\bar{e} < \tau e$. (Thus if $k = |e|$, $k+1 = |\tau e|$ and τe is obtained by taking the two natural copies of e in B. The picture is as follows:



We need only prove that $\tau e \in H_\alpha^\delta(B \oplus B)_p$. We first check that $\tau e \in \overline{(B \oplus B)}_p^\delta$. This is an evident consequence of the equalities

$$\left\| \sum_{t \in D_{k+1}} c(t)\tau e(t) \right\|^p = \left\| \sum_{t \in D_k} c(0,t)\tau e(0,t) \oplus \sum_{t \in D_k} c(1,t)\tau e(1,t) \right\|^p$$

$$= \left\| \sum_{t \in D_k} c(0,t)e(t) \right\|^p + \left\| \sum_{t \in D_k} c(1,t)e(t) \right\|^p$$

for all $c \in \mathbf{R}^{D_{k+1}}$ where $k = |e|$.

We now prove the statement:

$$e \in H_\alpha^\delta(B) \Rightarrow \tau e \in H_\alpha^\delta(B \oplus B)_p$$

by induction on α. The case $\alpha = 0$ is evident. Suppose $\alpha > 0$ and the statement is proved for all $\gamma < \alpha$. Then if α is a limit ordinal, $e \in H_\alpha^\delta(B) \Rightarrow e \in H_\gamma^\delta(B)$ for all $\gamma < \alpha \Rightarrow \tau e \in H_\gamma^\delta(B \oplus B)_p$ for all $\gamma < \alpha$ by the induction hypothesis $\Rightarrow \tau e \in H_\alpha^\delta(B \oplus B)_p$. Now suppose $\alpha = \beta + 1$. By definition, there exists a $d \in H_\beta^\delta(B)$ with $e < d$. By the "sub-tree" property mentioned after the definition of the H_α^δ's, we may assume that $|d| = |e| + 1$. By the induction hypothesis, we have that $\tau d \in H_\beta^\delta(B \oplus B)_p$. Thus, we need only verify that $\tau e < \tau d$, for by the sub-tree property, it then follows that τe is a non-maximal element of $H_\beta^\delta(B \oplus B)_p$.

Letting $t \in D_{|e|}$, we have that

$$(\tau e)(0,t) = e(t) \oplus 0 = \frac{d(t,0) \oplus 0 + d(t,1) \oplus 0}{2^{1/p}} = \frac{(\tau d)(0,t,0) + \tau d(0,t,1)}{2^{1/p}}$$

and similarly $(\tau e)(1,t) = ((\tau d)(1,t,0) + \tau d(1,t,1))/2^{1/p}$. Hence $\tau e(s) = (\tau d(s,0) + \tau d(s,1))/2^{1/p}$ for all $s \in D_{|e|+1}$, so $\tau e < \tau d$ and the lemma is proved.

We are now prepared for the precise definition of the spaces R^p_α and the verification of parts (1) and (2) of Theorem B of the introduction. By a "space of random variables" we mean a linear subspace of $L^0(\mu)$ for some probability space $(\Omega, \mathcal{S}, \mu)$; $L^0(\mu)$ denotes the space of all (equivalence classes of) real-valued measurable functions defined on Ω. Given a random variable x defined on Ω, dist x denotes the probability measure defined on the Borel subsets of the reals by $\operatorname{dist} x(S) = \mu\{\omega\colon x(\omega) \in S\}$ for all $S \in \mathcal{S}$. Given spaces of random variables X, Y on possibly different probability spaces, we say X and Y are *distributionally isomorphic* if there exists a linear bijection $T\colon X \to Y$ so that $\operatorname{dist} Tx = \operatorname{dist} x$ for all $x \in X$. It is not difficult to see that given such a map T, there exist σ-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of the measurable sets so that $x \in X$ ((resp. $y \in Y$) is $\mathcal{A}$-measurable) (resp. $\mathcal{B}$ measurable) and a map $\tilde{T}\colon L^0(\mathcal{A}) \to L^0(\mathcal{B})$ extending T. Of course, a distributional isomorphism preserves L^p-norms for all $0 < p \le \infty$. It is important for the inductive definition of the R^p_α's that they are "distributionally presented"; i.e. the isometric Banach space structure itself is not sufficient to define the family.

Given B a (closed linear) subspace of $L^p(\mu)$ for some probability space $(\Omega, \mathcal{S}, \mu)$, we let *the "$L^p$-disjoint sum"*, $(B \oplus B)_p$, denote a space of random variables distributionally isomorphic to the subspace of $\Omega \times \{0,1\}$ defined as

$$\{b(\omega,\varepsilon)\colon \text{there exist } b_\varepsilon \in B \text{ with } b(\omega,\varepsilon) = b_\varepsilon(\omega) \text{ for all } \omega \in \Omega, \varepsilon = 0 \quad\text{or}\quad 1\},$$

where, of course, $\{0,1\}$ is endowed with the fair probability assigning mass $\frac{1}{2}$ to each 0 and 1.

Given $B_1, B_2, \dots$ subspaces of $L^p(\Omega)$, we define the L^p-independent sum of the B_i's as follows: Let μ^N denote the product probability measure on $(\Omega^N, \mathcal{S}^N)$; for each i, let

$$\bar{B}_i = \{b \text{ on } \Omega^N\colon \exists f \in B_i \text{ with } b(\omega) = f(\omega_i) \text{ for all } \omega \in \Omega^N\}.$$

That is, $\bar{B}_i$ is simply a "copy" of B_i depending only on the i-th coordinate. *Then* $(\Sigma B_i)_{\mathrm{Ind},p}$, *the L^p-independent sum of the B_i*'s, denotes any space of random variables distributionally isomorphic to the closed linear span of the $\bar{B}_i$'s in $L^p(\Omega^N)$.

These notions may be "intrinsically" expressed as follows: Given B, a space of random variables Y on $(\Omega, \mathcal{S}, \mu)$ is distributionally isomorphic to $(B \oplus B)_p$

provided there exist sets $S_i \in \mathcal{S}$ with $\mu(S_i) = \frac{1}{2}$ $(i = 1,2)$, $S_i \cap S_2 = \varnothing$ and subspaces X_i of $L^p(2\mu|\mathcal{S} \cap S_i)$ each distributionally isomorphic to B, so that $Y = X_1 + X_2$ (where for $x \in X_i$, we regard x as a function on Ω, supported on S_i). Given $B_1, B_2, \ldots$, then Y is distributionally isomorphic to $(\Sigma B_i)_{\mathrm{Ind},\,p}$ provided there exist independent σ-subalgebras $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$ of $\mathcal{S}$ and spaces of random variables $\bar{B}_1, \bar{B}_2, \ldots$ with Y equal to the closed linear span of the $\bar{B}_i$'s in $L^p(\mu)$, so that for each i, every $b \in \bar{B}_i$ is $\mathcal{Q}_i$ measurable and $\bar{B}_i$, regarded as a subspace of $L^p(\mu|\mathcal{Q}_i)$, is distributionally isomorphic to B_i.

It is worth mentioning that if $\int b\,d\mu = 0$ for all i and $b \in B_i$, $(\Sigma B_i)_{\mathrm{Ind},\,p}$ has a natural unconditional Schauder decomposition, $\bar{B}_1, \bar{B}_2, \ldots$ in our above discussion. However if $1 \in B_i$ for all i, the independent sum is not even a direct sum. In this case, we simply let $B_i^0 = \{b \in B_i\colon \int b\,d\mu = 0\}$. Then $(\Sigma B_i)_{\mathrm{Ind},\,p} = (\Sigma B_i^0)_{\mathrm{Ind},\,p} + [1]$ ($[1]$ denotes the space of constant functions on Ω). We shall only deal with separable spaces of random variables; any such space is, of course, distributionally isomorphic to a space on $[0,1]$ under Lebesgue measure.

Definition 3. *Let* $1 \le p \le \infty$. *Let* $R_0^p = [1]$. *Let* β *be an ordinal with* $0 < \beta < \omega_1$ *and suppose* R_α^p *has been defined for all* $\alpha < \beta$. *If* $\beta = \alpha + 1$, *let* $R_\beta^p = (R_\alpha^p \oplus R_\alpha^p)_p$. *If* β *is a limit ordinal, let* $R_\beta^p = (\Sigma_{\alpha<\beta} R_\alpha^p)_{\mathrm{Ind},\,p}$.

We may now easily complete the proof of part (2) of Theorem B. We let $H_\alpha(R_\alpha^p) = H_\alpha^1(R_\alpha^p)$.

THEOREM 2.6. *Let* $1 \le p \le \infty, 0 \le \alpha < \omega_1$. *Then* $1 \in H_\alpha(R_\alpha^p)$.

Since $H_\alpha(R_\alpha^p) \neq \varnothing$, we thus obtain that $h_p(R_\alpha^p) \ge h_p(1, R_\alpha^p) \ge \alpha + 1$. We prove 2.6 by transfinite induction. The assertion is trivial for $\alpha = 0$. Suppose $0 < \alpha$ and the statement has been proved for all $\beta < \alpha$. If $\alpha = \beta + 1$, let us take the specific representation of $R_\alpha = (R_\beta \oplus R_\beta)_p$ given above. The element $\bar{1}$ of Lemma 2.5 is then precisely the function 1; thus $1 \in H_\alpha(R_\alpha^p) = H_{\beta+1}(R_\beta^p \oplus R_\beta^p)_p$ by 2.5. Now suppose α is a limit ordinal. Fix $\beta < \alpha$. It is evident that there exists a subspace $\bar{R}_\beta^p$ of R_α^p and a distributional isomorphism $T_\beta\colon R_\beta^p \to \bar{R}_\beta^p$. Thus T_β may be regarded as an isometry of R_β^p into R_α^p such that $T_\beta 1 = 1$. Define $\tau\colon \overline{R_\beta^p} \to \overline{R_\alpha^p}$ by

$$(\tau u)(x) = T_\beta u(x) \quad \text{for all} \quad u \in \left(R_\beta^p\right)^{\mathfrak{D}}, \qquad x \in D_{|u|}.$$

It is evident that τ is order preserving with $\tau 1 = 1$. Hence by Lemma 2.4, $1 = \tau 1 \in H_\beta(R_\alpha^p)$. Since this holds for all $\beta < \alpha$, $1 \in H_\alpha(R_\alpha^p)$, completing the proof of Theorem 2.6.

We do not require the local L^p-index to complete the proof of Theorem B(1).

PROPOSITION 2.7. *Let $0 \leq \alpha < \omega_1$. Then $L^p \not\hookrightarrow R^p_\alpha$ for $1 \leq p < \infty$. Moreover R^1_α has the Radon-Nikodym property and R^∞_α has both the Radon-Nikodym property and the Schur property.*

Proof. Let $0 < \alpha < \omega_1$ and suppose the result proved for all $\gamma < \alpha$. If $\alpha = \gamma + 1$, then if $L^p \hookrightarrow R^p_\alpha$, $L^p \overset{c}{\hookrightarrow} R^p_\alpha$ by Theorem 9.1 of [13]; hence $L^p \overset{c}{\hookrightarrow} (R^p_\gamma \oplus R^p_\gamma)_p \Rightarrow L^p \overset{c}{\hookrightarrow} R^p_\gamma$ by Theorem 1.1, for $1 < p < \infty$. The assertions for $p = 1$ and $p = \infty$ are trivial in this case. If α is a limit ordinal, let $\gamma_1, \gamma_2, \ldots$ be an enumeration of the ordinals $\gamma < \alpha$ and for each j, let Y_j be the mean-zero elements of $R^p_{\gamma_j}$. Now if $L^p \hookrightarrow R^p_\alpha$, $1 < p < \infty$, $L^p \overset{c}{\hookrightarrow} (\Sigma Y_j)_{\mathrm{Ind},\, p}$ and, of course, $L^p \not\hookrightarrow Y_j$ for all j. Let $(\overline{Y}_j)$ be the natural unconditional Schauder decomposition for $(\Sigma Y_j)_{\mathrm{Ind},\, p}$. Then by Theorem 1.1 there is a block basic sequence (z_j) of $(\overline{Y}_j)$ equivalent to the Haar basis of L^p. In particular L^p is isomorphic to $[z_j]$. Now (z_j) is a sequence of independent mean-zero random variables. It follows from the results of [19] and [20] (see also [23]) that $L^p \not\hookrightarrow [z_j]$. Let us see briefly why this is so. It is shown in [20] that there is a certain complemented subspace X_p of L^p, spanned by a sequence of independent mean-zero variables, so that $[z_j] \hookrightarrow X_p$ for any sequence of independent mean-zero variables (z_j); moreover X^*_p is isomorphic to X_q where $1/p + 1/q = 1$. Suppose $p > 2$. Then $L^p \hookrightarrow [z_j]$ implies $(l^2 \oplus l^2 \oplus \cdots)_p \hookrightarrow X_p$. But it is also shown in [19] that $X_p \hookrightarrow l^2 \oplus l^p$, hence $(l^2 \oplus l^2 \oplus \cdots)_p \hookrightarrow l^2 \oplus l^p$, proved impossible in [19]. If $1 < p < 2$ and $L^p \hookrightarrow [z_j]$, then $L^p \hookrightarrow X_p$ and hence by Theorem 9.1 of [13], $L^p \overset{c}{\hookrightarrow} X_p$ whence $L^q \overset{c}{\hookrightarrow} X_q$ where $1/p + 1/q = 1$, already proved impossible. (We have, of course, shown that $(l^2 \oplus l^2 \oplus \cdots)_p \hookrightarrow [z_j]$ is impossible for $p > 2$; the fact that this is impossible for $p < 2$ follows by the reproducibility of the natural basis for $(l^2 \oplus l^2 \oplus \cdots)_p$ and Proposition 2 of [23].)

Proposition 2.7 is now proved for $1 < p < \infty$, and, of course, the second assertion implies the first for the case $p = 1$. For any p, we have that a subspace of codimension one of R^p_α equals $(\Sigma Y_j)_{\mathrm{Ind},\, p}$ where each Y_j is isometric to a codimension-one subspace of R^p_γ for some $\gamma < \alpha$. Now unconditional decompositions in L^1 are boundedly complete. If $Z = [Z_i]$ where Z_i is a subspace of Z with the RNP for all i and (Z_i) is a boundedly complete Schauder decomposition of Z, then Z has the RNP. Hence R^1_α has the RNP. Finally, $(\Sigma Y_j)_{\mathrm{Ind},\, \infty}$ is isomorphic to $(\Sigma \oplus Y_j)_{l^1}$; hence again R^∞_α has the RNP and also the Schur property since all of its summands have this property.

Remark: As observed at the end of the next section, R^p_α is actually isometric to a separable dual space for $p = 1$ or ∞. Of course, the results of this section complete the proof of the proposition of the introduction; also we obtain that if B is separable and $R^\infty_\alpha \hookrightarrow B$ for all $\alpha < \omega_1$, $C([0,1]) \hookrightarrow B$.

We conclude Section 2 with a proof that the R_α^p's have unconditional bases for all $1 < p < \infty, \alpha < \omega_1$. (This is false for $p = 1$; see the remark at the end.)

PROPOSITION 2.8. *Let $\omega \leq \alpha < \omega_1$. There exists a sequence $(u_k^\alpha)_{k=1}^\infty$ so that u_k^α is $\{1,0,-1\}$-valued for all k, (u_k^α) is a martingale difference sequence, and the closed linear span of (u_k^α) in L^p equals R_α^p for all $1 \leq p \leq \infty$. Consequently (u_k^α) is an unconditional basis for R_α^p for all $1 < p < \infty$.*

Remarks. 1. A sequence (u_j) is a martingale difference sequence provided $\int_A u_j \, d\mu = 0$ for all measurable sets A depending on $\{u_1, \ldots, u_{j-1}\}$, $j = 2,3,\ldots$.

2. It is a theorem of Burkholder [4] that martingale difference sequences in L^p are unconditional, for $1 < p < \infty$.

3. We do not know the answer to the following questions: Let (u_j) be a $\{1,0,-1\}$-valued martingale difference sequence and $1 < p < \infty$, $p \neq 2$. Is $[u_j]_p$ complemented in L^p? Is $[u_j]_p$ an $\mathcal{L}_p$-space (an $\mathcal{L}_2$-space)?

Proof of Proposition 2.7. We shall, in fact, show the existence of (u_k^α) for all α, finite, of course, when $\alpha < \omega$, with $u_1^\alpha = 1$. So, the result trivially holds for $\alpha = 0$. Suppose the result proved for all $0 \leq \alpha < \beta$. If $\beta = \alpha + 1$, let $d_j = u_{j+1}^\alpha$ for $j = 1,2,\ldots$. Regarding R_α^p as a subspace of $L^p(\Omega, \mathcal{S}, \mu)$, we regard R_β^p as a subspace of $L^p(\Omega \times \{0,1\})$. It is then evident that defining r by

$$r(\omega, \varepsilon) = 1 \quad \text{if} \quad \varepsilon = 0, \quad r(\omega, \varepsilon) = -1 \quad \text{if} \quad \varepsilon = 1,$$

$$d_j^\varepsilon(\omega, \delta) = d_j(\omega) \quad \text{if} \quad \delta = \varepsilon \quad \text{and} \quad d_j^\varepsilon(\omega, \delta) = 0 \quad \textit{if} \quad \delta \neq \varepsilon,$$

$1, r, d_1^0, d_1^1, d_2^0, d_2^1, \ldots$ is a sequence whose closed linear span in L^p equals R_β^p for all $1 \leq p \leq \infty$, and, of course, this sequence is $\{1,0,-1\}$-valued since the original sequence (d_j) is. Let us check that this sequence is indeed a martingale difference sequence (m.d.s.). Evidently $(1, r)$ is an m.d.s. Fix $n \geq 0$ and suppose it has been verified that $1, r, d_1^0, d_1^1, \ldots, d_n^0, d_n^1$ is an m.d.s. Let $\mathcal{Q}_0$ denote the trivial algebra in Ω; for $1 \leq j \leq n$, let $\mathcal{Q}_j$ denote the algebra generated by $d_1, \ldots, d_j$. Suppose S is in the algebra generated by $1, r, \ldots, d_n^0, d_n^1$. Then it is evident that there exist sets $A_i \in \mathcal{Q}_n$ so that $S = A_1 \times \{0\} \cup A_2 \times \{1\}$. Then $\int_S d_{n+1}^0 = 1/2 \int_{A_1} d_{n+1} = 0$ since $1, d_1, \ldots, d_{n+1}$ is an m.d.s. Suppose S is in the algebra of sets generated by $1, r, \ldots, d_n^0, d_n^1, d_{n+1}^0$. Then it is evident that because d_{n+1}^0 vanishes on $\Omega \times \{1\}$, there is a set A in $\mathcal{Q}_n$ with $S \cap (\Omega \times \{1\}) = A \times \{1\}$. Hence $\int_S d_{n+1}^1 = 1/2 \int_A d_{n+1} = 0$.

Now suppose β is a limit ordinal. Let $\gamma_1, \gamma_2, \ldots$ be an enumeration of the ordinals less than β. Assuming that μ is an atomless probability measure, we may choose independent σ-subalgebras of $\mathcal{S}, \mathcal{Q}_1, \mathcal{Q}_2, \ldots$ and for each i, a sequence $(d_{ij})_{j=1}^\infty$ of $\{1,0,-1\}$-valued $\mathcal{Q}_i$-measurable functions so that $1, d_{i1}, d_{i2}, \ldots$ is an m.d.s. with closed linear span in L^p distributionally isomorphic to $R_{\gamma_i}^p$ for all

$1 \leq p \leq \infty$. Then evidently the closed linear span in L^p of $\{d_{ij}: 1 \leq i, j < \infty\} \cup \{1\}$ is distributionally isomorphic to R^p_β for all $1 \leq p \leq \infty$. We need only show that this set is an m.d.s. in a certain order. Of course, all the d_{ij}'s have mean zero; so we need only show that there is a bijection $\tau: N \to N \times N$ so that $(d_{\tau(j)})_{j=1}^\infty$ is an m.d.s.; then also $1, d_{\tau(1)}, d_{\tau(2)}, \ldots$ is an m.d.s. Order $N \times N$ by $(i,k) < (l,m)$ provided $i = l$ and $k < m$. Let τ be a bijection so that τ^{-1} is order preserving; that is, if $\tau(i) < \tau(j)$ then $i < j$. Fix $n \geq 1$ and let $\mathcal{G}$ equal the algebra of sets generated by $d_{\tau(1)}, \ldots, d_{\tau(n)}$. Let $\mathcal{A}$ equal the algebra of sets generated by

$$\{d_{\tau(l)}: \tau(l) < \tau(n+1)\}.$$

Let $\mathcal{B}$ equal the algebra of sets generated by

$$\{d_{\tau(l)}: \tau(l) \not< \tau(n+1) \quad \text{and} \quad 1 \leq l \leq n\}.$$

Then $\mathcal{A}$ and $\mathcal{B}$ are independent, and $\mathcal{G}$ is generated by $\mathcal{A}$ and $\mathcal{B}$. Moreover, letting $\tau(n+1) = (i,j)$ we have, since τ^{-1} is order-preserving, that $\mathcal{A}$ is contained in $\mathcal{H}$, the algebra generated by $\{d_{il}: 1 \leq l < j\}$ and $\mathcal{B}$ is, in fact, independent of $\mathcal{A}_i$. Now to show that $\int_G d_{\tau(n+1)} = 0$ for all $G \in \mathcal{G}$, it suffices to show that $\int_{A \cap B} d_{\tau(n+1)} = 0$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. Fixing such an A and B, $\int_{A \cap B} d_{\tau(n+1)} = \int_A d_{\tau(n+1)} \mu(B)$, by the independence of $\mathcal{A}$ and $\mathcal{B}$. In turn, $\int_A d_{\tau(n+1)} = \int_A d_{ij} = 0$ since $A \in \mathcal{H}$ and $(d_{ik})_{k=1}^\infty$ is an m.d.s. This completes the proof.

Remark. It is proved in [24] that the class of subspaces of L^1 with an unconditional basis has a universal element. Hence there must exist an α so that R^1_α has no unconditional basis. It would be interesting to find the least such α explicitly.

3. Tree subspaces of L^p

The main object of this section is to demonstrate that the R^p_α's of the introduction and Section 2 are all complemented in L^p for $1 < p < \infty$. Let $\mathcal{D}$ be the tree of all finite sequences of 0's and 1's; we obtain from Lemma 3.6 and Lemma 3.9 that R^p_α is isometric to a contractively complemented subspace of $X^p_{\mathcal{D}}$ for all $1 \leq p \leq \infty$ (where X^p_T is as defined in the introduction).

Thus $X^p_{\mathcal{D}}$ is the "natural" universal space for the R^p_α's. The meat of the proof that the R^p_α's are complemented is contained in the demonstration that $X^p_{\mathcal{D}}$ is complemented, Theorem 3.1. The needed inequalities used directly in the proof are given as Scholium 3.4 and Scholium 3.5, after which the proof of Theorem

3.1 is completed. An alternate description of the X_T^p's as translation-invariant subspaces of $L^p(\{0,1\}^N)$ is given at the end of the section.

We recall that $\mathfrak{D}$ denotes the set of all finite sequences of 0's and 1's; D_k denotes all such sequences of length k; thus $\mathfrak{D} = \cup_{k=0}^{\infty} D_k$. We now use the natural ordering on $\mathfrak{D}$; for α, $\beta \in \mathfrak{D}$, $\alpha < \beta$ provided, say $\alpha = (\alpha_1, \dots, \alpha_k)$, $\beta = (\beta_1, \dots, \beta_m)$, then $k < m$ and $\alpha_i = \beta_i$ for all $1 \le i \le k$. A finite branch in $\mathfrak{D}$ is simply the set of predecessors of some element of $\mathfrak{D}$. That is, the finite branch corresponding to $\alpha = (\alpha_1, \dots, \alpha_k)$ is simply the set of all $(\alpha_1, \dots, \alpha_j)$ for $0 \le j \le k$.

An infinite branch is then defined as a subset of $\mathfrak{D}$ order-isomorphic to N in its natural ordering. Of course, an infinite branch corresponds uniquely to an infinite sequence $(\alpha_j)_{j=1}^{\infty}$ of 0's and 1's; the branch then equals the set of all $(\alpha_1, \dots, \alpha_j)$ for all $0 \le j$.

Now our aim is to show that the R_α^p's are complemented in L^p. Of course, it suffices to work with $L^p(\{0,1\}^N)$ rather than $L^p[0,1]$. In fact, it is more convenient to work with $L^p\{0,1\}^{\mathfrak{D}}$. We say that a measurable function f on $\{0,1\}^{\mathfrak{D}}$ depends only on the coordinates $F \subset \mathfrak{D}$ provided $f(x) = f(y)$ for all x, $y \in \{0,1\}^{\mathfrak{D}}$ with $x(\gamma) = y(\gamma)$ for all $\gamma \notin F$. Of course, we say a set $S \subset \{0,1\}^{\infty}$ depends only on F if χ_S does.

We now arrive at a crucial definition.

Let $1 \le p \le \infty$; let $X_{\mathfrak{D}}^p$ denote the closed linear span in $L^p\{0,1\}^{\mathfrak{D}}$ over all finite branches Γ in $\mathfrak{D}$ of all those measurable functions which depend only on the coordinates of Γ.

(It is trivial that one can replace "finite" by "infinite" in this definition, and arrive at the same space.)

THEOREM 3.1. *$X_{\mathfrak{D}}^p$ is complemented in $L^p\{0,1\}^{\mathfrak{D}}$ for all $1 < p < \infty$.*

It is trivial that L^p is isometric to a contractively complemented subspace of $X_{\mathfrak{D}}^p$. Hence in view of the Pełczyński decomposition method, Theorem 3.1 *yields that $X_{\mathfrak{D}}^p$ is isomorphic to L^p, $1 < p < \infty$.*

We require some theorems concerning martingales and conditional expectations. For $\mathcal{Q}$ a σ-subalgebra of the measurable sets on a probability space, $\mathcal{E}_{\mathcal{Q}}$ denotes conditional expectation with respect to $\mathcal{Q}$. Let us now fix a probability space $(\Omega, \mathcal{S}, P)$. The next result is a special case of a result of Burkholder, Davis and Gundy [6].

LEMMA 3.2. *Let $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots$ be σ-subalgebras of $\mathcal{S}$, and let $f_1, f_2, \dots$ be non-negative measurable functions on Ω. Then $\|\Sigma_j \mathcal{E}_j f_j\|_p \le p \|\Sigma f_j\|_p$ for all $1 \le p < \infty$, where $\mathcal{E}_j = \mathcal{E}_{\mathcal{Q}_j}$ for all j.*

We present a simplified version of the proof in [11]. We first need the

SUBLEMMA. *Let $a_1, \ldots, a_n$ be non-negative numbers and $1 \leq p < \infty$. Then*

$$\left(\sum_{j=1}^{n} a_j\right)^p \leq p \sum_{k=1}^{n} \left(\sum_{j=1}^{k} a_j\right)^{p-1} a_k. \tag{3.1}$$

Proof. Let $s_j = \sum_{i=1}^{j} a_i$ with $s_0 = 0$; $0 \leq j \leq n$. Then, of course, $s_0 \leq s_1 \leq s_2 \leq \cdots \leq s_n$. Thus

$$s_n^p = p \int_0^{s_n} t^{p-1}\,dt = p \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} t^{p-1}\,dt \leq p \sum_{j=1}^{n} s_j^{p-1}(s_j - s_{j-1})$$

since t^{p-1} is increasing, proving the sublemma.

To prove 3.2, fix n. We recall that by the definition of conditional expectations, if g and f are non-negative measurable with g $\mathcal{Q}$-measurable, then

$$\int g \mathcal{E}_{\mathcal{Q}} f = \int g f. \tag{3.2}$$

Now fix n. Applying (3.1), we obtain immediately that

$$\left(\sum_{j=1}^{n} \mathcal{E}_j f_j\right)^p \leq p \sum_{k=1}^{n} \left(\sum_{j=1}^{k} \mathcal{E}_j f_j\right)^{p-1} \mathcal{E}_k f_k \quad \text{pointwise.} \tag{3.3}$$

Integrating this inequality and applying (3.2), we get

$$\int \left(\sum_{j=1}^{n} \mathcal{E}_j f_j\right)^p \leq p \sum_{k=1}^{n} \int \left(\sum_{j=1}^{k} \mathcal{E}_j f_j\right)^{p-1} \mathcal{E}_k f_k = p \sum_{k=1}^{n} \int \left(\sum_{j=1}^{k} \mathcal{E}_j f_j\right)^{p-1} f_k \tag{3.4}$$

by (3.2), since with k fixed, the fact that the algebras $\mathcal{Q}_i$ increase implies that $(\sum_{j=1}^{k} \mathcal{E}_j f_j)^{p-1}$ is $\mathcal{Q}_k$-measurable.

$$\begin{aligned} \sum_{k=1}^{n} \int \left(\sum_{j=1}^{k} \mathcal{E}_j f_j\right)^{p-1} f_k &\leq \sum_{k=1}^{n} \int \left(\sum_{j=1}^{n} \mathcal{E}_j f_j\right)^{p-1} f_k = \int \left(\sum_{j=1}^{n} \mathcal{E}_j f_j\right)^{p-1} \sum_{k=1}^{n} f_k \\ &\leq \left(\int \left(\sum \mathcal{E}_j f_j\right)^p\right)^{(p-1)/p} \left(\int \left(\sum f_k\right)^p\right)^{1/p} \end{aligned} \tag{3.5}$$

by Hölder's inequality.

Combining (3.4) and (3.5), we obtain

$$\left(\int \left(\sum \mathcal{E}_j f_j\right)^p\right)^{1/p} \leq p \left(\int \left(\sum f_k\right)^p\right)^{1/p},$$

proving Lemma 3.2.

Let us say that a sequence $(\mathcal{C}_j)$ of σ-subalgebras of $\mathcal{S}$ is *compatible* if for all i and j, $\mathcal{C}_i \subseteq \mathcal{C}_j$ or $\mathcal{C}_j \subseteq \mathcal{C}_i$. It is evident that Lemma 3.2 holds for compatible sequences $(\mathcal{C}_j)$ as well. Indeed, fix n and $f_1, \ldots, f_n$ non-negative measurable. Then the compatibility of the $\mathcal{C}_j$'s implies that there is a permutation σ of $\{1, \ldots, n\}$ with $\mathcal{C}_{\sigma(i)} \subseteq \mathcal{C}_{\sigma(j)}$ for all $1 \le i \le j \le n$. Hence

$$\left\| \sum_j \mathcal{E}_{\mathcal{C}_j} f_j \right\|_p = \left\| \sum_j \mathcal{E}_{\mathcal{C}_{\sigma(j)}} f_{\sigma(j)} \right\|_p \le p \left\| \sum_j f_{\sigma(j)} \right\|_p = p \left\| \sum_j f_j \right\|_p .$$

LEMMA 3.3. *Let m be a positive integer and $\mathcal{Y}_1, \ldots, \mathcal{Y}_m$ be σ-subalgebras of $\mathcal{S}$. Suppose there exist sequences $(\mathcal{B}_i)$, $(\mathcal{Z}_i)$, and $(\mathcal{W}_i)$ of σ-subalgebras with the following properties for all i, $1 \le i \le m$.*

(a) *Each sequence $(\mathcal{B}_i)$, $(\mathcal{Z}_i)$, and $(\mathcal{W}_i)$ is compatible.*

(b) *$\mathcal{E}_{\mathcal{B}_i}$, $\mathcal{E}_{\mathcal{Z}_i}$ and $\mathcal{E}_{\mathcal{W}_i}$ commute.*

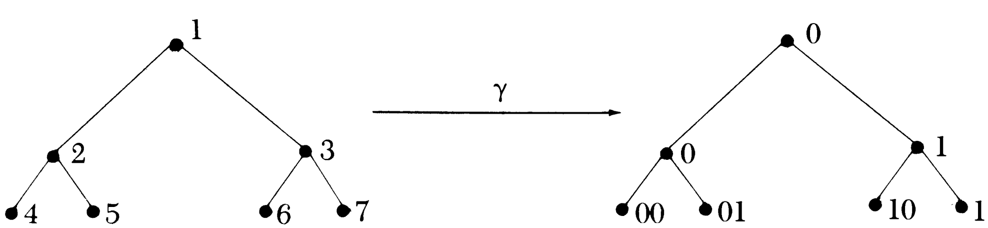
(c) $\mathcal{Y}_i = \mathcal{B}_i \cap \mathcal{Z}_i \cap \mathcal{W}_i$.

Then $\|\Sigma \mathcal{E}_{\mathcal{Y}_j} f_j\|_p \le p^3 \|\Sigma f_j\|_p$ for all non-negative measurable functions $f_1, \ldots, f_m$, $1 < p < \infty$.

Proof. The assumption (b) implies that $\mathcal{E}_{\mathcal{B}_i}\mathcal{E}_{\mathcal{Z}_i}\mathcal{E}_{\mathcal{W}_i} = \mathcal{E}_{\mathcal{B}_i \cap \mathcal{Z}_i \cap \mathcal{W}_i} = \mathcal{E}_{\mathcal{Y}_i}$ by (c). Lemma 3.3 then follows by our preceding remarks, i.e. applying Lemma 3.2 three times.

Remark. Of course, the analogous result holds for algebras equal to the intersection of a finite number of "commuting" compatible algebras; we only have need of the case of three such intersections. However, it seems natural to pose the following question: Let $\mathcal{C}_1, \mathcal{C}_2, \ldots$ be σ-subalgebras of $\mathcal{S}$. Under what (combinatorial) conditions on the $\mathcal{C}_j$'s, is it true that there exists a constant C_p so that $\|\Sigma \mathcal{E}_{\mathcal{C}_j} f_j\|_p \le C_p \|\Sigma f_j\|_p$ for all non-negative measurable functions $f_1, f_2, \ldots, 1 \le p < \infty$?

We now require an explicit order-preserving enumeration γ of $\mathcal{D}$. The enumeration between $\{1, \ldots, 7\}$ and $\cup_{j=0}^2 D_j$ is as follows:



In general, let n be a positive integer, and let $0 \le k$ and $t_1, \ldots, t_k$ be the unique choice of 0's and 1's so that

$$n = 2^k + \sum_{i=1}^{k} t_i 2^{k-i}.$$

(Thus the representation of n in dyadic notation is $1t_1 \cdots t_k$.) Let $\gamma(n) = (t_1, \ldots, t_k)$. Then γ: $N \to \mathcal{D}$ is a bijection and γ^{-1} is order-preserving; that is, if $\gamma(i) < \gamma(j)$, then $i < j$. Now for each j, let $\mathcal{Y}_j$ denote the family of all measurable subsets of $\{0,1\}^{\mathcal{D}}$ depending only on the coordinates $\{u \in \mathcal{D}: u \leq \gamma(j)\}$. (That is, $\mathcal{Y}_j$ is the "branch" algebra of sets determined by $\gamma(j)$.) We have arrived at a crucial step in the proof of Theorem 3.1.

SCHOLIUM 3.4. $\|\Sigma \mathcal{E}_{\mathcal{Y}_j} f_j\|_p \leq p^3 \|\Sigma f_j\|_p$ *for all non-negative measurable functions* $f_1, f_2, \ldots, 1 \leq p < \infty$.

Proof. Fix k and let $m = 2^{k+1} - 1$. We shall show that the hypotheses of Lemma 3.3 are valid. For F a subset of $\mathcal{D}$, let $\mathcal{A}(F)$ denote the σ-algebra of measurable sets depending only on the coordinates F. It is evident that if A and B are subsets of $\mathcal{D}$, then $\mathcal{E}_{\mathcal{A}(A)}$ and $\mathcal{E}_{\mathcal{A}(B)}$ commute. Of course, $\mathcal{A}(A) \cap \mathcal{A}(B) = \mathcal{A}(A \cap B)$. If $Y_j = \{u \in \mathcal{D}: u \leq \gamma(j)\}$, then, of course, $\mathcal{Y}_j = \mathcal{A}(Y_j)$.

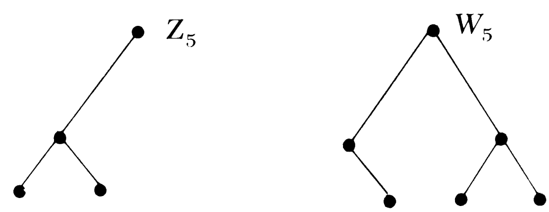
First fix n with $2^k \leq n < 2^{k+1}$ (thus $\gamma(n)$ is maximal in the partially ordered set $T_k = \{\gamma(i): i \leq m\}$.

Let

$$Z_n = \{u \in \mathcal{D}: u \leq \gamma(i) \quad \text{for some } i \text{ with } 2^k \leq i \leq n\}.$$

Let $$W_n = \{u \in \mathcal{D}: u \leq \gamma(i) \quad \text{for some } i \text{ with } n \leq i < 2^{k+1}\}.$$

For example, here is a picture, for $k = 2$, of Z_5 and W_5:



Then evidently the Z_n's increase, the W_n's decrease, and

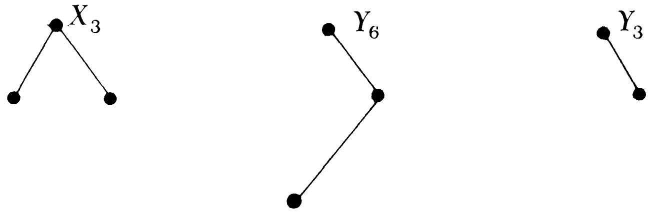
$$Y_n = Z_n \cap W_n \quad \textit{for all such} \quad n. \tag{3.6}$$

Now for each $1 \leq j \leq m$, let $X_j = \{\gamma(i): 1 \leq i \leq j\}$ and let $\mathcal{B}_j = \mathcal{A}(X_j)$. Finally, fix $j, 1 \leq j \leq m$, and let $n(j)$ be such that $\gamma(j) \leq \gamma(n(j))$ and $2^k \leq n(j) < 2^{k+1}$. Thus $\gamma(n(j))$ is a maximal element of our partially ordered set T_k containing $\gamma(j)$.

Then evidently

(3.7) $$Y_j = Y_{n(j)} \cap X_j.$$

We illustrate for the case $k = 2$, $j = 3$ and $n(j) = 6$.



We thus have by (3.6) and (3.7) that $Y_j = X_j \cap Z_{n(j)} \cap W_{n(j)}$.

We now simply set

$$\mathcal{B}_j = \mathcal{A}(X_j),$$

$$\mathcal{Z}_j = \mathcal{A}(Z_{n(j)}) \quad \text{and}$$

$$\mathcal{W}_j = \mathcal{A}(W_{n(j)}), \qquad 1 \le j \le m.$$

Thus the hypotheses of 3.3 are satisfied, so Scholium 3.4 is proved.

We finally need the following crucial martingale theorem of Burkholder [4].

SCHOLIUM 3.5. *Let $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$ be σ-algebras, $1 < p < \infty$, and let $b_1, \ldots, b_n, \ldots$ be functions in L^p so that for all j, b_j is $\mathcal{B}_j$-measurable with $\mathcal{E}_{\mathcal{B}_{j-1}} b_j = 0$ if $j > 1$. There exists a constant K_p depending only on p so that*

(3.7) $$K_p^{-1} \left\| \left(\sum b_j^2 \right)^{1/2} \right\|_p \le \left\| \sum b_j \right\|_p \le K_p \left\| \left(\sum b_j^2 \right)^{1/2} \right\|_p.$$

We are now prepared for the

Proof of Theorem 3.1. As in the proof of Scholium 3.4, we let $\mathcal{B}_j$ denote the algebra of measurable sets in $\{0,1\}^{\mathcal{D}}$ depending only on the coordinates $X_j = \{u \in \mathcal{D}: u \le \gamma(j)\}$. Also let $\mathcal{B}_0$ denote the trivial algebra.

For each $j \ge 0$, we let B_j denote the set of all functions f which are $\mathcal{B}_j$-measurable and $\mathcal{E}_{\mathcal{B}_{j-1}} f = 0$ if $j \leqslant 1$. We let $Y_0 = B_0$ (= the set of constant functions) and, for $1 \le j$, $Y_j = \{f \in B_j: f \text{ is } \mathcal{Y}_j\text{-measurable}\}$. Incidentally, (3.7) yields that $(B_j)_{j=0}^{\infty}$ is an unconditional Schauder decomposition of $L^p(\{0,1\}^{\mathcal{D}})$; in reality, it is simply the standard "dyadic martingale" decomposition of L^p. We next verify that $X^p_{\mathcal{D}}$ equals $[Y_j]$, the closed linear span of the Y_j's in L^p. It is trivial that $Y_j \subset X^p_{\mathcal{D}}$ for all j, hence $[Y_j] \subset X^p_{\mathcal{D}}$. For the reverse inclusion, suppose $n \in N$

is given and let x be $\mathcal{Y}_n$-measurable. Then

$$x = \left(\int x\right) + \sum_{j=1}^{n} (\mathcal{E}_{\mathcal{B}_j} - \mathcal{E}_{\mathcal{B}_{j-1}})x,$$

and, of course, with $1 \le j \le n, (\mathcal{E}_{\mathcal{B}_j} - \mathcal{E}_{\mathcal{B}_{j-1}})x \in B_j$. We need only verify that $(\mathcal{E}_{\mathcal{B}_j} - \mathcal{E}_{\mathcal{B}_{j-1}})x$ is $\mathcal{Y}_j$-measurable for all j with $1 \le j \le n$.

If $\gamma(j)$ and $\gamma(n)$ are not comparable with respect to the ordering of $\mathcal{D}$, then $\mathcal{B}_j \cap \mathcal{Y}_n = \mathcal{B}_{j-1} \cap \mathcal{Y}_n$. Hence since x is $\mathcal{Y}_n$-measurable, $(\mathcal{E}_{\mathcal{B}_j} - \mathcal{E}_{\mathcal{B}_{j-1}})x = 0$. If $\gamma(j)$ and $\gamma(n)$ are comparable, then $\gamma(j) \le \gamma(n)$ since γ^{-1} is order-preserving. Then $\mathcal{B}_j \cap \mathcal{Y}_n = \mathcal{Y}_j$. Indeed, if $i \le j$ and $\gamma(i) \le \gamma(n)$, then $\gamma(i)$ and $\gamma(j)$ must be comparable, whence $\gamma(i) \le \gamma(j)$ since γ^{-1} is order-preserving. Hence $\mathcal{B}_{j-1} \cap \mathcal{Y}_n \subset \mathcal{Y}_j$, so $(\mathcal{E}_{\mathcal{B}_j} - \mathcal{E}_{\mathcal{B}_{j-1}})x = (\mathcal{E}_{\mathcal{B}_j} - \mathcal{E}_{\mathcal{B}_{j-1}})\mathcal{E}_{\mathcal{Y}_n}x = (\mathcal{E}_{\mathcal{B}_j \cap \mathcal{Y}_n} - \mathcal{E}_{\mathcal{B}_{j-1} \cap \mathcal{Y}_n})x$, which is $\mathcal{Y}_j$-measurable. Thus $X^p_{\mathcal{D}} = [Y_j]$.

We shall prove that orthogonal projection P onto $X^2_{\mathcal{D}}$ yields a bounded linear projection onto $X^p_{\mathcal{D}}$ for all $1 < p < \infty$. Since P is "self-adjoint", it suffices to consider the case $p > 2$. Let $b \in L^p(\{0,1\}^{\mathcal{D}})$. There exists a unique sequence (b_j) with $b_j \in B_j$ for all j so that $b = \sum_{j=0}^{\infty} b_j$. Then

$$Pb = \sum_{j=0}^{\infty} \mathcal{E}_{\mathcal{Y}_j} b_j, \tag{3.8}$$

the series converging in L^2-norm. (We note that with j fixed, $\mathcal{E}_{\mathcal{B}_{j-1}}\mathcal{E}_{\mathcal{Y}_j} b_j = \mathcal{E}_{\mathcal{Y}_j}\mathcal{E}_{\mathcal{B}_{j-1}} b_j = 0$; hence $\mathcal{E}_{\mathcal{Y}_j} b_j$ indeed belongs to Y_j; that is, $\mathcal{E}_{\mathcal{Y}_j} | B_j$ is the orthogonal projection of B_j onto Y_j.)

With n fixed,

$$\left\| \sum_{j=0}^{n} \mathcal{E}_{\mathcal{Y}_j} b_j \right\|_p \le K_p \left\| \left(\sum_{j=0}^{n} (\mathcal{E}_{\mathcal{Y}_j} b_j)^2 \right)^{1/2} \right\|_p \quad \text{by (3.7)}$$

$$\le K_p \left\| \left(\sum_{j=0}^{n} \mathcal{E}_{\mathcal{Y}_j} b_j^2 \right)^{1/2} \right\|_p \le K_p \left(\frac{p}{2} \right)^{3/2} \left\| \left(\sum_{j=0}^{n} b_j^2 \right)^{1/2} \right\|_p$$

by Scholium 3.4 applied to $\frac{p}{2}$

$$\le K_p^2 \left(\frac{p}{2} \right)^{3/2} \left\| \sum_{j=0}^{n} b_j \right\|_p \quad \text{by (3.7)}.$$

Hence $\| P \| \le K_p^2 \left(\frac{p}{2} \right)^{3/2}$. This completes the proof of Theorem 3.1.

Remarks. 1. We are applying Scholium 3.4 to the sequence (b_j^2) and b_j^2 is, of course, $\mathcal{B}_j$-measurable. The proof of 3.4 then yields the sharper estimate $\| P \| \le pK_p^2/2$; only two intersections need be taken.

2. It is possible to deduce Theorem 3.1 by using an earlier result due to E. Stein; namely, *if* $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$ *are increasing σ-algebras and* $f_1, f_2, \ldots$ *are arbitrary measurable functions, then* $\|(\Sigma|\mathcal{E}_{\mathcal{Q}_j} f_j|^2)^{1/2}\|_p \leq A_p \|(\Sigma|f_j|^2)^{1/2}\|_p$ *for all* $1 < p < \infty$, *where* A_p *depends only on* p. (See Theorem 8, page 108 of [25].) The proof of Scholium 3.4 then yields that

$$\left\|\left(\sum|\mathcal{E}_{\mathcal{Y}_j} f_j|^2\right)^{1/2}\right\|_p \leq A_p^3 \left\|\left(\sum|f_j|^2\right)^{1/2}\right\|_p$$

for all measurable functions $f_1, f_2, \ldots, 1 < p < \infty$. This allows one to prove 3.1 for all $1 < p < \infty$ directly, without passing to the $p < 2$-case by duality. The above remark about estimates, however, remains exactly the same, since, in fact, A_p has order of magnitude $p^{1/2}$ as $p \to \infty$.

3. Our proof shows that $X^p_{\mathfrak{D}}$ has the following structure: there exist sub-algebras $\mathcal{Q}_j$ of $\mathcal{B}_j$ so that $\mathcal{E}_{\mathcal{Q}_j}$ and $\mathcal{E}_{\mathcal{B}_{j-1}}$ commute for all $j \geq 1$ and B equals the closed linear span in L^p of $\{f\colon f$ is $\mathcal{Q}_j$-measurable and $f \in B_j$, $j = 0, 1, 2, \ldots\}$ where $B = X^p_{\mathfrak{D}}$. Now given any such B, if the $\mathcal{Q}_j$'s satisfy the conclusion of Lemma 3.3, i.e. the inequality of the remark following 3.3, then B is indeed complemented in L^p, for all $1 < p < \infty$.

4. For $f \in L^1$, let $f = \Sigma_{j=0}^{\infty} b_j$ with $b_j \in B_j$ for all j and set $\|f\|_{H^1} = \|(\Sigma b_j^2)^{1/2}\|_1$; $H_1 = \{f \in L^1\colon \|f\|_{H_1} < \infty\}$. Let $X^{H^1}_{\mathfrak{D}}$ denote the closed linear span in H^1 of the functions depending on the coordinates of some finite branch in $\mathfrak{D}$. The second-named author has shown that the above orthonormal projection is *unbounded* from H^1 onto $X^{H^1}_{\mathfrak{D}}$ (in fact, $P\colon H^1 \to L^1$ is unbounded). This suggests that $X^{H^1}_{\mathfrak{D}}$ is uncomplemented in H^1; perhaps it is true that $X^{H^1}_{\mathfrak{D}}$ is not isomorphic to a complemented subspace of H^1.

Now let T be a subset of $\mathfrak{D}$. A subset Γ of T is called a *branch* of T if it contains the predecessors in T of all its elements; i.e. $\gamma \in \Gamma$, $\alpha \in T$ and $\alpha < \gamma \Rightarrow \alpha \in \Gamma$, where "$<$" is the natural order on $\mathfrak{D}$. *We define* X^p_T *as the closed linear span in* $L^p\{0,1\}^T$ *over all branches* Γ *of functions depending only on the coordinates of* Γ. We may and shall regard $L^p(\{0,1\}^T)$ as equal to the subspace of $L^p\{0,1\}^{\mathfrak{D}}$ consisting of those measurable functions f depending only on T.

LEMMA 3.6. *X^p_T is a contractively complemented subspace of $X^p_{\mathfrak{D}}$ for all* $1 \leq p < \infty$.

Proof. Let $P = \mathcal{E}_{\mathcal{Q}(T)}$; i.e. P is conditional expectation with respect to the algebra of measurable sets depending only on the coordinates T. Now every finite branch of T is contained in a finite branch of $\mathfrak{D}$. Indeed, let Γ be a finite non-empty branch of T and let m be its largest element; i.e. $m \in \Gamma$ and $\gamma \leq m$ for all $\gamma \in \Gamma$. Now let $\Lambda = \{d \in \mathfrak{D}\colon d \leq m\}$. Hence $\Lambda \supset \Gamma$. Then if f depends only on Γ, f depends only on Λ; this proves $X^p_T \subset X^p_{\mathfrak{D}}$; evidently $P|X^p_T = I|X^p_T$. On the other hand, let Λ be a finite branch of $\mathcal{D}$. Then $\Lambda \cap T$ is a branch of T. But if f depends only on Λ, Pf depends only on $\Lambda \cap T$, so $Pf \in X_T^p$. This proves $PX_{\mathcal{D}}^p = X_T^p$. Since $\|P\| = 1$, the lemma is proved.

Now the subsets of $\mathcal{D}$ in their inherited order may be described in the following abstract way: *A partially ordered set* $(T, <)$ *shall be called a tree provided it satisfies the following properties*:

(a) *The set of predecessors of an element of T is finite and linearly ordered*,

(b) *T is countable*.

(The more general definition used by logicians: (b) is not required and (a) is replaced by: the set of predecessors of an element is well-ordered. Thus, we are really just dealing with "countable trees of finite-ranked elements".)

Of course, $\mathcal{D}$ is a tree. So is $\mathcal{F}(N)$, the set of all finite sequences of positive integers, under the order $(t_1, \ldots, t_k) < (u_1, \ldots, u_m)$ if $k < m$ and $t_i = u_i$ for all $1 \leq i \leq k$. Any subset of a tree is also a tree in its inherited order. Given a tree T, we again say that $\Gamma \subset T$ is a branch if Γ contains the set of predecessors (in T) of all its elements. *A tree T is said to be well-founded if it has no infinite branches*. (Of course, a well-founded tree is a special case of a well-founded relation discussed in Section 2.) Given a tree T, we define X_T^p in exactly the same way we did preceding Lemma 3.6. Evidently X_T^p is isometrically and distributionally determined by the order type of T. Now it is a standard rather simple result in logic that $\mathcal{F}(N)$ is order isomorphic to a subset of $\mathcal{D}$ and every tree T is order isomorphic to a subset of $\mathcal{F}(N)$. That is, we have

LEMMA 3.7. *Every tree is order isomorphic to a subset of* $\mathcal{D}$.

THEOREM 3.8. *For every tree T and p with* $1 < p < \infty$, X_T^p *is complemented in* $L^p\{0,1\}^T$.

Proof. By the preceding result, we may assume that $T \subset \mathcal{D}$; we regard X_T^p as a subspace of $X_{\mathcal{D}}^p$ as in Lemma 3.6, and also $L^p\{0,1\}^T$ as a subspace of $L^p\{0,1\}^{\mathcal{D}}$. Then $X_{\mathcal{D}}^p$ is complemented in $L^p\{0,1\}^{\mathcal{D}}$ by Theorem 3.1. Thus the result follows immediately from 3.1 and Lemma 3.6.

Remark. It is possible to give a direct proof of 3.8, without passing through the dyadic tree $\mathcal{D}$. In particular, if we let $\gamma_1, \gamma_2, \ldots$ be the distinct finite branches of T and for each j, let U_j be the conditional expectation operator with respect to the algebra of sets depending only on the coordinates of γ_j (in $\{0,1\}^T$ of course), then we obtain again

$$\left\| \sum U_j f_j \right\|_p \leq p^3 \left\| \sum f_j \right\|_p$$

for all non-negative f_i's, $1 \leq p < \infty$, and

$$\left\| \left(\sum (U_i f_i)^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum (f_i)^2 \right)^{1/2} \right\|_p$$

for all measurable f_i's, $1 < p < \infty$, where C_p depends only on p, by using the result of Burkholder, Davis and Gundy [6] for the first inequality and that of Stein [25] for the second. The "purist" might thus prefer to cast this entire discussion in the language of "tree-martingales", that is, of martingales indexed by a partially ordered set.

We now complete the proof of Theorem B, part (3) of the introduction.

LEMMA 3.9. *Let $\alpha < \omega_1$. There exists a well-founded tree T_α so that R^p_α is distributionally isomorphic to $X^p_{T_\alpha}$, $1 \leq p < \infty$.*

Remark. It follows immediately that R^p_α is complemented in L^p for all $1 < p < \infty$ and $\alpha < \omega_1$.

Proof of 3.9. We establish the statement by induction on α. It trivially holds for $\alpha = 0$. Suppose $0 < \alpha$ and the result has been established for all $\gamma < \alpha$. If α is a successor ordinal, let γ be such that $\gamma + 1 = \alpha$. We may, of course, assume that $R^p_\gamma = X^p_{T_\gamma}$. Let $t \notin T_\gamma$, set $T_\alpha = T_\gamma \cup \{t\}$, and order T_α by $\prec$ defined as follows:

$$t \prec u \text{ for all } u \in T_\gamma;\text{ if } u, v \in T_\gamma,$$

then $u \prec v$ if and only if $u < v$, where "$<$" is the order on T_γ. (Thus t is simply a "top" node introduced above all of T_γ, where $u < v$ is visualized by "v is below u".) It is then trivial that T_α is also a well-founded tree. We must show that $R^p_\alpha = X^p_{T_\alpha}$. By definition, $R^p_\alpha = (R^p_\gamma \oplus R^p_\gamma)_p$. Define $e_i \in L^p\{0,1\}^{\{t\}}$ by

$$e_1(\varepsilon(t)) = \varepsilon(t), \quad e_2 = 1 - e_1, \text{ for } \varepsilon \in \{0,1\}^{\{t\}}.$$

Then

$$R^p_\alpha = \{b_1 \otimes e_1 + b_2 \otimes e_2 \colon b_i \in R^p_\gamma \text{ for } i = 1,2\} \tag{3.9}$$

where, of course, $(b_i \otimes e_i)(s, \varepsilon) = b_i(s)e_i(\varepsilon)$ for $s \in \{0,1\}^{T_\gamma}$, $\varepsilon \in \{0,1\}^{\{t\}}$, $i = 1,2$.

Now let f be a function on $\{0,1\}^{T_\alpha}$ which depends only on the coordinates of Γ, a branch of T_α. Since T_γ is non-empty, we can assume $\Gamma \cap T_\gamma = \Gamma \sim \{t\}$ is non-empty (by enlarging Γ if necessary). Then, of course, $\Gamma \cap T_\gamma$ is a branch of T_γ. We may then regard f as a function of two variables, s and ε, for $s \in \{0,1\}^{T\alpha}$ and $\varepsilon \in \{0,1\}^{\{t\}}$. Set

$$b_1(s) = f(s,0) \text{ and } b_2(s) = f(s,1) \text{ for all } s \in \Gamma_\gamma.$$

Then evidently b_i depends only on $\Gamma \cap \Gamma_\gamma$; hence $b_i \in X^p_{T_\gamma}$ for $i = 1,2$ and $f = b_1 \otimes e_1 + b_2 \otimes e_2$. Thus by (3.9), we have shown $X^p_{T_\alpha} \subset R^p_\alpha$. On the other hand, if Λ is a branch of T_γ and b depends only on Λ, $b \otimes e_i$ depends only on $\Lambda \cup \{t\}$, a branch of T_α; hence $b \otimes e_i \in X^p_{T_\alpha}$ for $i = 1,2$. Thus in view of (3.9), $X^p_{T_\alpha} = R^p_\alpha$.

Now suppose that α is a limit ordinal. We may choose trees T_γ such that $R^p_\gamma = X^p_{T_\gamma}$ for all $\gamma < \alpha$; *without loss of generality, we may assume that* $T_\gamma \cap T_{\gamma'} = \varnothing$ *for all* $\gamma \neq \gamma'$. *We then set* $T_\alpha = \bigcup_{\gamma<\alpha} T_\gamma$. Letting "$<_\gamma$" be the order relation on T_γ, we simply set $<_\alpha = \bigcup_{\gamma<\alpha} <_\gamma$. That is, for $u, v \in T_\alpha$, $u <_\alpha v$ if and only if $u, v \in T_\gamma$ for some γ and $u <_\gamma v$. (T_α may be visualized as simply setting the trees T_γ "side-by-side".) It is evident that T_α is well-founded since any branch of T_α must be contained in T_γ for some $\gamma < \alpha$. It is also clear that

$$X^p_{T_\alpha} = \Big(\sum_{\gamma<\alpha} X^p_{T_\gamma} \Big)_{\mathrm{Ind},\, p}. \tag{3.10}$$

Indeed, suppose f depends only on Γ, Γ a branch of T_α. Then as remarked above, $\Gamma \subset T_\gamma$ for some $\gamma < \alpha$; thus $f \in X^p_\gamma$. On the other hand, if Γ is a branch of T_γ for some $\gamma < \alpha$, then Γ is already a branch of T_α. Thus $X^p_{T_\alpha} = [X^p_{T_\gamma}]_{\gamma<\alpha}$. But the disjointness of the T_γ's implies $[X^p_{T_\gamma}]_{\gamma<\alpha} = [\Sigma_{\gamma<\alpha} X^p_{T_\gamma}]_{\mathrm{Ind},\, p}$. Thus (3.10) holds and the proof is complete, since

$$R^p_\alpha = \Big(\sum_{\gamma<\alpha} R^p_\gamma \Big)_{\mathrm{Ind},\, p}$$

by definition.

Remarks and open problems. 1. Let T be a tree. *Then there exists a subset* W_T *of the Walsh functions so that* $X^p_T = [w]_{w\in W_T}$ *in* L^p *for all* $1 \le p \le \infty$. That is, X^p_T is a closed translation invariant subspace of $L^p(G)$ where $G = \{0,1\}^N$. Let us see why this is so. Let β: $N \to T$ be a bijection; we then set

$$W_T = \{r_{n_1} \cdot \ldots \cdot r_{n_k} \colon k \ge 1, \beta(n_1) < \cdots < \beta(n_k)\} \cup \{1\}.$$

Here is an alternate description: For each $t \in T$, let $r_t \in L^p\{0,1\}^T$ be defined by $r_t(x) = (-1)^{x(t)}$. Then W_T equals the union over all branches Γ of the set of all finite products of Rademacher functions belonging to Γ; i.e.

$$W_T = \Big\{ w\colon \text{there exist } \Gamma \text{ a branch and } k, \text{ with } w = \prod_{i=1}^{k} r_{t_i} \text{ for } t_1, \ldots, t_k \in \Gamma \Big\}.$$

Now if Γ is a finite branch, then by standard properties of the Walsh functions, the span of the set of all products of Rademacher functions belonging to Γ equals $L^p\{0,1\}^\Gamma$; hence we obtain $[W_T] = X^p_T$. In particular, R^p_α may thus be regarded as a closed translation invariant subspace of $L^p(G)$ for all α. By a result of F. Lust [18], if a translation-invariant subspace of $L^p(G)$ has the RNP, it is isometric to a dual space for $p = 1$ or ∞. Thus, R^p_α is isometric to a dual space for all α, $p = 1$ or ∞. Consequently the proposition of the introduction may be strengthened as follows: *Let $\mathcal{C}$ denote the class of all subspaces of L^1 that are isometric to a dual space and let B be separable and universal for $\mathcal{C}$. Then $L^1 \hookrightarrow B$.*

2. The following question was suggested by A. Pełczyński: Let Γ be an infinite compact abelian group and $1 < p < \infty$, $p \neq 2$. Are there uncountably many non-isomorphic complemented translation-invariant subspaces of $L^p(\Gamma)$? What if $\Gamma = \Pi$, the circle group?

3. Let T be a well-founded tree. Is there an α so that X^p_T is isomorphic to R^p_α for all $1 < p < \infty$, $p \neq 2$?

4. Let B be an $\mathcal{L}_p$ space non-isomorphic to L^p, $1 < p < \infty$. Is there an α so that B embeds in R^p_α?

5. Are the R^p_α's isomorphically distinct over the family of limit ordinals α? Is it so that setting $\tau(\alpha) = \omega\alpha$, then $R^p_{\tau(\alpha+1)} \hookrightarrow R^p_{\tau(\alpha)}$ for all α? What is the explicit value of $h_p(R^p_\alpha)$ for all α? For $\alpha = \omega$?

6. Does there exist an α such that R^p_α contains uncountably many non-isomorphic $\mathcal{L}_p$-spaces, $1 < p < \infty$, $p \neq 2$? Of course, our results yield that there exists an α and a λ_p such that R^p_α contains infinitely many non-isomorphic $\mathcal{L}_{p,\lambda_p}$-spaces.

7. Let W be the class of all separable $\mathcal{L}_1$-spaces B so that $L^1 \hookrightarrow B$ and let X be separable and universal for W. Does $L^1 \hookrightarrow X$?*

8. Let $1 \leq p < \infty$, $p \neq 2$, X, Y be Banach spaces, and suppose $L^p \hookrightarrow X \oplus Y$. Does $L^p \hookrightarrow X$ or $L^p \hookrightarrow Y$? This problem was posed in [21]; a possible approach to the problem: is there a function f_p: $\omega_1 \times \omega_1 \to \omega_1$ so that $h_p(X \oplus Y) \leq f_p(h_p(X), h_p(Y))$ provided $L^p \not\hookrightarrow X$ and $L^p \not\hookrightarrow Y$? Can f_p be chosen to be addition? Although the basic problem stated has an affirmative answer for $p = 2$ or $p = \infty$, we do not know if such a function f_p exists for $p = 2$ or for $p = \infty$ (where one replaces "L^p" by "$C([0,1])$").

FIRST AUTHOR:
UNIVERSITY OF PARIS VI, FRANCE
FREE UNIVERSITY OF BRUSSELS, BELGIUM

SECOND AUTHOR:
UNIVERSITY OF PARIS VI, FRANCE
UNIVERSITY OF TEXAS AT AUSTIN

THIRD AUTHOR:
ECOLE POLYTECHNIQUE, PARIS, FRANCE
OHIO STATE UNIVERSITY AT COLUMBUS

*The first author has answered this in the affirmative; see "A new class of L-spaces" (to appear).

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(Received September 21, 1979)