On certain classes of Baire-1 functions
with applications to Banach space theory

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Abstract

Certain subclasses of $B_1(K)$, the Baire-1 functions on a compact metric space $K$, are defined and characterized. Some applications to Banach spaces are given.

0. Introduction.

Let $X$ be a separable infinite dimensional Banach space and let $K$ denote its dual ball, $Ba(X^*)$, with the weak$^*$ topology. $K$ is compact metric and $X$ may be naturally identified with a closed subspace of $C(K)$. $X^{**}$ may also be identified with a closed subspace of $A_\infty(K)$, the Banach space of bounded affine functions on $K$ in the sup norm. Our general objective is to deduce information about the isomorphic structure of $X$ or its subspaces from the topological nature of the functions $F \in X^{**} \subseteq A_\infty(K)$. A classical example of this type of result is: $X$ is reflexive if and only if $X^{**} \subseteq C(K)$.

A second example is the following theorem. ($B_1(K)$ is the class of bounded Baire-1 functions on $K$ and $DBSC(K)$ is the subclass of differences of bounded semicontinuous functions on $K$. The precise definitions appear below in §1.) We write $Y \hookrightarrow X$ if $Y$ is isomorphic to a subspace of $X$.

Theorem A. Let $X$ be a separable Banach space and let $K = Ba(X^*)$ with the weak$^*$ topology.

a) $[35] \ell_1 \hookrightarrow X$ iff $X^{**} \setminus B_1(K) \neq \emptyset$.

b) $[7] c_0 \hookrightarrow X$ iff $[X^{**} \cap DBSC(K)] \setminus C(K) \neq \emptyset$.

Theorem A provides the motivation for this paper: What can be said about $X$ if $X^{**} \cap [B_1(K) \setminus DBSC(K)] \neq \emptyset$? To study this problem we consider various subclasses of $X^{**}$...
$B_1(K)$ for an arbitrary compact metric space $K$. J. Bourgain has also used this approach and some of our results and techniques overlap with those of [8,9,10]. In a different direction, generalizations of $B_1(K)$ to spaces where $K$ is not compact metric with ensuing applications to Banach space theory have been developed in [22].

In §1 we consider two subclasses of $B_1(K)$ denoted $B_{1/4}(K)$ and $B_{1/2}(K)$ satisfying

\[(0.1) \quad C(K) \subseteq DBSC(K) \subseteq B_{1/4}(K) \subseteq B_{1/2}(K) \subseteq B_1(K).\]

Our interest in these classes stems from Theorem B (which we prove in §3).

**Theorem B.** Let $K$ be a compact metric space and let $(f_n)$ be a uniformly bounded sequence in $C(K)$ which converges pointwise to $F \in B_1(K)$.

a) If $F \notin B_{1/2}(K)$, then $(f_n)$ has a subsequence whose spreading model is equivalent to the unit vector basis of $\ell_1$.

b) If $F \in B_{1/4}(K) \setminus C(K)$, there exists $(g_n)$, a convex block subsequence of $(f_n)$, whose spreading model is equivalent to the summing basis for $c_0$.

Theorem B may be regarded as a local version of Theorem A (see Corollary 3.10). In fact the proof is really a localization of the proof of Theorem A. In Theorem 3.7 we show that the converse to a) holds and thus we obtain a characterization of $B_1(K) \setminus B_{1/2}(K)$ in terms of $\ell_1$ spreading models. We do not know if the condition in b) characterizes $B_{1/4}(K)$ (see Problem 8.1).

Given that our main objective is to deduce information about the subspaces of $X$ from the nature of $F \in X^{**} \cap B_1(K)$, it is useful to introduce the following definition.

Let $\mathcal{C}$ be a class of separable infinite-dimensional Banach spaces and let $F \in B_1(K)$. $F$ is said to govern $\mathcal{C}$ if whenever $(f_n) \subseteq C(K)$ is a uniformly bounded sequence converging pointwise to $F$, then there exists a $Y \in \mathcal{C}$ which embeds into $[(f_n)]$, the closed linear span of $(f_n)$. We also say that $F$ strictly governs $\mathcal{C}$ if whenever $(f_n) \subseteq C(K)$ is a uniformly bounded sequence converging pointwise to $F$, there exists a convex block subsequence $(g_n)$ of $(f_n)$ and a $Y \in \mathcal{C}$ with $[(g_n)]$ isomorphic to $Y$.

Theorem A (b) can be more precisely formulated as: if $F \in DBSC(K) \setminus C(K)$, then $F$ governs $\{c_0\}$. (In fact Corollary 3.5 below yields that $F \in B_1(K) \setminus C(K)$ strictly
governs \( \{ c_0 \} \) if and only if \( F \in DBSC(K) \). In §4 we prove that the same result holds if \( F \in DSC(K) \setminus C(K) \). (A more general result, with a different proof, has been obtained by Elton [13].) We also note in §4 that there are functions that govern \( \{ c_0 \} \) but are not in \( DSC(K) \).

In §6 we give a characterization of \( B_{1/4}(K) \) (Theorem 6.1) and use it to give an example of an \( F \in B_{1/4}(K) \setminus C(K) \) which does not govern \( \{ c_0 \} \). Thus Theorem B (b) is best possible.

In §7 we note that there exists a \( K \) and an \( F \in B_{1/2}(K) \) which governs \( \{ \ell_1 \} \). We also give an example of an \( F \in B_{1/2}(K) \) which governs \( C = \{ X : X \text{ is separable and } X^* \text{ is nonseparable} \} \) but does not govern \( \{ \ell_1 \} \).

§1 contains the definitions of the classes \( DBSC(K) \), \( DSC(K) \), \( B_{1/2}(K) \) and \( B_{1/4}(K) \). At the end of §1 we briefly recall the notion of spreading model. In §2 we recall some ordinal indices which are used to study \( B_1(K) \). A detailed study of such indices can be found in [25]. Our use of these indices and many of the results of this paper have been motivated by [8,9,10]. Proposition 2.3 precisely characterizes \( B_{1/2}(K) \) in terms of our index.

In §5 we show that the inclusions in (0.1) are, in general, proper. We first deduce this from a Banach space perspective. Subsequently, we consider the case where \( K \) is countable. Proposition 5.3 specifies precisely how large \( K \) must be in order for each separate inclusion in (0.1) to be proper.

In §8 we summarize some problems raised throughout this paper and raise some new questions regarding \( B_{1/4}(K) \).

We are hopeful that our approach will shed some light on the central problem: if \( X \) is infinite dimensional, does \( X \) contain an infinite dimensional reflexive subspace or an isomorph of \( c_0 \) or \( \ell_1 \)? A different attack has been mounted on this problem in the last few years by Ghoussoub and Maurey. The interested reader should also consult their papers (e.g., [18,19,20,21]). Another fruitful approach has been via the theory of types ([26], [24], [38]). We wish to thank S. Dilworth and R. Neidinger for useful suggestions.
1. Definitions.

In this section we give the basic definitions of the Baire-1 subclasses in which we are interested. Let $K$ be a compact metric space. $B_1(K)$ shall denote the class of bounded Baire-1 functions on $K$, i.e., the pointwise limits of (uniformly bounded) pointwise converging sequences $(f_n) \subseteq C(K)$. $DBSC(K) = \{ F : K \to \mathbb{R} \mid \text{there exists } (f_n)_{n=0}^{\infty} \subseteq C(K) \text{ and } C < \infty \text{ such that } f_0 \equiv 0, (f_n) \text{ converges pointwise to } F \}$ and

$$\sum_{n=0}^{\infty} |f_{n+1}(k) - f_n(k)| \leq C \text{ for all } k \in K.$$ \hspace{1cm} (1.1)

If $F \in DBSC(K)$ we set $|F|_D = \inf \{ C \mid \text{there exists } (f_n)_{n=0}^{\infty} \subseteq C(K) \text{ converging pointwise to } F \text{ satisfying (1.1) with } f_0 \equiv 0 \}$. $DBSC(K)$ is thus precisely those $F$’s which are the “difference of bounded semicontinuous functions on $K$.” Indeed if $(f_n)$ satisfies (1.1), then $F = F_1 - F_2$ where $F_1(k) = \sum_{n=0}^{\infty} (f_{n+1} - f_n)^+(k)$ and $F_2(k) = \sum_{n=0}^{\infty} (f_{n+1} - f_n)^-(k)$ are both (lower) semicontinuous. The converse is equally trivial.

It is easy to prove that $(DBSC(K), | \cdot |_D)$ is a Banach space by using the series criterion for completeness. The fact that $\|F\|_{\infty} \leq |F|_D$ but the two norms are in general not equivalent on $DBSC(K)$, leads naturally to the following two definitions.

$$B_{1/2}(K) = \{ F \in B_1(K) \mid \text{there exists a sequence } (F_n) \subseteq DBSC(K) \text{ converging uniformly to } F \}$$

and

$$B_{1/4}(K) = \{ F \in B_1(K) \mid \text{there exists } (F_n) \text{ converging uniformly to } F \text{ with } \sup_n |F_n|_D < \infty \}.$$ 

It can be shown that $DBSC(K)$ is a Banach algebra under pointwise multiplication, and hence $B_{1/2}(K)$ can be identified with $C(\Omega)$, where $\Omega$ is the “structure space” or “maximal ideal space” of $\Omega$. Thus $B_{1/4}(K)$ also has a natural interpretation in the general context of commutative Banach algebras.

There is a natural norm on $B_{1/4}(K)$ given by

$$|F|_{1/4} = \inf \{ C : \text{there exists } (F_n) \text{ converging uniformly with } \sup_n |F_n|_D \leq C \}$$

Furthermore $(B_{1/4}(K), | \cdot |_{1/4})$ is a Banach space. One way to see this is to use the following elementary
Lemma 1.1. Let $(M, d_1)$ be a complete metric space and let $d_2$ be a metric on $M$ with $d_1(x, y) \leq d_2(x, y)$ for all $x, y \in M$. If all $d_2$-closed balls in $M$ are also $d_1$-closed, then $(M, d_2)$ is complete.

The hypotheses of the lemma apply to $M = \{F : |F|_{1/4} \leq 1\}$ and $d_1, d_2$ given, respectively, by $\|\cdot\|_\infty$ and $|\cdot|_{1/4}$.

Remark 1.2. While we shall confine our attention to $B_{1/2}$ and $B_{1/4}$, one could of course continue the game, defining

\[ B_{1/8}(K) = \{ F \in B_1(K) \mid \text{there exists } (F_n) \subseteq DBSC(K) \with |F_n - F|_{1/4} \to 0 \} \] 

\[ B_{1/16}(K) = \{ F \in B_1(K) \mid \text{there exists } F_n \with \sup_n |F_n|_D < \infty \text{ and } |F_n - F|_{1/4} \to 0 \} \, . \]

This could be continued obtaining

\[ DBSC(K) \subseteq \cdots \subseteq B_{1/2^{2n}}(K) \subseteq B_{1/2^{2n-1}}(K) \subseteq \cdots \subseteq B_{1/2}(K) \]

with $B_{1/2^{2n}}(K)$ having a norm $|\cdot|_{1/2^{2n}}$ which, using Lemma 1.1, is easily seen to be complete.

There is another class of Baire-1 functions that shall interest us, the differences of (not necessarily bounded) semi-continuous functions on $K$.

\[ DSC(K) = \{ F : K \to \mathbb{R} \mid \text{there exists a uniformly bounded sequence} \] 

\[ (f_n)_{n=0}^{\infty} \subseteq C(K) \text{ converging pointwise to } F \text{ with} \] 

\[ \sum_{n=0}^{\infty} |f_{n+1}(k) - f(k)| < \infty \text{ for } k \in K \} \, . \]

An interesting subclass of $DSC(K)$, is $PS(K)$, the pointwise limits of pointwise stabilizing (pointwise ultimately constant) sequences.

\[ PS(K) = \{ F \in B_1(K) \mid \text{there exists a uniformly bounded sequence} \] 

\[ (f_n) \subseteq C(K) \text{ with the property that for all } k \in K \text{ there exists} \] 

\[ m \in \mathbb{N} \text{ such that } f_n(k) = F(k) \text{ for } n \geq m \} \, . \]
Remark 1.3. We discuss $PS(K)$ in Proposition 4.9. Both of these classes were considered in [10], and as noted there, if an indicator function $1_A \in B_1(K)$, then $1_A \in PS(K)$. Indeed $A$ must be both $F_{\sigma}$ and $G_\delta$ (cf. Proposition 2.1 below) and so we can write $A = \bigcup_n F_n = \bigcap_n G_n$ where $F_1 \subseteq F_2 \subseteq \cdots$ are closed sets and $G_1 \supseteq G_2 \supseteq \cdots$ are open sets. Then by the Tietze extension theorem, for each $n$ choose $f_n \in BA(C(K))$ with $f_n$ identically 1 on $F_n$ and identically 0 on $K \setminus G_n$. Thus for all $k \in K$, $(f_n(k))_n$ is ultimately $1_A(k)$.

The *summing basis* $(s_n)$ for (an isomorph of) $c_0$ is characterized by

$$\|\sum a_n s_n\| = \sup_k \|\sum_{i=1}^k a_i\|.$$  

Let $(x_n)$ be a seminormalized basic sequence. A basic sequence $(e_n)$ is said to be a *spreading model* of $(x_n)$ if for all $k \in \mathbb{N}$ and all $\varepsilon > 0$ there exist $N$ so that if $N < n_1 < n_2 < \cdots < n_k$ and $(a_i)_1^k \subseteq \mathbb{R}$ with $\sup_i |a_i| \leq 1$, then

$$\left|\|\sum_{i=1}^k a_i x_{n_i}\| - \|\sum_{i=1}^k a_i e_i\|\right| < \varepsilon.$$  

For further information on spreading models see [4].

We recall that if $(f_n) \subseteq BA(C(K))$ converges pointwise to $F \in B_1(K) \setminus C(K)$ then there exists a $C = C(F)$ such that $(f_n)$ has a basic subsequence $(f'_n)$ with basis constant $C$ which $C$-dominates $(s_n)$. Thus $C\|\sum a_n f'_n\| \geq \|\sum a_n s_n\|$, for all $(a_n) \subseteq \mathbb{R}$ (see e.g., [31]). Furthermore $(f'_n)$ can be taken to have a spreading model [4]. The constant $C$ depends only on sup\{osc$(F,k)$ | $k \in K$\} (see §2 for the definition of osc$(F,k)$).

Finally we recall that a sequence $(g_n)$ in a Banach space is a *convex block subsequence* of $(f_n)$ if $g_n = \sum_{i=p_n+1}^{p_{n+1}} a_i f_i$ where $(p_n)$ is an increasing sequence of integers, $(a_i) \subseteq \mathbb{R}^+$ and for each $n$, $\sum_{i=p_n+1}^{p_{n+1}} a_i = 1$.  

6
2. Ordinal Indices for $B_1(K)$.

Let $(K, d)$ be a compact metric space and let $F : K \to \mathbb{R}$ be a bounded function. The Baire characterization theorem [3] states that $F \in B_1(K)$ iff for all closed nonempty $L \subseteq K$, $F|_L$ has a point of continuity (relative to the compact space $(L, d)$). This leads naturally to an ordinal index for Baire-1 functions which we now describe.

For a closed set $L \subseteq K$ and $\ell \in L$ let the oscillation of $F|_L$ at $\ell$ be given by

$$\text{osc}_L(F, \ell) = \lim_{\epsilon \downarrow 0} \sup \{ f(\ell_1) - f(\ell_2) \mid \ell_i \in L \text{ and } d(\ell_i, \ell) < \epsilon \text{ for } i = 1, 2 \}.$$ 

We define the oscillation of $F$ over $L$ by

$$\text{osc}_L(F) = \sup \{ F(\ell_1) - F(\ell_2) \mid \ell_1, \ell_2 \in L \}.$$

For $\delta > 0$, let $K_0(F, \delta) = K$ and if $\alpha < \omega_1$ let

$$K_{\alpha+1}(F, \delta) = \{ k \in K_\alpha(F, \delta) \mid \text{osc}_{K_\alpha(F, \delta)}(F, k) \geq \delta \}.$$

For limit ordinals $\alpha$, set

$$K_\alpha(F, \delta) = \bigcap_{\beta < \alpha} K_\beta(F, \delta).$$

Note that $K_\alpha(F, \delta)$ is always closed and $K_\alpha(F, \delta) \supseteq K_\beta(F, \delta)$ if $\alpha < \beta$. The index $\beta(F, \delta)$ is given by

$$\beta(F, \delta) = \inf \{ \alpha < \omega_1 \mid K_\alpha(F, \delta) = \emptyset \}$$

provided $K_\alpha(F, \delta) = \emptyset$ for some $\alpha < \omega_1$ and $\beta(F, \delta) = \omega_1$ otherwise. Since $K$ is separable, the transfinite sequence $(K_\alpha(F, \delta))_{\alpha < \omega_1}$ must stabilize: there exists $\beta < \omega_1$ so that $K_\alpha(F, \delta) = K_\beta(F, \delta)$ for $\beta \geq \alpha$.

The Baire characterization theorem yields that $\beta(F, \delta) < \omega_1$ for all $\delta > 0$ iff $F \in B_1(K)$. In fact we have the following proposition. In its statement $\mathcal{A}$ denotes the algebra of ambiguous subsets of $K$. Thus $A \in \mathcal{A}$ iff $A$ is both $F_\sigma$ and $G_\delta$. Also we write $[F \leq a]$ for the set $\{ k \in K \mid F(k) \leq a \}$.

**Proposition 2.1.** Let $F : K \to \mathbb{R}$ be a bounded function on the compact metric space $K$. The following are equivalent.

1) $F \in B_1(K)$.
2) $\beta(F, \delta) < \omega_1$ for all $\delta > 0$.
3) For $a$ and $b$ real, $[F \leq a]$ and $[F \geq b]$ are both $G_\delta$ subsets of $K$. 

4) For $U$ an open subset of $\mathbb{R}$, $F^{-1}(U)$ is an $F_\sigma$ subset of $K$.

5) For $a < b$, $[F \leq a]$ and $[F \geq b]$ may be separated by disjoint sets in $\mathcal{A}$. Equivalently, there exists $A \in \mathcal{A}$ with $[F \leq a] \subseteq A$ and $A \cap [F \geq b] = \emptyset$.

6) $F$ is the uniform limit of a sequence of $\mathcal{A}$-simple functions ($\mathcal{A}$-measurable functions with finite range).

7) $F$ is the uniform limit of a sequence $(g_n) \subseteq DSC(K)$.

8) $F$ is the uniform limit of a sequence $(g_n) \subseteq PS(K)$.

The proof is standard and can be compiled from [23]. We are more interested in an analogous characterization of $B_{1/2}(K)$. Before stating that proposition we need a few more definitions.

$\mathcal{D}$ shall denote the algebra of all finite unions of differences of closed subsets of $K$. $\mathcal{D}$ is easily seen to be a subalgebra of $\mathcal{A}$.

One of the statements in our next proposition involves another ordinal index for Baire-1 functions, $\alpha(F; a, b)$, which as we shall see is closely related to our index. For $a < b$, let $K_0(F; a, b) = K$ and for any ordinal $\alpha$, let

$$K_{\alpha+1}(F; a, b) = \{k \in K_\alpha(F; a, b) \mid \text{for all } \varepsilon > 0 \text{ and } i = 1, 2, \text{there exist } k_i \in K_\alpha(F; a, b) \text{ with } d(k_i, k) \leq \varepsilon, \quad F(k_1) \geq b \text{ and } F(k_2) \leq a\}.$$ 

Equivalently, $K_{\alpha+1} = K_\alpha \cap [F \leq a] \cap K_\alpha \cap [F \geq b]$. At limit ordinals $\alpha$ we set

$$K_\alpha(F; a, b) = \bigcap_{\beta < \alpha} K_\beta(F; a, b).$$

As before these sets are closed and decreasing. We let $\alpha(F; a, b) = \inf\{\gamma < \omega_1 \mid K_\gamma(F; a, b) = \emptyset\}$ if $K_\gamma(F; a, b) = \emptyset$ for some $\gamma < \omega_1$ and let $\alpha(F; a, b) = \omega_1$ otherwise.

**Remark 2.2.** The index $\alpha(F; a, b)$ is only very slightly different from the index $L(F, a, b)$ considered by Bourgain [8]. $L(F, a, b) = \inf\{\eta < \omega_1 \mid \text{there exists a transfinite increasing sequence of open sets } (G_\alpha)_{\alpha \leq \eta} \text{ with } G_0 = \emptyset, \ G_\eta = K, \ G_{\alpha+1} \setminus G_\alpha \text{ is disjoint from either } [F \leq a] \text{ or } [F \geq b] \text{ for all } \alpha < \eta \text{ and } G_\gamma = \bigcup_{\alpha < \gamma} G_\alpha \text{ if } \gamma \leq \eta \text{ is a limit ordinal}\}$. In fact one can show that if $\alpha(F; a, b) = \eta + n$ where $\eta$ is a limit ordinal and $n \in \mathbb{N}$, then
$L(F, a, b) \in \{\eta + 2n, \eta + 2n - 1\}$. In Proposition 2.3 we shall show that $\alpha(F; a, b) < \omega$ for all $a < b$ iff $\beta(F, \delta) < \omega$ for all $\delta > 0$. We note that a more general result has subsequently been obtained in [25]. Indeed if we define $\beta(F) = \sup \{\beta(F; \delta) \mid \delta > 0\}$ and $\alpha(F) = \sup \{\alpha(F; a, b) \mid a < b \text{ rational}\}$ then Kechris and Louveau have shown that $\beta(F) \leq \omega^\xi$ iff $\alpha(F) \leq \omega^\xi$.

Also we note that the following result follows from [8]. Let $X$ be a separable Banach space not containing $\ell_1$. Let $K = Ba(X^*)$ in its weak* topology. Then

$$\sup \{\beta(x^{**}\mid_K) : x^{**} \in X^{**}\} < \omega_1.$$ 

**Proposition 2.3.** Let $F : K \to \mathbb{R}$ be a bounded function on the compact metric space $K$. The following are equivalent

1) $F \in B_{1/2}(K)$.  
2) $F$ is the uniform limit of $\mathcal{D}$-simple functions on $K$.  
3) For $a < b$, $[F \leq a]$ and $[F \geq b]$ may be separated by disjoint sets in $\mathcal{D}$.  
4) $\beta(F) \leq \omega$.  
5) $\alpha(F; a, b) < \omega$ for all $a < b$.

**Proof.**

4) $\Rightarrow$ 5). This follows from the elementary observation that for all ordinals $\alpha$ and reals $a < b$, $K_\alpha(F; a, b) \subseteq K_\alpha(F, b - a)$, and the fact that 4) holds if and only if $\beta(F, \delta) < \omega$ for all $\delta > 0$.

5) $\Rightarrow$ 3). Let $K_i = K_i(F; a, b)$. Thus $K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n = \emptyset$ where $n = \alpha(F; a, b)$. Let

$$D = \bigcup_{i=1}^n (F \leq a \cap K_{i-1}) \setminus ([F \geq b] \cap K_{i-1}) \in \mathcal{D}.$$
Since $K_i = ([F \leq a] \cap K_{i-1}) \cap ([F \geq b] \cap K_{i-1})$, 

$$D = \bigcup_{i=1}^{n} ([F \leq a] \cap K_{i-1} \setminus K_i)$$ 

$$\supseteq \bigcup_{i=1}^{n} \left( ([F \leq a] \cap K_{i-1}) \setminus K_i \right)$$ 

$$= \bigcup_{i=1}^{n} ([F \leq a] \cap (K_{i-1} \setminus K_i)) = [F \leq a].$$

Since $K_{i-1}$ is closed, 

$$D \subseteq \bigcup_{i=1}^{n} (K_{i-1} \setminus [F \geq b] \cap K_{i-1})$$ 

$$\subseteq \bigcup_{i=1}^{n} \left( K_{i-1} \setminus ([F \geq b] \cap K_{i-1}) \right)$$ 

$$= \bigcup_{i=1}^{n} (K_{i-1} \setminus [F \geq b]) = K \setminus [F \geq b].$$

3) $\Rightarrow$ 2). This is a standard exercise in real analysis.

2) $\Rightarrow$ 1). Since every $D$-simple function can be expressed in the form $\sum_{i=1}^{k} a_i 1_{L_i}$ where the $L_i$‘s are closed sets and $DBSC(K)$ is a linear space it suffices to recall that $1_L \in DBSC(K)$ whenever $L$ is closed. In fact $1_L$ is upper semicontinuous.

1) $\Rightarrow$ 4). Let $F$ be the uniform limit of $(F_n) \subseteq DBSC(K)$. For $\delta > 0$ and $n$ sufficiently large, $\beta(F, 2\delta) \leq \beta(F_n, \delta)$ and thus is suffices to prove that for $G \in DBSC(K)$, $\beta(G, \delta) < \omega$ for $\delta > 0$. This is immediate from the following

**Lemma 2.4.** If $m \in \mathbb{N}$, $\delta > 0$ and $G : K \to \mathbb{R}$ is such that $K_m(G, \delta) \neq \emptyset$, then $|G|_{D} \geq m\delta/4$.

**Proof.** Let $(g_n) \subseteq C(K)$ converge pointwise to $G$. It suffices to show that there exist integers $n_1 < n_2 < \cdots < n_{m+1}$ and $k \in K$ such that $|g_{n_i+1}(k) - g_{n_i}(k)| > \delta/4$ for $1 \leq i \leq m$.

Let $n_1 = 1$, $k_0 \in K_m(G, \delta)$ and let $U_0$ be a neighborhood of $k_0$ for which $\text{osc}_{U_0} g_{n_1} < \delta/8$. Choose $k_{01}^1$ and $k_{02}^2$ in $U_0 \cap K_{m-1}(G, \delta)$ with $G(k_{01}^1) - G(k_{02}^2) > 3\delta/4$. Then choose
\[ n_2 > n_1 \text{ such that } g_{n_2}(k_0^1) - g_{n_2}(k_0^2) > 3\delta/4. \] Thus there is a nonempty neighborhood \( U_1 \subseteq U_0 \) of either \( k_0^1 \) or \( k_0^2 \) such that for \( k \in U_1 \), \( |g_{n_2}(k) - g_{n_1}(k)| > \delta/4. \)

Similarly we can find a neighborhood \( U_2 \subseteq U_1 \) of a point in \( K_{m-1}(G, \delta) \) and \( n_3 > n_2 \) so that for \( k \in U_2 \), \( |g_{n_3}(k) - g_{n_2}(k)| > \delta/4, \) etc.

### Remarks 2.5.
1. Of course by using a bit more care one can show that \( |G|_D \geq m\delta/2 \) whenever \( K_m(G, \delta) \neq \emptyset \).

2. Following [25] we say that for \( F \in B_1(K) \), \( F \in B_1^\xi(K) \) iff \( \beta(F) \leq \omega^\xi \). Thus \( B_{1/2}(K) \equiv B_1^1(K) \) by Proposition 2.3, a result also observed in [25].

3. We do not yet have an index characterization of \( B_{1/4}(K) \), however we have a necessary condition (which may be sufficient). To describe this we first must generalize our index above. Let \( F : K \to \mathbb{R} \) and let \( (\delta_i)_{i=1}^\infty \) be positive numbers. Set \( K_0(F, (\delta_i)) = K \) and for \( 0 \leq i \)

\[
K_{i+1}(F, (\delta_j)) = \left\{ k \in K_i(F, (\delta_j)) \mid \text{osc}_{K_i(F, (\delta_j))}(F, k) \geq \delta_{i+1} \right\}.
\]

### Proposition 2.6.
Let \( F \in B_{1/4}(K) \). Then there exists an \( M < \infty \) so that if \( K_n(F, (\delta_i)) \neq \emptyset \), then \( \sum_{i=1}^n \delta_i \leq M. \)

**Proof.** Let \( F \) be the uniform limit of \( (G_n) \) with \( |G_n|_D \leq C < \infty \) for all \( n \). Suppose that \( K_n(F, (\delta_i)) \neq \emptyset \) for some sequence \( (\delta_i)_{i=1}^\infty \subseteq \mathbb{R}^+ \). Since \( K_n(F, (\delta_i)) \subseteq K_n(G_m, (\delta_i/2)) \) for large \( m \), the latter set is non-empty as well. The proof of Lemma 2.4 yields

\[
\begin{align*}
\text{(2.1)}
\end{align*}
\]

Thus by (2.1) we have, for large \( m \), \( C \geq |G_m|_D \geq 4^{-1} \sum_{i=1}^n \delta_i \)
and so \( \sum_{i=1}^n \delta_i \leq 4C. \)

We shall explore in greater detail in §3 and §8 some questions related to the problem of an index characterization of Baire-1/4. The following proposition gives a sufficient index criterion for a function to be Baire-1/4. It also shows (via Proposition 2.3) that if \( F \in B_{1/2}(K) \setminus B_{1/4}(K) \), then \( \beta(F) = \omega. \)
Proposition 2.7. Let $F \in B_1(K)$. If $\beta(F) < \omega$, then $F \in B_{1/4}(K)$.

Proof. Without loss of generality let $F : K \to [0, 1]$ with $\beta(F) \leq n$. Thus $\alpha(F; a, b) \leq n$ for all $a < b$. It follows from the proof of 5) $\Rightarrow$ 3) in Proposition 2.3 that for all $0 < a < b < 1$ there exists a $D \in \mathcal{D}$ with $|1_D|_D \leq 2n$, $[F \leq a] \subseteq D$ and $[F \geq b] \cap D = \emptyset$. Thus for all $m < \infty$ there exist sets $D_1 \supseteq D_2 \supseteq \cdots \supseteq D_m$ in $\mathcal{D}$ with $F \leq i/m \subseteq D_i$, $F \leq (i-1)/m \cap D_i = \emptyset$ and $|1_{D_i}|_D \leq 2n$ for $i \leq m$. In particular if $G = \sum_{i=1}^m m^{-1}1_{D_i}$, then $\|F - G\|_{\infty} \leq m^{-1}$ and $|G|_D \leq 2n$. 

The following proposition is related to work of A. Sersouri [39]. It is of interest to us because it shows that a separable Banach space $X$ can have functions of large index in $X^{**}$ and yet be quite nice. In fact it shows there are Baire-1 functions of arbitrarily large index which strictly govern the class of quasireflexive (order 1) Banach spaces. Our proof was motivated by discussions with A. Pełczyński.

Proposition 2.8. For all $\gamma < \omega_1$ there exists a quasireflexive (of order 1) Banach space $Q_\gamma$ such that $Q_{\gamma}^{**} = Q_\gamma \oplus \langle F_\gamma \rangle$ where $\beta(F_\gamma) > \gamma$.

(The index $\beta(F_\gamma)$ is computed with respect to $Ba(Q^{**}_\gamma)$.)

Remark 2.9. In §6 we shall show the existence of a quasireflexive space whose new functional (in the second dual) is Baire-1/4.

Proof of Proposition 2.8. We use interpolation, namely the method of [12]. (This has also been used in [19] in a slightly different manner to produce a quasireflexive space from a weak* convergent sequence.)

To begin let $\gamma < \omega_1$ be any ordinal and choose a compact metric space $K$ containing an ambiguous set $A_\gamma$ with $\alpha(1_{A_\gamma} ; \frac{1}{4}, \frac{3}{4}) > \gamma$. (For example $1_{A_\alpha}$ could be taken to be one of the functions $F_\delta$ described in §5 with $\delta > \omega^\gamma$.) Choose a sequence $(f_n) \subseteq Ba(C(K))$ converging pointwise to $1_{A_\gamma}$ such that $(1_{A_\gamma}, f_1, f_2, \ldots)$ is basic in $C(K)^{**}$. Let $W$ be the closed convex hull of $\{\pm f_n\}_{n=1}^\infty$ in $C(K)$. Let $Q_\gamma$ be the Banach space obtained from $W \subseteq Ba(C(K))$ by [DFJP]-interpolation. Thus for all $n \in \mathbb{N}$, $\| \cdot \|_n$ is the gauge of $U_n = 2^nW + 2^{-n}Ba(C(K))$, and $Q_\gamma = \{x \in C(K) : \| x \| \equiv (\sum_n \| x \|_n^2)^{1/2} < \infty\}$. Following the notation of [12], we let $C = Ba(Q_\gamma) = \{x \in C(K) : \| x \| \leq 1\}$ and let
$j : Q_\gamma \to C(K)$ be the natural semiembedding.

We first observe that $Q_\gamma$ is quasireflexive of order 1. Indeed it is easy to check that

\[ \tilde{W}, \text{the weak* closure of } W \text{ in } C(K)^{**} \text{ is just} \]

\[ \tilde{W} = \left\{ \sum_{i=1}^{\infty} a_i f_i + a_\infty 1_{A_\gamma} : |a_\infty| + \sum_{i=1}^{\infty} |a_i| \leq 1 \right\}. \]

Furthermore $\tilde{C} \subseteq \tilde{W}$ ([12], Lemma 1(v)) which has the basis $(1_{A_\gamma}, f_1, f_2, \ldots)$. Now $j^{**} : Q^{**}_\gamma \to C(K)^{**} \text{ is one-to-one and } (j^{**})^{-1}(C(K)) = Q_\gamma$ (Lemma 1(iii)). Thus if $F_\gamma \in Q^{**}_\gamma$ satisfies $j^{**}F_\gamma = 1_{A_\gamma}$, then $Q^{**}_\gamma = Q_\gamma \oplus (F_\gamma)$. Of course $F_\gamma$ must be the weak* limit of $(j^{-1}(f_n))_n$ in $Q^{**}_\gamma$.

It remains to show that $\beta(F_\gamma) \geq \gamma$. We shall prove

\[ (2.2) \quad \overline{\sigma}(F_\gamma ; \frac{1}{4}, \frac{3}{4}) \geq \alpha(1_{A_\gamma} ; \frac{1}{4}, \frac{3}{4}) \]

where $\overline{\beta}$ is the index computed with respect to $F_\gamma \in B_1(3Ba(Q^*_\gamma))$. Since $\beta(F_\gamma) \geq \alpha(F_\gamma ; \frac{1}{12}, \frac{1}{4}) \geq \overline{\sigma}(F_\gamma ; \frac{1}{4}, \frac{3}{4})$, the result follows.

Since $\|j\| \leq 3$, if $K_0 = 3Ba(Q^*_\gamma)$ and $H_0 = Ba(C(K)^*)$, then $j^*H_0 \subseteq K_0$. More generally if $K_{\beta+1} = \{y^* \in K_\beta \mid \text{for all non-empty relative weak* neighborhoods } U \text{ of } y^* \text{ in } K_\beta \text{ there exists } y_1^*, y_2^* \in U \text{ with } F_\gamma(y_1^*) \geq \frac{3}{4} \text{ and } F_\gamma(y_2^*) \leq \frac{1}{4} \}$ and $H_{\beta+1}$ is defined similarly in terms of $1_{A_\gamma}$, then $j^*H_{\beta+1} \subseteq K_{\beta+1}$ for all $\beta$, since $j^*$ is $\omega^*$-continuous and

\[ F_\gamma(j^*x^*) = (j^{**}F_\gamma)x^* = 1_{A_\gamma}(x^*). \]

This proves (2.2).
Lemma 3.1. Let $\bigcap_{i \in I} A_i \cap \bigcap_{i \notin I} B_i \neq \emptyset$. The proof is similar to that of Lemma 2.4 and is actually a local version of the proof of the main result of [35] (see [8] for a more general discussion of the consequences of $K_\beta(F; a, b) \neq \emptyset$).

Proof. The proof is similar to that of Lemma 2.4 and is actually a local version of the proof of the main result of [35] (see [8] for a more general discussion of the consequences of $K_\beta(F; a, b) \neq \emptyset$).

We first show how to choose a finite subsequence $(f_{n_i})_{i=1}^m$ of $(f_n)$ so that $(A_{n_i}, B_{n_i})_{i=1}^m$ is independent, where $A_{n_i} = [f_{n_i} \leq a']$ and $B_{n_i} = [f_{n_i} \geq b']$. Let $k_1 \in K_m(F; a, b)$. Thus there exist $k_0$ and $k_1$ in $K_{m-1}(F; a, b)$ with $F(k_0) \leq a$ and $F(k_1) \geq b$. Choose $n_1$ and neighborhoods $U_0$ and $U_1$ of $k_0$ and $k_1$, respectively, so that $f_{n_1} < a'$ on $U_0$ and $f_{n_1} > b'$ on $U_1$. Let $k_{\varepsilon_1, \varepsilon_2} \in U_{\varepsilon_1} \cap K_{m-2}(F; a, b)$ for $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ with $F(k_{\varepsilon_1, 0}) \leq a$ and $F(k_{\varepsilon_1, 1}) \geq b$ for $\varepsilon_1 \in \{0, 1\}$. Choose $n_2 > n_1$ and neighborhoods $U_{\varepsilon_1, \varepsilon_2} \subseteq U_{\varepsilon_1}$ of $k_{\varepsilon_1, \varepsilon_2}$ so that $f_{n_2} < a'$ on $U_{\varepsilon_1, 0}$ and $f_{n_2} > b'$ on $U_{\varepsilon_1, 1}$ (for $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$). Continue up to $f_{n_m}$. The sets $(A_{n_i}, B_{n_i})_{i=1}^m$ are then independent since for $I \subseteq \{1, \ldots, m\}$, $\bigcap_{i \in I} A_{n_i} \cap \bigcap_{i \notin I} B_{n_i} \supseteq U_{\varepsilon_1, \ldots, \varepsilon_m} \neq \emptyset$ where $\varepsilon_i = 0$ if $i \in I$ and $\varepsilon_i = 1$ if $i \notin I$.

Now the existence of an infinite subsequence $(f'_{n_i})$ satisfying the conclusion of 3.1 follows immediately from Ramsey’s theorem. Indeed, by the latter, there exists $(f'_{n_i})$ a subsequence of $(f_n)$ so that $(f'_{n_i})$ satisfies the conclusion, or such that for all $n_1 < \cdots < n_m$, $(A'_{n_i}, B'_{n_i})_{i=1}^m$ is not independent. But we have proved that the second alternative is impossible.

Proof of Theorem B(a). $(f_n)$ is a bounded sequence in $C(K)$ converging pointwise to $F \notin B_{1/2}(K)$. By Proposition 2.3 there exists $a < b$ so that $K_m(F; a, b) \neq \emptyset$ for all $m \in \mathbb{N}$. By passing to a subsequence we may assume $(f_n)$ has a spreading model. Furthermore, by Lemma 3.1, passing to subsequences and diagonalization we may assume that for some
\( a < a' < b' < b, (A_{n_i}, B_{n_i})_{i=1}^m \) is independent whenever \( m \leq n_1 < n_2 < \cdots < n_m \) and \( A_{n_i} = [f_{n_i} \leq a'], B_{n_i} = [f_{n_i} \geq b'] \). By Proposition 4 of [36] it follows that there exists \( C < \infty \) so that \((f_{n_i})_{i=1}^m\) is \( C \)-equivalent to the unit vector basis of \( \ell_1^m \) whenever \( m \leq n_1 < \cdots < n_m \).

The proof of Theorem B(b) will require a more precise version of Theorem A(b) and the following elementary lemma (which follows easily from the Hahn-Banach theorem). If \( C \) is a subset of a Banach space \( X \), \( \bar{C} \) denotes the \( w^* \)-closure of \( C \) in \( X^{**} \).

**Lemma 3.3.** Let \( C \) and \( D \) be convex subsets of \( X \). Then \( \text{md}(C, D) = \text{md}(\bar{C}, \bar{D}) \). By \( \text{md}(C, D) \) we mean the minimum distance,

\[
\inf \{ \|c - d\| \mid c \in C, d \in D \}.
\]

The variant of Theorem A(b) which we need is

**Lemma 3.4.** Let \( F : K \to \mathbb{R} \) be bounded and let \((f_n) \subseteq C(K)\) converge pointwise to \( F \) with \( \sum_{n=0}^{\infty} |f_{n+1}(k) - f_n(k)| \leq M \) for all \( k \in K \) \( (f_0 \equiv 0) \). Suppose \( \text{osc}(F, k) > \delta \) for some \( \delta > 0 \). Then there exists a subsequence \((f'_n)\) of \((f_n)\) which is \( C = C(M, \delta) \) equivalent to the summing basis.

Let \( F \in B_1(K) \setminus C(K) \). It is evident that if \( F \) strictly governs \( \{c_0\} \), then \( F \in DBSC(K) \). The next result shows that the converse is true.

**Corollary 3.5.** Let \( F \in DBSC(K) \) and let \((f_n)\), \( M \) and \( \delta \) be as in the hypothesis of Lemma 3.4. Let \( (g_n) \subseteq C(K) \) converge pointwise to \( F \) with \( \sup_n \|g_n\|_\infty < \infty \). Then there exists \((h_n)\), a convex block subsequence of \((g_n)\), which is \( C(M, \delta) \)-equivalent to the summing basis.

The proof is straightforward from Lemmas 3.3 and 3.4.

**Proof of Theorem B(b).** Let \( F \in B_{1/4}(K) \setminus C(K) \) and let \((f_n) \subseteq C(K)\) be a bounded sequence converging pointwise to \( F \). Choose \((F_n) \subseteq DBSC(K)\) which converges uniformly to \( F \) so that \( \sup_n |F_n|_D < M < \infty \). For each \( n \in \mathbb{N} \), choose \((f^n_i)_{i=0}^{\infty} \subseteq C(K)\), \( f^n_0 \equiv 0 \), which converges pointwise to \( F_n \) and satisfies \( \sum_{i=0}^{\infty} |f^n_{i+1}(k) - f^n_i(k)| \leq M \) for \( k \in K \).
Since \( F \notin C(K) \) we may assume there exists \( \delta > 0 \) so that for all \( n, \text{osc}_K(F_n, k) > \delta > 0 \) for some \( k \in K \). Thus, by Lemma 3.4, we may suppose for all \( n, (f^n)_{i=1}^\infty \) is \( C = C(M, \delta) \)-equivalent to the summing basis. We may also assume \( \|F_n - F_{n+1}\|_\infty < \varepsilon_n \) where \( \varepsilon_n \downarrow 0 \) and for all \( n \in \mathbb{N}, \sum_{i=n+1}^{\infty} \varepsilon_i < \varepsilon_n \).

By induction and Lemma 3.3 we may replace each sequence \( (f^n)_{i=1}^\infty \) by a convex block subsequence \( (g^n)_{i=1}^\infty \) such that for \( n > 1 \),

\[
\begin{align*}
\text{there exists a convex block subsequence } & (h^n_i)_{i=1}^\infty \text{ of } (g^n_{i-1})_{i=1}^\infty \\
\text{with } & \|g^n_i - h^n_i\|_\infty < \varepsilon_{n-1} \text{ for } i \in \mathbb{N}.
\end{align*}
\]

Let \( (g^n)_{i=1}^\infty \) be the diagonal sequence. Clearly \( (g^n) \) converges pointwise to \( F \). Also by (\ast\) for \( n > k, \text{mod}(g^n_n, \text{co}(g^n_j)_{j=1}^\infty) < \sum_{j=k}^n \varepsilon_j < \varepsilon_{k-1} \). In fact for \( k \) fixed, there exists a convex block subsequence \( (d^n_k)_{n>k} \) of \( (g^n_j)_{j=1}^\infty \) with \( \|g^n_n - d^n_k\|_\infty < \varepsilon_{k-1} \) for \( n > k \). Thus for any \( k, (g^n_n)_{n>k} \) is an \( \varepsilon_{k-1} \)-perturbation of a sequence \( (d^n_k)_{n>k} \) which is \( C' \)-equivalent to the summing basis where \( C' \) depends solely on \( C \).

By Lemma 3.3 applied to \( \text{co}(f_n) \) and \( \text{co}(g^n_n) \), there are convex block subsequences \( (g_n) \) of \( (f_n) \) and \( (\bar{g}_n) \) of \( (g^n_n) \) with \( \|g_n - \bar{g}_n\|_\infty \rightarrow 0 \). Since \( (\bar{g}_n)_{n>i} \) is an \( \varepsilon_{i-1} \)-perturbation of a sequence which is \( C' \)-equivalent to the summing basis, \( (\bar{g}_n) \) and hence \( (g_n) \) has a subsequence which has spreading model equivalent to the summing basis.

\( \blacksquare \)

**Remark 3.6.** The constant of equivalence of the spreading model of \( (g_n) \) with the summing basis depends solely upon \( \sup_{k \in K} \text{osc}_K(F, k) \) and \( |F|_{1/4} \).

Our next theorem is a converse to Theorem B(a).

**Theorem 3.7.** Let \( F \in B_1(K) \). Assume that whenever \( (f_n) \subseteq C(K) \) is a uniformly bounded sequence converging pointwise to \( F \), then any spreading model of \( (f_n) \) is equivalent to the unit vector basis of \( \ell_1 \). Then \( F \notin B_{1/2}(K) \).

**Lemma 3.8.** Let \( F \in B_{1/2}(K) \setminus C(K), \|F\|_\infty \leq 1 \). Then there exists \( (f_n) \subseteq C(K) \) converging pointwise to \( F \) with spreading model \( (e_n) \) and a function \( M : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) satisfying

\[
\|\sum a_n e_n\| \leq M(\varepsilon)\|\sum a_n s_n\| + \varepsilon\sum |a_n| \tag{3.1}
\]
for all \((a_n) \subseteq \mathbb{R}\) and \(\varepsilon > 0\).

**Proof.** Let \((g_n) \subseteq Ba(C(K))\) converge pointwise to \(F\) and let \(\varepsilon_n \downarrow 0\). By the proof of Theorem B(b) we can choose \((f_n)\), a convex block subsequence of \((g_n)\) such that for all \(m\), \((f_n)_{n=m}^\infty\) is an \(\varepsilon_m\)-perturbation of a sequence which is \(M(\varepsilon_m, F)\)-equivalent to the summing basis.

**Proof of Theorem 3.7.** This is immediate from Lemma 3.8, since if \((e_n)\) satisfies (3.1), then

\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=1}^{n} (-1)^i e_i \right\| = 0.
\]

In particular \((e_i)\) is not equivalent to the unit vector basis of \(\ell_1\).  

The proof of Theorem 3.7 combined with Theorem B(a) yields the following result. Let \(F \in B_1(K)\). Then \(F \notin B_{1/2}(K)\) if and only if there exists \((f_n) \subseteq C(K)\), a uniformly bounded sequence converging pointwise to \(F\), so that if \((g_n)\) is a convex block subsequence of \((f_n)\), then some subsequence of \((g_n)\) has the unit vector basis of \(\ell_1\) as a spreading model.

We do not know if the converse to Theorem B(b) is valid.

**Problem 3.9.** Let \(F \in B_1(K)\) and \(C < \infty\) be such that whenever \((f_n)\) is a uniformly bounded sequence in \(C(K)\) converging pointwise to \(F\), then there exists \((g_n)\), a convex block subsequence of \((f_n)\) with spreading model \(C\)-equivalent to the summing basis. Is \(F \in B_{1/4}(K)\)?

We now turn to the Banach space implications of Theorem B. Let \(K\) be compact metric and let \(X\) be a closed subspace of \(C(K)\). For example, \(K\) could be \(Ba(X^*)\) but we do not require this. \(X^{**}\) is naturally isometric to \(X^{\perp\perp} \subseteq C(K)^{**}\). In this setting it can be shown (see [35]) that if \(B_1(X) = \{x^{**} \in X^{**} : \text{there exists } (x_n) \subseteq X \text{ with } (x_n) \text{ converging weak* in } X^{**} \text{ to } x^{**}\}\), then \(B_1(X) \subseteq B_1(C(K))\) and \(B_1(C(K))\) is naturally identified with \(B_1(K)\).

**Corollary 3.10.** Let \(K\) be compact metric and let \(X\) be a closed subspace of \(C(K)\).

a) If \(X^{**} \cap [B_1(K) \setminus B_{1/2}(K)] \neq \emptyset\), then \(X\) contains a basic sequence with spreading model equivalent to the unit vector basis of \(\ell_1\).
b) If $[X^{**} \cap B_{1/4}(K)] \setminus X \neq \emptyset$ then $X$ contains a basic sequence with spreading model equivalent to the summing basis.

**Remark 3.11.** This corollary has immediate purely local consequences. Thus if $X$ and $K$ are as above and $X$ does not contain $\ell_1^n$'s uniformly, then $X^{**} \cap B_{1/4}(K) \subset X$. Moreover if $X$ is $B$-convex, i.e., does not contain $\ell_1^n$'s uniformly, then $X^{**} \setminus X \subset B_{1/2}(K) \setminus B_{1/4}(K)$.

**4. DSC($K$).**

**Theorem 4.1.** Let $K$ be compact metric and let $F \in DSC(K) \setminus C(K)$. Then $F$ governs $\{c_0\}$.

**Remark 4.2.** If $X$ is a separable Banach space, $K = Ba(X^*)$ in its weak* topology and $F \in X^{**}$, then if $F \in DSC(K)$, $F \in DBSC(K)$ (and hence for such functions Theorem 4.1 follows from Theorem A). To see this assertion, first choose $(f_n)$ uniformly bounded in $C(K)$ so that $f_n \to F$ pointwise and $\sum_{n=1}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty$ for all $k \in K$. Now since $F \in B_1(X)$, we may choose $(g_j)$ a convex block subsequence of $(f_j)$ and $(x_j)$ a sequence in $X$ with $\|g_j - x_j\| < 2^{-j}$ for all $j$. But then it follows that $x_j \to F$ pointwise and moreover $\sum_{j=1}^{\infty} |x_{j+1}(k) - x_j(k)| < \infty$ for all $k \in K$. Thus by the uniform boundedness principle,

$$\sup_{k \in K} \sum_{j=1}^{\infty} |x_{j+1}(k) - x_j(k)| < \infty,$$

so $F \in DBSC(K)$.

Theorem 4.1 follows from the stronger result of Elton [13] which was motivated by work of Fonf [16].

**Theorem.** [13]. Let $X$ be a Banach space and let $\mathcal{E}$ be the set of extreme points of $Ba(X^*)$. Let $(x_i)$ be a normalized basic sequence in $X$ such that $\sum_{i=1}^{\infty} |x^*(x_i)| < \infty$ for all $x^* \in \mathcal{E}$. Then $c_0 \hookrightarrow [(x_i)]$.

Theorem 4.1. can be phrased in this way provided $\mathcal{E}$ is replaced by $\overline{\mathcal{E}}$. However we wish to present a separate proof of our weaker result which seems to be of interest in its own right. The main step is given by the following lemma. If $Y$ is a subspace of $C(K)$, $U \subseteq K$ and $r > 0$, we say $U$ $r$-norms $Y$ if $\|y\|_{U} \geq r\|y\|$ for all $y \in Y$. 18
Lemma 4.3. Let $L$ be a compact metric space and let $(f_i)$ be a normalized basic sequence in $C(L)$. If $c_0 \not\hookrightarrow [(f_i)]$, then there exists a nonempty compact set $K \subseteq L$ and a normalized block basis $(g_i)$ of $(f_i)$ so that

$$\begin{cases} 
\text{for any nonempty relatively open subset } U \text{ of } K \text{ there are an } \\
\text{ } r > 0 \text{ and an } n_0 \in \mathbb{N} \text{ such that } U \text{ r-norms } [(g_n)_{n=n_0}^\infty].
\end{cases}$$

Remark 4.4. It can be deduced from [36] that $[(x_n)]$ contains an isomorph of $\ell_1$ iff there exists a compact set $K \subseteq L$ such that (4.1) holds for some fixed $r > 0$ independent of $U$.

Proof of Lemma 4.3. Let $(U_m)_{m=1}^\infty$ be a base of open sets for $L$. We inductively construct for each $m$ a normalized block basis $(f_i^m)_{i=1}^\infty$ of $(f_i)$ and a certain subsequence $M$ of $\mathbb{N}$.

Let $(f_i^0) = (f_i)$ and suppose $(f_i^m)_{i=1}^\infty$ has been chosen. There are two possibilities.

(i) There is a normalized block basis $(g_i)$ of $(f_i^m)_{i=1}^\infty$ with $\|g_i\|_{U_m} \to 0$ as $i \to \infty$.

(ii) There exists no such sequence.

If (i) holds, choose $(f_i^{m+1})_{i=1}^\infty$ to be a normalized block basis of $(f_i^m)_{i=1}^\infty$ with

$$\|f\|_{U_m} < 2^{-k}\|f\|_\infty \text{ for all } f \in [(f_i^{m+1})_{i=1}^\infty]$$

and put $m$ in $M$. If (ii) holds let $(f_i^{m+1})_{i=1}^\infty = (f_i^m)_{i=1}^\infty$ and put $m$ in $\mathbb{N} \setminus M$. Let $K = L \setminus \bigcup_{m \in M} U_m$ and for all $n \in M$ let $g_n = f_{n+1}^m$. We may assume $M$ is infinite or else the conclusion of the lemma is satisfied with $K = L$ and $g_i = f_i^m (m = \max M$ or 0 if $M = \emptyset$).

First we check that $K \neq \emptyset$. If $K = \emptyset$, then $L \subseteq \bigcup_{n \in M} U_n$. By compactness there exists $n_1 \in M$ so that $L \subseteq \bigcup_{n \in M, n \leq n_1} U_n$. But then $\|g_{n_1}\|_{U_n} < 2^{-(n_1+1)}$ for $n \in M$ with $n \leq n_1$, we have $\|g_{n_1}\|_\infty < 1$, a contradiction.

We claim that $K$ and $(g_n)$ satisfy (4.1). If not there exist $(h_n)$, a normalized block basis of $(g_n)$ and a $U_m$ such that $K \cap U_m \neq \emptyset$ and so $m \notin M$ yet $\|h_i\|_{K \cap U_m} < 2^{-i}$ for all $i$. Indeed there must exist $m' \in M$ with $K \cap U_{m'} \neq \emptyset$ and $(h_i)$, a normalized block basis of $(g_n)$, with $\|h_i\|_{K \cap U_{m'}} < 2^{-i}$. Then choose $m \in \mathbb{N}$ so that $\bigcup_m \subseteq U_{m'}$ and $K \cap U_m \neq \emptyset$. Let $j_0 = m$ and if $j_i$ is defined choose $j_{i+1} > j_i$ so that

$$\bigcup_m \cap [h_{j_i} \geq 2^{-i}] \subseteq \bigcup_{n \in M, n \leq j_{i+1}} U_n.$$
This can be done since \( \overline{U}_m \cap [h_{j_i} \geq 2^{-i}] \subseteq \overline{U}_m \cap [h_{j_i} \geq 2^{-j_i}] \subseteq L \setminus K = \bigcup_{n \in M} U_n \). This completes the definition of \( j_1, j_2, \ldots \). Now for \( t \in U_m \), \( |h_{j_{i_0}}(t)| \geq 2^{-i_0} \) (if such an \( i_0 \) exists). Then \( t \in \bigcup_{n \in M} U_n \) and for \( i > i_0 \), \( h_{j_i} \) is a normalized element in \( [(g_j)_{j \geq j_i, j \in M}] = [(f_{j_i+1})_{j \geq j_i, j \in M}] \subseteq [(f_{p+1})_{p \geq j_i+1}] \). Thus if \( t \in U_n \) with \( n \leq j_{i_0+1}, n \in M \), then \( h_{j_i} \in [(f_{p+1})_{p \geq j_i+1}] \) and so by (4.2), \( |h_{j_i}(t)| \leq \|h_{j_i}\|_{U_n} < 2^{-j_i} \leq 2^{-i} \).

Thus \( \sum_{i=1}^{\infty} |h_{j_i}(t)| \leq 2 \) for all \( t \in \overline{U}_m \). Since \( \overline{U}_m \) norms \( |h_{j_i}| \), it follows from [7] that \( c_0 \hookrightarrow [h_{j_i}], \) a contradiction.

\textbf{Proof of Theorem 4.1.} Let \( (f_n) \) be a bounded sequence in \( C(K) \) converging pointwise to \( F \). By Lemma 3.3 and passing to a convex block subsequence of \( (f_n) \), if necessary, we may suppose that \( \sum_{n=1}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty \) for all \( k \in K \). Also since \( F \not\in C(K) \), by passing to a subsequence \( (f_n') \subseteq (f_n) \) we may assume that \( (h_n) \equiv (f_{2n} - f_{2n+1}) \) is a seminormalized basic sequence satisfying \( \sum_{n=1}^{\infty} |h_n(k)| < \infty \) for all \( k \in K \). If \( c_0 \nrightarrow [(h_n)] \), then by Lemma 4.3 there exist \( (g_n) \), a normalized block basis of \( (h_n) \), and a closed nonempty set \( K_0 \subseteq K \) satisfying (4.1) (with \( K \) replaced by \( K_0 \)).

For \( m \in \mathbb{N} \) set \( K_m = \{ k \in K_0 : \sum_{n=1}^{\infty} |g_n(k)| \leq m \} \). Since \( (g_n) \) is a normalized block basis of \( (h_n) \), \( \sum_{n=1}^{\infty} |g_n(k)| < \infty \) for all \( k \in K \) and thus \( \bigcup_{m=1}^{\infty} K_m = K_0 \). By the Baire category theorem there exists \( m_0 \) so that \( K_{m_0} \) has nonempty interior \( U \) (relative to \( K_0 \)). Choose \( n_0 \) and \( r > 0 \) so that \( U \) \( r \)-norms \( [(g_n)_{n \geq n_0}] \). Since \( \sum |g_n| \leq m_0 \) on \( U \), \( (g_n) \) is equivalent to the unit vector basis of \( c_0 \) [7], a contradiction.

A natural problem is to classify those functions \( F \in B_1(K) \) which govern \( \{c_0\} \). We do not know how to do this, but it is easy to see that this class is strictly larger than \( DSC(K) \).

\textbf{Example 4.5.} Let \( L \) be a countable compact metric space, large enough so that there exists an \( F \in B_1(L) \setminus DBSC(L) \) (see Proposition 5.3). Choose a bounded sequence \( (f_n) \subseteq C(L) \) which converges pointwise to \( F \) and let \( X = [(f_n)] \). \( C(L) \) is \( c_0 \)-saturated (every infinite dimensional subspace of \( C(L) \) contains \( c_0 \) isomorphically) and thus \( X \) is \( c_0 \)-saturated. Thus \( F \) governs \( \{c_0\} \) by Lemma 3.3. Let \( K = Ba(X^*) \). \( F \not\in DSC(K) \) or otherwise (Remark 4.2) \( F \in DBSC(K) \) and hence \( F \in DBSC(L) \). Using this example, it can be shown that if
$K$ is any uncountable compact metric space, there exists an $F \in B_1(K) \setminus DSC(K)$ which governs $\{c_0\}$.

**Question 4.6.** Let $F \in B_1(K)$. If $F$ governs $\{c_0\}$ does there exist a bounded sequence $(f_n) \subseteq C(K)$ converging pointwise to $F$ and a $u^*$-closed set $L \subseteq Ba[(f_n)]^*$ such that $L$ norms $[(f_n)]$ and $F|_L \in DSC(L)$? (Could $L$ be taken to be countable?)

**Question 4.7.** Let $F \in B_1(K)$. Suppose there exists $(f_n) \subseteq C(K)$, a bounded sequence converging pointwise to $F$ and satisfying $\sum_{n=1}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty$ for all $k$ in some residual set (complement of a first category set). Does $F$ govern $\{c_0\}$?

We should also mention the following result of Bourgain which gives some global information about the class $DSC(K)$.

**Proposition 4.8.** [10] Let $F \in DSC(K) \setminus C(K)$ and let $(f_n)$ be a bounded sequence in $C(K)$ converging pointwise to $F$ with $\sum_{n=1}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty$ for all $k \in K$. Then there exists a subsequence $(f'_n)$ of $(f_n)$ with $[(f'_n)]^*$ separable.

It follows that if $F \in DSC(K) \setminus C(K)$, then $F$ strictly governs the class $C$ of infinite dimensional Banach spaces with separable duals. However we don’t know that if $F$ governs $\{c_0\}$, then $F$ strictly governs $C$. (A negative answer, of course, would give a negative answer to 4.6.)

We give a somewhat different proof than that of [10].

**Proof.** We may assume that $\|f_n\| = 1$ for all $n$. As mentioned in the introduction there exists a subsequence $(f'_n)$ of $(f_n)$ which is basic and $C_1$-dominates the summing basis for some $C_1 < \infty$. It follows that $(h_n)_{n=1}^{\infty}$ is seminormalized basic where $h_1 = f'_1$ and $h_n = f'_n - f'_{n-1}$ for $n > 1$. [Indeed let $(a_i)_{i=1}^{m}$ be given and let $1 \leq n < m$ with $\|\sum_{i=1}^{n} a_i h_i \| = 1$. $\sum_{i=1}^{n} a_i h_i = (a_1 - a_2) f'_1 + \cdots + (a_{n-1} - a_n) f'_{n-1} + a_n f'_n \equiv f + a_n f'_n$. If $\|f\| \geq \frac{1}{2}$, then $\|\sum_{i=1}^{m} a_i h_i \| \geq C_2^{-1} \|f\| \geq 2^{-1} C_2^{-1}$ where $C_2$ is the basis constant of $(f'_n)$. Otherwise]
\[ |a_n| \geq \frac{1}{2} \] and so

\[
\left\| \sum_{i=1}^{m} a_i h_i \right\| = \left\| \sum_{i=1}^{m-1} (a_i - a_{i+1}) f_i' + a_m f_m' \right\|
\]

\[
\geq (C_2 + 1)^{-1} \left\| \sum_{i=n}^{m-1} (a_i - a_{i+1}) f_i' + a_m f_m' \right\|
\]

\[
\geq (C_2 + 1)^{-1} C_1^{-1} \left\| \sum_{i=n}^{m-1} (a_i - a_{i+1}) s_i + a_m s_m \right\|
\]

\[
\geq (C_2 + 1)^{-1} C_1^{-1} \left\| \sum_{i=n}^{m-1} (a_i - a_{i+1}) + a_m \right\|
\]

\[
= (C_2 + 1)^{-1} C_1^{-1} |a_n| \geq 2^{-1} (C_2 + 1)^{-1} C_1^{-1} .
\]

Also for \( k \in K \), \( \sum_{n=1}^{\infty} |h_n(k)| < \infty \). Thus \( (h_n) \) is shrinking. Indeed if \( (h_n) \) has basis constant \( C \) and \( g_n = \sum_{i=p_n+1}^{p_{n+1}} a_i h_i \) is a normalized block basis, then for \( k \in K \)

\[
|g_n(k)| \leq \left( \max_{p_{n+1} \leq i \leq p_{n+1}} |a_i| \right) \sum_{i=p_n+1}^{p_{n+1}} |h_i(k)|
\]

\[
\leq (C + 1) \min_i \| h_i \|^{-1} \sum_{i=p_n+1}^{p_{n+1}} |h_i(k)|
\]

which goes to 0 as \( n \to \infty \).

The following proposition characterizes the subclass \( PS(K) \) of \( DSC(K) \) which was defined in §1.

**Proposition 4.9.** Let \( F \in B_1(K) \). The following are equivalent.

a) \( F \in PS(K) \).

b) For all closed \( L \subseteq K \), \( F|_L \) is continuous on a relatively open dense subset of \( L \).

c) There exists \( \eta < w_1 \) and a family \( (K_\alpha)_{\alpha \leq \eta} \) of closed subsets of \( K \) with \( K_0 = K \), \( K_\eta = \emptyset \), \( K_\gamma = \bigcap_{\alpha < \gamma} K_\alpha \) if \( \gamma \) is a limit ordinal and \( K_\alpha \supseteq K_\beta \) if \( \alpha < \beta \), such that \( F|_{K_\alpha \setminus K_{\alpha+1}} \) is continuous for all \( \alpha \).

d) There exists a sequence \( (K_n) \) of closed subsets of \( K \) with \( K_n \subseteq K_{n+1} \) for all \( n \) such that \( K = \bigcup_n K_n \) and \( F|_{K_n} \) is continuous for all \( n \).
Remark 4.10. Property (c) suggests the following index for $P S(K)$:

$$I(F) = \inf \{ \eta < w_1 : \exists (K_\alpha)_{\alpha \leq \eta} \text{ satisfying (c)} \}.$$ 

Proof of 4.9. d) $\Rightarrow$ a): Let $(K_n)$ be as in d) and for $n \in \mathbb{N}$ let $f_n \in C(K_n)$ be given by $f_n = F|_{K_n}$. By the Tietze extension theorem there exists an extension of $f_n$, $\tilde{f}_n \in C(K)$, with $\|\tilde{f}_n\|_\infty \leq \|F\|_\infty$. Clearly $(\tilde{f}_n)$ is pointwise stabilizing and has limit $F$.

a) $\Rightarrow$ b): For $n \in \mathbb{N}$ set

$$L_n = \{ k \in L : f_m(k) = F(k) \text{ for } m \geq n \}$$

where $(f_n) \subseteq C(K)$, $\|f_n\| \leq \|F\|$ and $(f_n)$ is pointwise stabilizing with limit $F$. Let $G = \bigcup_n \text{int}(L_n)$. Thus $G$ is open in $L$. Also by the Baire Category theorem, $G$ is dense in $L$.

b) $\Rightarrow$ c): Let $K_0 = K$ and let $K_1 = \sim G_0$ where $G_0$ is a dense open subset of $K$ and $F$ is continuous on $G_0$. Now if $K_\alpha$ is defined choose $G_\alpha$, a dense open subset of $K_\alpha$, so that $F|_{K_\alpha}$ is continuous on $G_\alpha$ and set $K_{\alpha+1} = K_\alpha \setminus G_\alpha$. At limit ordinals $\gamma$, set $K_\gamma = \bigcap_{\alpha < \gamma} K_\alpha$. Since $K$ is a separable metric space, $K_\eta = \emptyset$ for some $\eta < w_1$.

c) $\Rightarrow$ d): Let $(K_\alpha)_{\alpha \leq \eta}$ be as in c). Let $\mathcal{E}_n \downarrow 0$ and for each $n$ set $K_{\alpha,n} = \{ k \in K_\alpha : d(k, K_{\alpha+1}) \geq \mathcal{E}_n \}$ where $d$ is the metric on $K$. Let $K_n = \bigcup_{\alpha < \eta} K_{\alpha,n}$. We note that $K_n$ is closed. Indeed let $(k_i) \subseteq K_n$ converge to $k$. Then there exists $\alpha < \eta$ so that $k \in K_\alpha$ but $k \notin K_{\alpha+1}$. We claim that $k_i \in K_{\alpha,n}$ for sufficiently large $i$ and thus $k \in K_{\alpha,n}$ since $K_{\alpha,n}$ is closed. To see this note first that if $k_i \notin K_\alpha$, then $d(k_i, k) \geq \mathcal{E}_n$. Thus for large $i$, $k_i \in K_\alpha$ and (since $k \notin K_{\alpha+1}$) $k_i \notin K_{\alpha+1}$. Hence $k_i \in K_{\alpha,n}$ for large $i$ (since the $K_{\alpha,n}$’s are disjoint in $n$).

Finally $F|_{K_n}$ is continuous, for if $(k_i) \subseteq K_n$ and $(k_i)$ converges to $k \in K_{\alpha,n}$, then by the above argument $k_i \in K_{\alpha,n}$ for large $i$ and $F|_{K_{\alpha,n}}$ is continuous.

We end this section with an improvement of Proposition 4.8 in a special case.

Proposition 4.11. Let $K$ be a compact metric space and let $F$ be a simple Baire-1 function on $K$. Then there exists $(f_n) \subseteq C(K)$ converging pointwise to $F$ such that $[(f_n)]$ embeds into $C(L)$ for some countable compact space $L$.  

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Proof. First we consider the case where \( K \) is totally disconnected. Choose \( \mathcal{E}_0 > 0 \) so that if \( F(k_1) \neq F(k_2) \), then \( |F(k_1) - F(k_2)| > \mathcal{E}_0 \). Let \( K_\alpha = K_\alpha(F, \mathcal{E}_0) \) for \( \alpha \leq \eta \) with \( K_\eta = \emptyset \).

By our choice of \( \mathcal{E}_0 \), \( F|_{K_\alpha \setminus K_{\alpha+1}} \) is continuous (with respect to \( K_\alpha \)) for all \( \alpha < \eta \).

Choose a countable partition \((D_j)\) of \( K \) into closed sets with the following properties.

a) \( \text{diam } D_j \to 0 \)

b) for each \( j \), \( D_j \) is a relatively clopen subset of \( K_\alpha \setminus K_{\alpha+1} \) for some \( \alpha < \eta \) such that \( F|_{D_j} \) is constant.

This can be done as follows. For each \( \alpha \) choose a finite partition of relatively clopen subsets of \( K_\alpha \setminus K_{\alpha+1} \) such that \( F \) is constant on each set of the partition. Each such set is relatively open in \( K_\alpha \) and thus may in turn be partitioned into a countable number of relatively clopen subsets of \( K_\alpha \). List all the sets thus obtained for all \( \alpha < \eta \) as \((C_i)_{i=1}^\infty\).

Each \( C_i \) is closed in \( K \) and thus may in turn be partitioned into a finite number of closed subsets of diameter not exceeding \( 1/i \). We list all these sets as \((D_j)_{j=1}^\infty\).

Let \( L = K/\{D_j\} \) be the quotient space of \( K \). Since each \( D_j \) is closed and \( \text{diam } D_j \to 0 \), \( L \) is compact metric. For \( n \in \mathbb{N} \) choose \( \hat{f}_n \in C(L) \) with \( \|\hat{f}_n\|_\infty \leq \|F\|_\infty \) and \( \hat{f}_n(D_j) \) equal to the constant value of \( F|_{D_j} \) for \( j \leq n \). Let \( \phi : K \to L \) denote the quotient map and let \( f_n = \hat{f}_n \circ \phi \). Clearly \( f_n \in C(K) \), \( \|f_n\| \leq \|F\| \) and \((f_n)\) converges pointwise to \( F \). Also \([\{f_n\}] \) is isometric to \([\{\hat{f}_n\}] \subseteq C(L)\).

For the general case let \( \phi : \Delta \to K \) be a continuous surjection and let \( F \) be a simple Baire-1 function on \( K \). By the first part of the proof there exist \((f_n) \subseteq C(\Delta)\) converging pointwise to \( F \circ \phi \) and a countable compact metric space \( L \) such that \([\{f_n\}] \hookrightarrow C(L)\). Let \((g_n)\) be a bounded sequence in \( C(K) \) converging pointwise to \( F \). By Lemma 3.3 there exist convex block subsequences \((h_n)\) and \((d_n)\) of \((g_n)\) and \((f_n)\), respectively, such that \( \sum \|g_n \circ \phi - d_n\| < \infty \). Thus \([\{g_n\}] \cong [(g_n \circ \phi)] \hookrightarrow C(L)\).

Question 4.12. Does Proposition 4.11 remain true if we only assume \( F \in PS(K) \) or even \( F \in DSC(K) \)? Note that if \( F \) satisfies the conclusion of 4.11, \( F \) strictly governs the class of \( c_0 \)-saturated spaces, while it is not clear that \( DSC \) functions have this property.
5. The Baire-1 Solar System.

In this section we shall examine the relationships between the various classes of Baire-1 functions which we have defined. We begin with a result which follows easily from the Banach space theory — that developed above and some examples presented in later sections.

**Proposition 5.1.** Let $K$ be an uncountable compact metric space. Then

\[(5.1) \quad C(K) \subsetneq DBSC(K) \subsetneq B_{1/4}(K) \subsetneq B_{1/2}(K) \subsetneq B_1(K).\]

**Proof.** Since $C(K)$ and $C(K')$ are isomorphic whenever $K$ and $K'$ are both uncountable compact metric spaces [29], it suffices to separately consider each of the inclusions in (5.1). Thus if we show $C(K') \neq DBSC(K')$ for some uncountable compact metric space $K'$, then $C(K) \neq DBSC(K)$ as well. Indeed if $j : C(K) \to C(K')$ is an onto isomorphism, then $\tilde{j} = j^{**}|_{B_1(K)} : B_1(K) \to B_1(K')$. is an onto isomorphism satisfying $\tilde{j}(DBSC(K)) = DBSC(K')$, $\tilde{j}(B_{1/4}(K)) = B_{1/4}(K')$ and $\tilde{j}(B_{1/2}(K)) = B_{1/2}(K')$.

For the first inclusion, $C(K) \subsetneq DBSC(K)$, let $X = c_0$. Then $K = (Ba(X^*), w^*)$ is uncountable compact metric and, as is well known, $X^{**} \subseteq DBSC(K)$. In particular if $F \in X^{**} \setminus X$, then $F \in DBSC(K) \setminus C(K)$.

The fact that $B_{1/4}(K) \subsetneq DBSC(K)$ follows from Theorem A(b) and our example in §6 where we produce a nonreflexive separable Banach space $X$ not containing $c_0$ such that $X^{**} \subseteq B_{1/4}(K)$, where $K = Ba(X^*)$.

For the next inclusion let $X = J$, the James space. $J$ is not reflexive and has no spreading model isomorphic to $c_0$ or $\ell_1$ [1]. Thus if $K = (Ba(J^*), w^*)$, then $X^{**} \setminus X \subseteq B_{1/2}(K) \setminus B_{1/4}(K)$ by virtue of Theorem B.

For the last inclusion let $Y$ be the quasi-reflexive space of order 1 (see the proof of Proposition 6.3) whose dual is $J(e_i)$, where $(e_i)$ is the unit vector basis of Tsirelson’s space. It is proved in [32] that the only spreading models of $Y$ are isomorphic to $\ell_1$. Thus by Theorem 3.7, if $Y^{**} = Y \oplus \langle F \rangle$ and $K = Ba(Y^*)$, then $F \notin B_{1/2}(K)$. An alternative method would be to consider the quasi-reflexive spaces $Q_γ$ constructed in Proposition 2.8.

**Remark 5.2.** How does the class $DSC(K)$ relate to the classes in (5.1)? Of course we always have $DBSC(K) \subseteq DSC(K)$ and in fact for $K$ an uncountable compact metric

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space we have the following diagram.

Thus \( DSC(K) \) is an asteroid in the Baire-1 solar system. Indeed our proof of Proposition 5.1 along with Theorem 4.1 yields that \( B_{1/4}(K) \setminus DSC(K) \neq \emptyset, B_{1/2}(K) \setminus [DSC(K) \cup B_{1/4}(K)] \neq \emptyset \) and \( B_1(K) \setminus [DSC(K) \cup B_{1/2}(K)] \neq \emptyset \). The fact that \( DSC(K) \cap B_{1/4}(K) \setminus B_{1/2}(K), DSC(K) \cap B_{1/2}(K) \setminus B_{1/4}(K) \) and \( DSC(K) \cap B_{1/4}(K) \setminus DBSC(K) \) are all nonempty follows from Proposition 5.3 below.

We now turn to the case where \( K \) is a countable compact metric space. In this setting we have, of course, \( DSC(K) = B_1(K) \). However if \( K \) is large enough, the classes in (5.1) are still distinct. Since every countable compact metric space is homeomorphic to some countable ordinal, given the order topology [30], we confine ourselves to this setting.

**Proposition 5.3.**

a) If \( K = \omega^\omega + \), then \( B_{1/4}(K) \setminus DBSC(K) \neq \emptyset \).

b) If \( K = \omega^+ \), then \( B_{1/2}(K) \setminus B_{1/4}(K) \neq \emptyset \).

c) If \( K = \omega^+ \), then \( B_1(K) \setminus B_{1/2}(K) \neq \emptyset \).

d) If \( K = \omega^+ \), then \( DBSC(K) \setminus C(K) \neq \emptyset \).

Before proving this proposition we need some terminology. Recall that an indicator function \( 1_A \) is Baire-1 iff \( A \) is ambiguous (simultaneously \( F_\sigma \) and \( G_\delta \)). Thus if \( A \subseteq K \) where \( K \) is countable compact metric, then \( 1_A \in B_1(K) \). We begin with a discussion of such functions.

Let \( \delta \) be a countable compact ordinal space (in its order topology). Recursively we define \( I_0 = \emptyset, I_1 = \{ x \in \delta : x \) is an isolated point of \( \delta \} \), and for \( \alpha > 1, I_\alpha = \)
\( \{ x \in \delta \setminus \bigcup_{\beta < \alpha} I_\beta : x \text{ is an isolated point of } \delta \setminus \bigcup_{\beta < \alpha} I_\beta \} \). The \( I_\alpha \)'s are just the relative complements of the usual derived sets.

Let us say an ordinal is \textit{even} if it is of the form \( \gamma + 2n \) for some \( n \in \mathbb{N} \) where \( \gamma = 0 \) or \( \gamma \) is a limit ordinal. Let \( F_\delta = 1_{A_\delta} \) where \( A_\delta = \bigcup_{\alpha \text{ even}} I_\alpha \). We have

\( ^\circ 1) \| F_{\omega^m+} \|_\infty = 1 \) and \( |F_{\omega^m+}|_D = n \).
\( ^\circ 2) |F_\delta|_D = \infty \) if \( \delta \geq \omega^\omega \).

\( ^\circ 1) \) implies \( ^\circ 2) \) trivially. To see \( ^\circ 1) \), one first notes that \( K_n(F_{\omega^m+}, 1) \neq \emptyset \). Indeed, \( K_\alpha(F_\delta, 1) \) is just the \( \alpha^{th} \) derived set of \( \delta \). Hence \( |F_{\omega^m+}|_D \geq n \) by the proof of Lemma 2.4. We leave the reverse inequality to the reader.

\textit{Definition.} We say that a function \( F : \omega^m+ \to \mathbb{R} \) is of \textit{type} 0 if \( F = n^{-1}F_{\omega^m+} \). The domain of \( F \), \( \omega^m+ \), is called a \textit{space of type} 0.

Thus if \( F \) is a function of type 0 with domain \( \omega^m+ \), \( |F|_D = 1 \) and \( \|F\|_\infty = n^{-1} \).

More generally for \( n \in \mathbb{N} \) we have the

\textit{Definition.} A class of real valued functions \( \mathcal{F}_n \) defined on countable compact metric spaces is said to be of \textit{type} \( n \) if

a) For \( F \in \mathcal{F}_n \), \( |F|_D \geq n \).

b) For \( F \in \mathcal{F}_n \), \( F \) is the uniform limit of \( (F_m) \) with \( \sup_m |F_m|_D \leq 1 \).

c) For each \( \varepsilon > 0 \), there is an \( F \in \mathcal{F}_n \) with \( \|F\|_\infty < \varepsilon \).

The domain of \( F \in \mathcal{F}_n \) is called a \textit{space of type} \( n \).

\textbf{Lemma 5.4.} For \( n \in \mathbb{N} \cup \{0\} \) there exists a class \( \mathcal{F}_n \) of functions of type \( n \).

\textit{Proof.} We have seen that \( \mathcal{F}_0 \) exists. Suppose \( \mathcal{F}_n \) exists. To obtain functions \( F \in \mathcal{F}_{n+1} \) we begin with a function \( G \in \mathcal{F}_0 \) defined on a set \( K \). Let \( (t_i)_{i=1}^\infty \) be a list of the isolated points of \( K \). We enlarge \( K \) as follows. To each \( t_i \) we adjoin a sequence of disjoint clopen sets \( K_1^{i}, K_2^{i}, \ldots \) clustering only at \( t_i \). Each of the \( K_j^{i} \)'s is a space of type \( n \) supporting a function \( F_j^{i} \) of type \( n \) with \( \|F_j^{i}\|_\infty \leq (i + j + m)^{-1} \). Here \( m \in \mathbb{N} \) is arbitrary but fixed. \( K_{n+1} \), the new space of type \( n + 1 \), is this enlarged space. Set

\[
F(t) = \begin{cases} 
G(t), & t \in K \\
F_j^{i}(t), & t \in K_{ij}
\end{cases}
\]
Let $\mathcal{F}_{n+1}$ be the set of all such $F$’s thusly obtained. We must check that $\mathcal{F}_{n+1}$ satisfies a) and b) with $n$ replaced by $n+1$ (c) is immediate). b) holds since $F$ is the uniform limit of $(F_k)$ where

$$F_k(t) = \begin{cases} G(t), & t \in K \\ F^i_j(t), & t \in K^i_j \text{ with } i + j \leq k \\ 0, & \text{otherwise} \end{cases}$$

and each $F_k$ is the uniform limit of $(F_{k,n})_{n=1}^{\infty}$ where $|F_{k,n}|_D \leq 1$ for all $n$.

Finally we check a). Let $(f_m)_{0}^{\infty} \subseteq C(K_{n+1})$, $f_0 \equiv 0$, converge pointwise to $F$. Since $F|_K = G$ and $|G|_D = 1$, for $\varepsilon > 0$ there exist $t_{i_0} \in K$ and $k \in \mathbb{N}$ with $\sum_{i=0}^{k-1} |f_{i+1}(t_{i_0}) - f_i(t_{i_0})| > 1 - \varepsilon$. Moreover by the nature of $G$ we may assume $|G(t_{i_0})| < \varepsilon$. Since the $K^{i_0}_j$’s cluster at $t_{i_0}$ and each $f_i$ is continuous there exists $j_0 \in \mathbb{N}$ so that for $t \in K^{i_0}_{j_0}$, $\sum_{i=0}^{k-1} |f_{i+1}(t) - f_i(t)| > 1 - \varepsilon$ and $|f_k(t)| < \varepsilon$. But on $K^{i_0}_{j_0}$, $(f_m)$ converges pointwise to $F_{i_0}^{j_0}$ and $|F_{i_0}^{j_0}|_D \geq n$. Thus there exists $t \in K^{i_0}_{j_0}$ with

$$|f_{k+1}(t)| + \sum_{i>k} |f_{i+1}(t) - f_i(t)| > n - \varepsilon.$$

It follows that

$$\sum_{i=0}^{\infty} |f_{i+1}(t) - f_i(t)| > n + 1 - 3\varepsilon$$

which proves a).

Remark 5.4. Our proof yields that the spaces of type-$n$ can be constructed within $\omega^{\omega^{(n+1)+}}$.

Proof of Proposition 5.3. a) Let $K = \omega^{\omega^2}+$ and choose (by Remark 5.4) a sequence $(K_n)_{n=0}^{\infty}$ of disjoint clopen subspaces of $K$ with $K_n$ of type-$n$. Let $F_n$ be a function of type-$n$ supported on $K_n$ with $\|F_n\|_\infty \to 0$ and let $F$ be the sum of the $F_n$’s. Clearly $|F|_D = \infty$ since $|F_n|_D \geq n$. Yet $F$ is the uniform limit of a sequence of functions with $|\cdot|_D$ not exceeding 1.

b) Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of disjoint clopen subspaces of type-0 of $\omega^{\omega^+} + K$ such that $K_n$ supports a function $F_n$, which is a multiple of a function of type-0, with $\|F_n\|_\infty \leq n^{-1}$ and $|F_n|_D \geq n$. Define

$$F(t) = \begin{cases} F_n(t), & \text{if } t \in K_n \\ 0, & \text{otherwise} \end{cases}$$
Clearly $F \in B_{1/2}(K) \setminus B_{1/4}(K)$.

c) The type-0 function $F_{\omega^+}$ is not Baire-1/2.

d) $F_{\omega^+}$ is DBSC.

It is easy to check that the results of Proposition 5.3 are best possible.

6. A Characterization of $B_{1/4}(K)$ and an Example.

In this section we give an example which shows that functions of class Baire-1/4 need not govern $\{e_0\}$. Thus Theorem B(b) is best possible. Before giving the example we give a sufficient (and necessary) criterion for a function to be Baire-1/4.

**Theorem 6.1.** Let $K$ be a compact metric space and let $F \in B_1(K)$. Then $F \in B_{1/4}(K)$ iff there exists a $C < \infty$ such that for all $\varepsilon > 0$ there exists a sequence $(S_n)_{n=0}^{\infty} \subseteq C(K)$, $S_0 \equiv 0$, with $S_n(k) \to F(k)$ for all $k \in K$ and such that for all subsequences $(n_i)$ of $\{0\} \cup \mathbb{N}$ and $k \in K$,

\[
(6.1) \quad \sum_{j \in B((n_i), k)} |S_{n_{j+1}}(k) - S_{n_j}(k)| \leq C .
\]

Here $B((n_i), k) = \{j : |S_{n_{j+1}}(k) - S_{n_j}(k)| \geq \varepsilon\}$.

**Proof.** First assume $F \in B_{1/4}(K)$, let $\varepsilon > 0$ and let $\varepsilon_n \downarrow 0$. By the proof of Theorem B(b) there exists $(f_n)^{\infty}_{n=0} \subseteq C(K)$, $f_0 \equiv 0$, converging pointwise to $F$ with the following property. For each $m \in \mathbb{N}$, there exists $(h_j^m)_{j=0}^{\infty} \subseteq C(K)$ with $h_0^m \equiv 0$ and

\[
(6.2) \quad \sum_{j=0}^{\infty} |h_{j+1}^m(k) - h_j^m(k)| \leq M \equiv 2|F|_{1/4} , \quad \text{for } k \in K .
\]

Furthermore $\|h_j^m - f_j\|_{\infty} \leq \varepsilon_m$ for $j \geq m$.

Let $\varepsilon > 0$ and fix $m$ with $4\varepsilon_m < \varepsilon$. Let $(S_n)^{\infty}_{n=0} = (0, f_m, f_{m+1}, \ldots)$, and let $(n_i)$ be a subsequence of $\{0\} \cup \mathbb{N}$ and let $k \in K$ be fixed. Then

\[
(6.3) \quad \sum_{j \in B((n_i), k)} |S_{n_{j+1}}(k) - S_{n_j}(k)| \leq \sum_{j=0}^{\infty} |h_{j+1}^m(k) - h_j^m(k)| + 2\varepsilon_m \#B((n_i), k) .
\]

Since $|f_p(k) - f_q(k)| \geq \varepsilon$ implies for $p > q \geq m$ or $q = 0$ that $|h_p^m(k) - h_q^m(k)| \geq \varepsilon - 2\varepsilon_m > \varepsilon/2$, (6.2) yields that $\#B((n_i), k) \leq 2M/\varepsilon$. Thus (6.3) yields (6.1) with $C = 2M = 4|F|_{1/4}$.
For the converse, let $C > \varepsilon > 0$ and let $(S_n)_0^\infty \subseteq C(K)$, $S_0 \equiv 0$, converge pointwise to $F$ and satisfy (6.1) for any subsequence $(n_i)$ of $\{0, 1, 2, \ldots\}$ and any $k \in K$. For $k \in K$ we linearly extend the sequence $(S_n(k))_0^\infty$ to $(S_r(k))_{r \geq 0}$. Precisely, if $r = \lambda n + (1 - \lambda)(n + 1)$ we set $S_r(k) = \lambda S_n(k) + (1 - \lambda)S_{n+1}(k)$. Since the $S_n$’s are continuous, $S_r \in C(K)$ as well. Furthermore, if $0 \leq r_1 < r_2 < r_3 < \ldots$, $k \in K$ and $B = B((r_i), k) = \{j : |S_{r_{j+1}}(k) - S_{r_j}(k)| \geq \varepsilon\}$, then

\begin{equation}
\sum_{j \in B} |S_{r_{j+1}}(k) - S_{r_j}(k)| \leq 3C.
\end{equation}

Indeed if $J_n = \{j \in \mathbb{N} : n \leq r_j < r_{j+1} \leq n + 1\} \neq \emptyset$, then \(\varepsilon \leq \sum_{j \in J_n} |S_{r_{j+1}}(k) - S_{r_j}(k)| \leq |S_{n+1}(k) - S_n(k)|\). If $j \in B \setminus \bigcup_n J_n$, there exists integers $\ell_j$ and $m_j$ with $\ell_j - 1 \leq r_j < \ell_j \leq m_j < r_{j+1} \leq m_{j+1}$. Thus by linearity for some choice of $p_j \in \{\ell_j - 1, \ell_j\}$ and $q_j \in \{m_j, m_j + 1\}$ we have \(\varepsilon \leq |S_{r_{j+1}}(k) - S_{r_j}(k)| \leq |S_{q_j}(k) - S_{p_j}(k)|\). Thus

\begin{align*}
\sum_{j \in B} |S_{r_{j+1}}(k) - S_{r_j}(k)| & \leq \sum_{\{n : J_n \neq \emptyset\}} |S_{n+1}(k) - S_n(k)| \\
& + \sum_{2j \in B \setminus \bigcup_n J_n} |S_{q_{2j}}(k) - S_{p_{2j}}(k)| + \sum_{2j+1 \in B \setminus \bigcup_n J_n} |S_{q_{2j+1}}(k) - S_{p_{2j+1}}(k)| \leq 3C.
\end{align*}

We shall construct a sequence $(f_n)_n^\infty \subseteq C(K)$, $f_0 \equiv 0$, such that for $k \in K$,

\begin{equation}
\sum_{n=0}^\infty |f_{n+1}(k) - f_n(k)| \leq 4C \quad \text{and}
\end{equation}

\begin{equation}
\text{if } H \text{ is the pointwise limit of } (f_n) \text{ then } \|H - F\|_\infty \leq 5\varepsilon.
\end{equation}

This will complete the proof.

Each $f_n$ shall be an average of functions $S_t$ where $t : K \to [0, \infty)$ is continuous and $S_t(k) \equiv S_{t(k)}(k)$ for $k \in K$. Let $f_0 = S_0 \equiv 0$. Let $\alpha_1^1 : [0, \infty) \to [0, 1]$ be identically 0 on $[0, \varepsilon]$, identically 1 on $[3\varepsilon/2, \infty)$ and linear on $[\varepsilon, 3\varepsilon/2]$. Let $\alpha_2^1 : [0, \infty) \to [0, 1]$ be identically 0 on $[0, 3\varepsilon/2]$, identically 1 on $[2\varepsilon, \infty)$ and linear on $[3\varepsilon/2, \varepsilon]$. For $i = 1, 2$ let $t_i(k) = \alpha_i^1(|S_1(k)|)$. Let $f_1 = 2^{-1}(S_{t_1} + S_{t_2})$. We next define continuous functions $t_{i,j}$ for $i = 1, 2$ and $j = 1, 2, 3, 4$ by $t_{i,j}(k) = t_{i,k} + \alpha_2^2(|S_2(k) - S_{t_i}(k)|)(2 - t_i(k))$. Here $\alpha_2^2 : [0, \infty) \to [0, 1]$ is identically 0 on $[0, (4 + j - 1)\varepsilon/4]$, identically 1 on $[(4 + j)\varepsilon/4, \infty)$ and linear on $[(4 + j - 1)\varepsilon/4, (4 + j)\varepsilon/4]$. Set $f_2 = 8^{-1}\sum_{i=1}^2 \sum_{j=1}^4 S_{t_{i,j}}$. [30]
In general if \( f_n = 2^{-1}2^{-2} \cdots 2^{-n} \sum S_{t_1, \ldots, t_n} \), where the indices of summation range over \( \{(i_1, \ldots, i_n) : 1 \leq i_j \leq 2^j \} \). We define \( t_{i_1, \ldots, i_{n+1}} \) for \( 1 \leq i_{n+1} \leq 2^{n+1} \) by

\[
t_{i_1, \ldots, i_{n+1}}(k) = t_{i_1, \ldots, i_n}(k) + \alpha^{n+1}_{i_{n+1}}(k - S_{t_{i_1, \ldots, i_n}}(k))(n + 1 - t_{i_1, \ldots, i_n}(k)).
\]

The functions \( \alpha^{n+1}_{i_1} \) for \( 1 \leq j \leq 2^{n+1} \) are defined as before to be identically 0 on \([0, \varepsilon + (j - 1)\varepsilon 2^{-n-1}]\), identically 1 on \([\varepsilon + j\varepsilon 2^{-n-1}], \varepsilon + j\varepsilon 2^{-n-1}]\) and linear on \([\varepsilon + (j - 1)\varepsilon 2^{-n-1}, \varepsilon + j\varepsilon 2^{-n-1}]\).

The point of the construction is this. For \( k \in K \) and \( (i_1, \ldots, i_n) \) fixed, \( |S_{t_{i_1, \ldots, i_n+1}}(k) - S_{t_{i_1, \ldots, i_n}}(k)| \) is either 0 or a number exceeding \( \varepsilon \) for all but perhaps one choice of \( i_{n+1} \). [This is because the nonconstant parts of the \( \alpha^{n+1}_{i_j} \)'s are disjointly supported.] Also except for at most one value of \( i_{n+1} \), if \( |S_{t_{i_1, \ldots, i_n+1}}(k) - S_{t_{i_1, \ldots, i_n}}(k)| \geq \varepsilon \) then \( S_{t_{i_1, \ldots, i_n+1}}(k) = S_{n+1}(k) \).

We next check (6.5). Fix \( k \in K \) and \( m \in \mathbb{N} \). A simple calculation using the triangle inequality shows that

\[
\sum_{n=0}^{m} |f_{n+1}(k) - f_n(k)| \leq \text{AVE} \sum_{n=0}^{m} |S_{t_{i_1, \ldots, i_{n+1}}}(k) - S_{t_{i_1, \ldots, i_n}}(k)|
\]

where the average is taken over \( \{(i_1, \ldots, i_m) : 1 \leq i_j \leq 2^j \text{ for all } j \} \). If we fix \( (i_1, \ldots, i_m) \) and let

\[
B = \{ n \leq m \mid |S_{t_{i_1, \ldots, i_{n+1}}}(k) - S_{t_{i_1, \ldots, i_n}}(k)| \geq \varepsilon \}
\]

then

\[
\sum_{n \in B} |S_{t_{i_1, \ldots, i_{n+1}}}(k) - S_{t_{i_1, \ldots, i_n}}(k)| \leq 3C
\]

by (6.4).

Now for \( 1 \leq n \leq m \) fixed, the percentage of terms in the “AVE” of (6.7) for which \( 0 < |S_{t_{i_1, \ldots, i_{n+1}}}(k) - S_{t_{i_1, \ldots, i_n}}(k)| < \varepsilon \) is at most \( 2^{-n-1} \). It follows that

\[
\text{AVE} \sum_{n=0}^{m} |S_{t_{i_1, \ldots, i_{n+1}}}(k) - S_{t_{i_1, \ldots, i_n}}(k)| \leq 3C + 2^{-1}\varepsilon + \cdots + 2^{-m-1}\varepsilon
\]

and (6.5) follows from this since \( \varepsilon < C \).

(6.5) implies \( (f_n) \) is pointwise convergent to some function \( H \). For fixed \( k \in K \) choose \( m \in \mathbb{N} \) so that \( 2^{-m}C < \varepsilon \), \( |S_m(k) - F(k)| < \varepsilon \) and \( |f_m(k) - H(k)| < \varepsilon \). We claim that \( |f_m(k) - S_m(k)| < 3\varepsilon \), which proves (6.6). Indeed

\[
f_m(k) = \text{AVE} S_{t_{i_1, \ldots, i_m}}(k) \quad \text{and} \quad C \geq |S_{t_{i_1, \ldots, i_m}}(k) - S_m(k)| \geq 2\varepsilon
\]
for at most $2^{-m} \# \{(i_1, \ldots, i_m) : i_j \leq 2^j \}$ choices of $(i_1, \ldots, i_m)$. Thus $|f_m(k) - S_m(k)| \leq 2 \varepsilon + 2^{-m}C < 3 \varepsilon$.

**Remark 6.2.** Let $F \in B_1(K)$. Our proof shows that $F \in B_{1/4}(K)$ iff there exists $C < \infty$ and $(S_n)_{n=0}^{\infty} \subseteq C(K)$, $S_0 \equiv 0$, converging pointwise to $F$ such that for all $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that if $(n_i)$ is any subsequence of $\{0, m, m + 1, \ldots \}$ then (6.1) holds.

**Proposition 6.3.** There exists a compact metric space $K$ and $F \in B_{1/4}(K)$ which does not govern $\{c_0\}$.

**Proof.** Let $(e_i)$ be the unit vector basis of the Tsirelson space $T$ constructed in [17] (see also [11]) and let $X = J(e_i)$ be its “Jamesification” as described in [6]. For completeness we recall the definition of $X$. Let $c_{oo}$ be the linear space of all finitely supported functions $x : \mathbb{N} \rightarrow \mathbb{R}$ and for $n \in \mathbb{N}$ define $S_n : c_{oo} \rightarrow \mathbb{R}$ by $S_n(x) = \sum_{i=1}^{n} x(i)$. Let $S_0 \equiv 0$. For $x \in c_{oo}$ let

$$\|x\| = \sup \left\{ \left\| \sum_{i=1}^{m} (S_{n_i} - S_{p_i-1})(x)e_{p_i} \right\|_T : 1 \leq p_1 \leq n_1 < p_2 \leq n_2 < \cdots < p_m \leq n_m \right\}.$$

Let $X$ be the completion of $(c_{oo}, \| \cdot \|)$.

As shown in [6], the unit vectors $(u_i)$ form a boundedly complete normalized basis for $X$. Thus $X = Y^*$ (where $Y = [(u_i^*)] \subseteq X^*$). Furthermore it was shown that $Y$ is quasi-reflexive and $Y^{**}$ has a basis given by $\{S, u_1^*, u_2^*, \ldots \}$, where

$$S \left( \sum a_i u_i \right) = \sum_{1}^{\infty} a_i.$$

Of course $(u_i^*)$ are the biorthogonal functionals to $(u_i)$ and $S$ is the weak* limit in $Y^{**}$ of $(S_n)$.

Let $K = Ba(X) = Ba(Y^*)$ in the weak* topology (of $Y^*$). Since $Y$ does not contain $c_0$, our example will be complete if we can prove that $S \in B_{1/4}(K)$. By Theorem 6.1 it suffices to prove that if $\varepsilon > 0$ then for $m \in \mathbb{N}$ with $m > 2/\varepsilon$, if $x \in Ba(X)$ and $(n_i)$ is a subsequence of $\{m, m + 1, m + 2, \ldots \}$, then

$$\sum_{j \in B} |S_{n_{j+1}}(x) - S_{n_j}(x)| \leq 2.$$
where

\[ B = \{ j : |S_{n_{j+1}}(x) - S_{n_j}(x)| \geq \varepsilon \} . \]

We first note that \( \#B < m \). Indeed if \( \#B \geq m \), then by the properties of \( T \),

\[
1 \geq \|x\| \geq \left\| \sum_{j \in B} (S_{n_{j+1}}(x) - S_{n_j}(x))e_{n_j} \right\|_T,
\]

\[
\geq 2^{-1}m\varepsilon ,
\]
a contradiction. The last inequality is due to the fact that \( \| \sum_{A} a_i e_i \|_T \geq 2^{-1} \sum_{A} |a_i| \) provided \( \min A \leq \#A \).

Thus \( m \leq \min B \leq \#B \) and so

\[
\sum_{j \in B} |S_{n_{j+1}}(x) - S_{n_j}(x)| \leq 2 \left\| \sum_{j \in B} (S_{n_{j+1}}(x) - S_{n_j}(x))e_{n_j} \right\|
\]

\[
\leq 2\|x\| \leq 2 .
\]

Remark 6.4. Our proof of Proposition 6.3 shows that there exists a quasi-reflexive (of order one) Banach space \( Y \) such that if \( K = Ba(Y^*) \) then \( Y^{**} \setminus Y \subseteq B_{1/4}(K) \). In particular, it follows that there exists an \( F \in B_{1/4}(K) \setminus C(K) \) which strictly governs the class of quasi-reflexive Banach spaces.
7. Some Bad Baire-1/2 Functions.

In this section we show that functions of class Baire-1/2 need not be that nice.

**Proposition 7.1.** There exists a compact metric space $K$ and $F \in B_{1/2}(K)$ which governs $\{\ell_1\}$.

**Remark 7.2.** The first example of an $F \in B_1(K)$ which governs $\{\ell_1\}$ was due to Bourgain [9,10]. His ingenious construction forms the motivation behind our next example (Proposition 7.3). Another example of such an $F$ appears in [2]. While the example of [2] can be shown to be Baire-1/2, we prefer to present a very slight modification.

**Proof.** Let $(e_n)$ be the unit vector basis of a Lorentz sequence space $d_{w,1}$ (see e.g., [27]). Let $J(e_i)$ be the Jamesification of $(e_n)$ (see [6]) and let $(u_i)$ be the unit vector basis of $J(e_i)$. Thus

$$
\left\| \sum_{i=1}^{k} a_i u_i \right\| = \sup \left\{ \left\| \sum_{i=1}^{p} \left( \sum_{j=n_i}^{m_i} a_j \right) e_i \right\|_{d_{w,1}} \mid 1 \leq n_1 \leq m_1 < n_2 \leq m_2 < \cdots < n_p \leq m_p \right\}.
$$

$(u_i)$ is a normalized spreading basis for $J(e_i)$ which is not equivalent to the unit vector basis of $\ell_1$ and thus by [36], $(u_i)$ is weak Cauchy. Furthermore by standard block basis arguments one can show that $J(e_i)$ is hereditarily $\ell_1$. Also if $F$ is defined by $u_i \rightarrow F$ weak* then $F \in B_{1/2}(K)$ where $K = Ba(J(e_i)^*)$. But this is immediate by Theorem B(a) since $(u_i)$, being its own spreading model, does not have $\ell_1$ as a spreading model. The fact that $F$ governs $\ell_1$ follows from Lemma 3.3. Indeed if $(f_n)$ is a bounded sequence in $C(K)$ converging pointwise to $F$, then some convex block subsequence of $(f_n)$ is a basic sequence equivalent to some convex block subsequence of $(u_i)$. Since $[(u_i)]$ is hereditarily $\ell_1$, $\ell_1 \hookrightarrow [(f_n)]$.

**Proposition 7.3.** There exists a compact metric space $K$ and $F \in B_{1/2}(K)$ such that $F$ does not govern $\{\ell_1\}$ yet $F$ strictly governs $\{X : X$ is separable and $X^*$ is not separable$\}$.

**Remark 7.4.** In [33] a function $F \in B_1(K) \setminus B_{1/2}(K)$ was constructed satisfying the conclusion of Proposition 7.3. The construction we now present will be a modification of that example.
Proof of Proposition 7.3. We begin by defining a Banach space $Y$. (The space $Y$ was first defined in [34]) Let $D = \{\phi\} \cup \bigcup_n \{0,1\}^n$ be the dyadic tree with its natural order (see Remark 4.2) and let $(K_\alpha)_{\alpha \in D}$ be the natural clopen base for the Cantor set $\Delta$. For $f \in C(K_\alpha)$ we let $\tilde{f} \in C(\Delta)$ be given by $\tilde{f}(t) = f(t)$ for $t \in K_\alpha$ and $\tilde{f}(t) = 0$ otherwise. Let

$$
Y = \left\{ (f_\alpha)_{\alpha \in D} \mid f_\alpha \in C(K_\alpha) \text{ for all } \alpha \in D \right\}
$$

and for $f \in C(K_\alpha)$ we let

$$
\|f\|_{Y} \equiv \sup \left\{ \left( \sum_{k=1}^\ell \sum_{\alpha \in S_k} \|\tilde{f}_\alpha\|_\infty^2 \right)^{1/2} : (S_k)_{k=1}^\ell \text{ are disjoint segments in } D \right\} < \infty.
$$

$Y$ is a Banach space under the given norm.

We shall construct a weak Cauchy sequence $(g_n) \subseteq Y$ with weak* limit $F$ such that

(7.1) \quad $\ell_1 \not\hookrightarrow [(g_n)]$,

(7.2) \quad $[(h_n)]^*$ is nonseparable for every convex block subsequence $(h_n)$ of $(g_n)$ and there exists a weak* closed set $K \subseteq Ba(Y^*)$ such that

(7.3) \quad $K$ norms $[(g_n)]$ and $F|_K \in B_{1/2}(K)$.

The proposition follows immediately from (7.1)–(7.3). Indeed to see that $F$ governs $\{X : X$ is separable and $X^*$ is nonseparable\}, let $(f_n)$ be a bounded sequence in $C(K)$ converging pointwise to $F$. By Lemma 3.3 there exist convex block subsequences $(d_n)$ and $(h_n)$ of $(f_n)$ and $(g_n)$, respectively, such that $\|d_n - h_n\|_{C(K)} \to 0$. Since $[(h_n)]^*$ is nonseparable, so is $[(d_n)]^*$.

Our construction of $(g_n)$ depends upon the following (which in turn follows from our discussion of functions of type-0 in §5): for $n \in \mathbb{N}$ there exists $F_n \in B_1(\Delta)$ such that

(7.4) \quad $\|F_n\|_\infty = 1$ and $|F_n|_D = n$. Moreover if $(h_i) \subseteq C(\Delta)$ converges pointwise to $F_n$ then

(7.5) \quad there exists $k \in \Delta$, integers $\ell_1 < \ell_2 < \cdots < \ell_{n+1}$ and $\varepsilon_i = \pm 1$ $(1 \leq i \leq n)$ such that $\sum_{i=1}^n \varepsilon_i (h_{\ell_{i+1}} - h_{\ell_i})(k) > n - 1$. 

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Actually our $F_n$’s are indicator functions whose domains are countable compact metric spaces $K$. Of course one can embed $K$ into $\Delta$ and the corresponding extended indicator functions have the desired properties (7.4) and (7.5).

We use “$<_L$” for the natural linear order on $D$. Thus $\phi < 0 < 1 < 00 < 01 < 10 < 11 < 000 < \cdots$. For each $\alpha \in D$ choose $n_\alpha \in \mathbb{N}$ and $c_\alpha \in \mathbb{R}^+$ satisfying the following seven properties:

i) $\sum_{\beta \in D} c_\beta \leq 1$.

ii) $c_\alpha^{-1} n_\alpha^{-1} \sum_{\beta < \alpha} n_\beta < 1/10$.

iii) $2c_\alpha^{-1} \sum_{\beta > \alpha} c_\beta < 1/10$.

iv) $1 - n_\alpha^{-1} > 9/10$.

v) $2c_\alpha c_0^{-1} < 1/10$ if $\alpha <_L \alpha_0$.

vi) $c_\alpha c_0^{-1} n_\alpha n_\alpha^{-1} < 1/10$ if $\alpha_0 <_L \alpha$.

vii) $\sum_{\beta \in D} b_\beta^2 < \infty$ where $b_\beta = \sum_{\gamma \geq \beta} c_\gamma$.

Of course we could trim this list somewhat, but we prefer to list the properties in the form in which they are used. The $c_\alpha$’s and $n_\alpha$’s can be chosen as follows. Let $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ be a listing of $D$ in the linear order. Let $c_{\alpha_j} = (22)^{-j}$. It is quickly checked that properties i), iii), v) and vii) hold. We then choose $n_{\alpha_j}$ inductively to be an increasing sequence of positive integers with $n_{\alpha_1} = 11$ (so that iv) holds). If $n_{\alpha_j}$ is picked, choose $n_{\alpha_{j+1}}$ to satisfy ii) and vi) for $\alpha = \alpha_{j+1}$. For each $\alpha \in D$, let $F_{n_\alpha} \in B_1(K_\alpha)$ satisfy (7.4) and (7.5) (with $\Delta$ replaced by $K_\alpha$ and $n$ replaced by $n_\alpha$).

For each $\alpha \in D$ choose $(f^n_{\alpha})_{n=1}^{\infty} \subseteq C(K_\alpha)$, $f^n_{\alpha} \geq 0$ and $\|f^n_{\alpha}\| = 1$, so that $(f^n_{\alpha})_{n=1}^{\infty}$ converges pointwise to $F_{n_\alpha}$ and is equivalent to $(s_n)$ with

\[ |f^1_{\alpha}(k)| + \sum_{n=1}^{\infty} |f^{n+1}_{\alpha}(k) - f^n_{\alpha}(k)| \leq n_\alpha \quad \text{for all } k \in K_\alpha. \tag{7.6} \]

Let $g_n = (c_\alpha f^n_{\alpha})_{\alpha \in D}$. Clearly $g_n \in Y$ since $\|g_n\| \leq \sum_{\alpha \in D} c_\alpha \leq 1$ by i). Furthermore $\ell_1 \not\hookrightarrow [(g_n)]$ by the following lemma and the fact that for all $\alpha$, $\ell_1 \not\hookrightarrow [f^n_{\alpha} : n \in \mathbb{N}]$.

**Lemma 7.5.** For all $\alpha \in D$, let $Y_\alpha$ be a closed subspace of $C(K_\alpha)$ which does not contain $\ell_1$. Let

\[ \tilde{Y}_\alpha = \{(h_\beta)_{\beta \in D} \in Y : h_\alpha \in Y_\alpha \text{ and } h_\beta \equiv 0 \text{ if } \alpha \neq \beta\}. \]
Let $Z$ be the closed linear span of $\{\tilde{\gamma}_\alpha : \alpha \in D\}$. Then $Z$ does not contain $\ell_1$.

**Proof.** It is shown in [34] that $Y$ does not contain a sequence $(h_n)_{n=1}^\infty = ((h_\alpha^n)_{\alpha \in D})_{n=1}^\infty$ which is both equivalent to the unit vector basis of $\ell_1$ and has the following property: for all $\alpha_0 \in D$ there exists $m_0 \in \mathbb{N}$ so that for $m \geq m_0$ and $\alpha \leq_L \alpha_0$, $h_\alpha^m \equiv 0$.

But if $Z$ contains $\ell_1$, then $Y$ must contain such a sequence $(h_n)$. This follows easily from the fact that if $(f_n)_{n=1}^\infty$ is an $\ell_1$-basis in $Z$, then for all $\varepsilon > 0$ and $\alpha_0 \in D$, there exists a normalized block basis $(d_n)_{n=1}^\infty = ((d_\alpha^n)_{\alpha \in D})_{n=1}^\infty$ of $(f_n)$ with $\|d_\alpha^n\|_{C(K_{\alpha_0})} < \varepsilon$ for all $n$.

Thus by [36] we may pass to a subsequence of $(g_n)$ which is weak Cauchy. By relabeling we assume that $(g_n)$ itself is weak Cauchy and converges weak* to $F \in Y^{**}$.

We next verify (7.2). Let $(h_n)$ be a convex block subsequence of $(g_n)$. For $k \in \Delta$ and $h = (h_\alpha)_{\alpha \in D} \in Y$, define $\delta_k(h) = \sum_{\alpha \in \gamma_k} \tilde{h}_\alpha(k)$ where $\gamma_k = \{\alpha \in D : k \in K_\alpha\}$. Clearly $\delta_k$ is a normalized element of $Y^*$. We shall show that

\[
\begin{align*}
\text{for all } \alpha \in D \text{ there exists } k_\alpha \in K_\alpha \text{ and } h = (h_\beta) \in Ba[(h_n)] \\
such that } \delta_{k_\alpha}(h) > 7/10 \text{ and } \delta_k(h) < 3/10 \text{ if } k \in \Delta \setminus K_\alpha.
\end{align*}
\]

(7.7)

As in [33] this implies $[(h_n)]^*$ is nonseparable. Indeed by (7.7) we can choose $(h_\alpha^\alpha)_{\alpha \in D} \subseteq Ba[(h_n)]$ and a collection of basic clopen sets $(K'_\alpha)_{\alpha \in D}$ in $\Delta$ such that for all $\alpha \in D$,

a) $K'_{\alpha,0} \cap K'_{\alpha,1} = \emptyset$,

b) $K'_{\alpha,\varepsilon} \subseteq K'_{\alpha}$ for $\varepsilon = 0, 1$ and

c) $\delta_k(h_\alpha) > 7/10$ for $k \in K'_\alpha$ and

$\delta_k(h_\alpha) < 3/10$ for $k \notin K'_\alpha$.

For each branch (a maximal subset linearly ordered by $\prec$) $\gamma$ in $D$ choose $k_\gamma \in \bigcap_{\alpha \in \gamma} K'_\alpha$. By a) and b) $k_\gamma$ is well defined and $k_\gamma \neq k_{\gamma'}$ if $\gamma \neq \gamma'$. By c), $\|\delta_{k_\gamma} - \delta_{k_{\gamma'}}\|_{[(h_n)]} > 2/5$ if $\gamma \neq \gamma'$.

We return to the proof of (7.7). Fix $\alpha \in D$ and set $h_\alpha = (h_\beta^n)_{\beta \in D}$. Since $(h_\alpha^n)_{n=1}^\infty$ is a convex block subsequence of $(c_\alpha f_\alpha^n)_{n=1}^\infty$, $(h_\alpha^n)_{n=1}^\infty$ converges pointwise to $c_\alpha F_{\alpha}$. Thus by (7.5) and (7.6) we may assume (by passing to a subsequence and relabeling, if necessary) that there exist $\varepsilon_i = \pm 1$ $(1 \leq i \leq n_\alpha)$ and $k_\alpha \in K_\alpha$ such that

$$n_\alpha \geq \sum_{i=1}^{n_\alpha} c_\alpha^{-1} \varepsilon_i (h_\alpha^{i+1} - h_\alpha^i)(k_\alpha) > n_\alpha - 1.$$
Let \( h = n^{-1}_\alpha c^{-1}_\alpha \sum_{i=1}^{n_\alpha} \varepsilon_i (h_{i+1} - h_i) \equiv (h_\beta)_{\beta \in D} \). Thus \( 1 \geq h_\alpha(k_\alpha) > 1 - n^{-1}_\alpha > 9/10 \) by iv).

Furthermore by applying (7.6) to each \( \beta < \alpha \) we have from ii)
\[
\sum_{\beta < \alpha} \tilde{h}_\beta(k_\alpha) \leq \sum_{\beta < \alpha} n^{-1}_\alpha c^{-1}_\alpha c_\beta n_\beta \\
\leq c^{-1}_\alpha n^{-1}_\alpha \sum_{\beta < \alpha} n_\beta < 1/10 .
\]

By the triangle inequality and the definition of \( h \),
\[
\sum_{\beta > \alpha} \tilde{h}_\beta(k_\alpha) \leq c^{-1}_\alpha n^{-1}_\alpha \sum_{\beta > \alpha} 2c_\beta n_\alpha \\
= 2c^{-1}_\alpha \sum_{\beta > \alpha} c_\beta < 1/10 \ (\text{by iii}) .
\]
Thus \( \delta_{k_\alpha}(h) > 9/10 - 2/10 = 7/10 \) which proves the first part of (7.7).

Let \( k \in \Delta \setminus K_\alpha \) be fixed. There exists a unique \( \alpha_0 \in D \ (\alpha_0 \neq \alpha) \) with the same length as \( \alpha_0, |\alpha| = |\alpha_0| \), such that \( k \in K_{\alpha_0} \). The calculations above yield \( \sum_{\beta < \alpha_0} \tilde{h}_\beta(k) + \sum_{\beta > \alpha_0} \tilde{h}_\beta(k) < 2/10 \). If \( \alpha_0 \leq L \alpha \) then by (7.6)
\[
0 \leq h_{\alpha_0}(k) = n^{-1}_\alpha c^{-1}_\alpha \sum_{i=1}^{n_\alpha} \varepsilon_i (h_{\alpha_0}^{i+1} - h_{\alpha_0}^i)(k) \\
\leq n^{-1}_\alpha c^{-1}_\alpha c_\alpha n_{\alpha_0} \leq 1/10 \ (\text{by vi}) .
\]
If \( \alpha < L \alpha_0 \) then we have (from the equality above) that
\[
0 \leq h_{\alpha_0}(k) \leq n^{-1}_\alpha c^{-1}_\alpha c_\alpha n_\alpha = 2c_\alpha c^{-1}_\alpha < 1/10
\]
by v). It follows that \( \delta_k(h) < 3/10 \) which completes the proof of (7.7).

Finally, we verify (7.3). Let \( S = [\alpha, \beta] \equiv \{ \gamma \in D \mid \alpha \leq \gamma \leq \beta \} \) be a finite segment in \( D \). For \( k \in K_\beta \) and \( f \in Y \) we set \( \delta_{S,k}(f) = \sum_{\gamma \in S} \tilde{f}_\gamma(k) \). \( \delta_{S,k}(f) \) is defined similarly if \( S = [\alpha, \infty) \equiv \{ \gamma \in D : \alpha \leq \gamma \} \) is an infinite segment and \( k \in \bigcap_{\beta \in S} K_\beta \). Define
\[
K = \left\{ \sum_{i=1}^{\infty} a_i \delta_{S_i,k_i} : (a_i)_{i=1}^{\infty} \in Ba(\ell_2), (S_i)_{i=1}^{\infty} \text{ are disjoint segments and } k_i \in \bigcap_{\beta \in S_i} K_\beta \text{ for every } i \right\}.
\]

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From the definition of the norm in $Y$ it is clear that $K \subseteq \text{Ba}(Y^*)$. Furthermore it is easy to check that $K$ is weak* closed and $K$ 1-norms $Y$.

It remains to show that $F|_K \in B_{1/2}(K)$. For $m, n \in \mathbb{N}$ let $g(n, m) \in Y$ be given by

$$g(n, m) = (g(n, m)_\beta)_{\beta \in D}$$

where

$$g(n, m)_\beta = \begin{cases} g^n_\beta & \text{if } |\beta| \leq m \\ 0 & \text{otherwise.} \end{cases}$$

Let $y^* = \sum_{i=1}^\infty a_i \delta_{S_i, k_i} \in K$. Then for $m$ fixed,

$$\sum_{n=1}^\infty \left| [g(n + 1, m) - g(n, m)](y^*) \right| = \sum_{n=1}^\infty \left| \sum_{i=1}^\infty a_i \sum_{\gamma \in S_i, |\gamma| \leq m} \left[ \tilde{g}^{n+1}_\gamma(k_i) - \tilde{g}^n_\gamma(k_i) \right] \right|$$

$$\leq \sum_{i=1}^\infty |a_i| \sum_{\gamma \in S_i, |\gamma| \leq m} \sum_{n=1}^\infty |\tilde{g}^{n+1}_\gamma(k_i) - \tilde{g}^n_\gamma(k_i)|$$

$$\leq \sum_{i=1}^\infty |a_i| \sum_{\gamma \in S_i, |\gamma| \leq m} c^n_\gamma$$

(by (7.6) )

$$\leq \sum_{|\gamma| \leq m} c^n_\gamma < \infty.$$ 

In particular $(g(n, m))_{n=1}^\infty$ converges pointwise on $K$ to a function $G_m \in DBSC(K)$.

All that remains is to show that $\|G_m - F\|_K \to 0$ as $m \to \infty$. Let $m \in \mathbb{N}$ be fixed and let $y^* = \sum_{i=1}^\infty a_i \delta_{S_i, k_i} \in K$. Then

$$|G_m(y^*) - F(y^*)| = \left| \sum_{i=1}^\infty a_i \sum_{\gamma \in S_i, |\gamma| > m} c^n_\gamma \tilde{F}_{n, \gamma}(k_i) \right|$$

$$\leq \sum_{i=1}^\infty |a_i| \left( \sum_{\gamma \in S_i, |\gamma| > m} c^n_\gamma \right).$$

For each $i$ set

$$b^n_i = \sum_{\gamma \in S_i, |\gamma| > m} c^n_\gamma.$$ 

Thus

$$|G_m(y^*) - F(y^*)| \leq \left( \sum_{i=1}^\infty (b^n_i)^2 \right)^{1/2}$$

by Hölder’s inequality. The latter goes to 0 as $m \to \infty$ by vii).
8. Problems.

We have previously raised two problems concerning $B_{1/4}(K)$.

**Problem 8.1.** Let $F \in B_1(K)$ and $C < \infty$ be such that if $(f_n) \subseteq C(K)$ is a bounded sequence converging pointwise to $F$, then there exists $(g_n)$, a convex block subsequence of $(f_n)$, with spreading model $C$-equivalent to the summing basis. Is $F \in B_{1/4}(K)$?

**Problem 8.2.** Let $F \in B_1(K)$ and assume there exists a $C < \infty$ such that if $(\varepsilon_i) \subseteq IR^+$ and $K_n(F, (\varepsilon_i)) \neq \emptyset$, then $\sum_i^n \varepsilon_i \leq C$. Is $F \in B_{1/4}(K)$?

These problems lead naturally to the following definitions. Let $F \in B_1(K)$.

$$|F|_I = \max \left\{ \sup \left\{ \sum_{i=1}^m \delta_i : K_m(F, (\delta_i)) \neq \emptyset \right\}, \|F\|_\infty \right\}.$$  

$$|F|_{I'} = \max \left\{ \sup \left\{ m\delta : K_m(F, \delta) \neq \emptyset \right\}, \|F\|_\infty \right\}.$$  

$$|F|_S = \inf \left\{ C : \text{there exist } (f_n) \subseteq C(K) \text{ converging pointwise to } F \right\}$$

with for all $(a_i)_{1}^{1/k} \subseteq IR$, 

$$\lim_{n_1, \ldots, n_k \to \infty} \left\| \sum_{i=1}^k a_i f_{n_i} \right\| \leq C \left\| \sum_{i=1}^k a_i s_i \right\|.$$  

**Remark 8.3.** We do not know if $|F|_I$ or $|F|_{I'}$ are norms. It is clear that $|F|_S$ is a norm and also that

$$\|F\|_\infty \leq |F|_{I'} \leq |F|_I \leq |F|_S \leq |F|_{1/4} \leq |F|_D$$

($|F|_S \leq |F|_{1/4}$ follows from the proof of Theorem B.) Furthermore, using the series criterion for completeness, it is easy to show that $(\{F \in B_1(K) : |F|_S < \infty\}, \cdot |_S)$ is a Banach space.

**Problem 8.4.** Are $\cdot |_I$ and $\cdot |_{I'}$ equivalent? Are $\cdot |_S$ and $\cdot |_{1/4}$ equivalent?

The solution of Problem 8.4 would of course solve Problems 8.1 and 8.2. Furthermore an affirmative answer to Problem 8.2 would yield an affirmative answer to Problems 8.1 and 8.4.

**Proposition 8.5.** $\cdot |_I$ and $\cdot |_{I'}$ are not (in general) equivalent.

**Proof.** Define $F : [0, 1]^\omega \to IR$ as follows:
If \( t_0 \neq 0 \) let
\[
F(t_0, t_1, \ldots) = \sin t_0^{-1}.
\]

If \( t_0 = t_1 = \cdots = t_r = 0 \neq t_{r+1} \), set
\[
F(t_0, t_1, \ldots) = \frac{1}{r+2} \sin t_r^{-1}.
\]

It’s easy to see that \( \text{osc}(F; (0, t_1, t_2, \ldots)) = 2 \) for all \( t_1, t_2, \ldots \in [0, 1] \) and so
\[
K_1(F, \varepsilon) = \{0\} \times [0, 1]^{\omega \setminus \{0\}}
\]
whenever \( 0 < \delta < 2 \). Similar calculations show that if \( r = \lceil \frac{2}{\varepsilon} \rceil \) then
\[
K_r(F, \varepsilon) = \{0\}^r \times [0, 1]^{\omega \setminus r}
\]
and \( K_{r+1}(F, \varepsilon) = \emptyset \). Thus \( K_m(F, \varepsilon) \neq \emptyset \) implies \( m\varepsilon \leq 2 \). On the other hand, for \( m \geq 1 \),
\[
K_m \left( F; \left( 2, 1, \frac{2}{3}, \ldots, \frac{2}{m} \right) \right) = \{0\}^m \times [0, 1]^{\omega \setminus m}.
\]

We conclude by mentioning some further problems for study, some of which have been raised above.

**Problem 8.6.** Classify (or give useful sufficient conditions) for a function \( F \in B_1(K) \) to govern \( \{X : X^* \text{ is separable and } \dim X = \infty\} \). In particular is \( F \in B_{1/4}(K) \setminus C(K) \) a sufficient condition?

**Problem 8.7.** Classify those \( F \in B_1(K) \) which govern \( \{\ell_1\} \), which govern \( \{c_0\} \), which govern \( \{X : X \text{ is reflexive}\} \) or which govern \( \{X : X \text{ is quasi-reflexive}\} \).

We note that if \( X \) is a Polish Banach space (i.e., \( Ba(X) \) is Polish in the weak topology) then Edgar and Wheeler [14] have shown that \( X \) is hereditarily reflexive (see also [37] and [18]). Bellenot [5] and Finet [15] have independently extended this result by showing that whenever \( X \) is Polish, if \( x^{**} \in X^{**} \setminus X \) then \( x^{**}|_{Ba(X^*)} \) strictly governs the class of quasi-reflexive spaces of order 1.

**References**

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38. **********, *Some remarks concerning unconditional basic sequences*, Longhorn Notes, University of Texas, (1982-83), 15–48.