On certain classes of Baire-1 functions with applications to Banach space theory

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Abstract

Certain subclasses of $B_1(K)$, the Baire-1 functions on a compact metric space K, are defined and characterized. Some applications to Banach spaces are given.

0. Introduction.

Let X be a separable infinite dimensional Banach space and let K denote its dual ball, $Ba(X^*)$, with the weak* topology. K is compact metric and X may be naturally identified with a closed subspace of $C(K)$. X^{**} may also be identified with a closed subspace of $A_\infty(K)$, the Banach space of bounded affine functions on K in the sup norm. Our general objective is to deduce information about the isomorphic structure of X or its subspaces from the topological nature of the functions $F \in X^{**} \subseteq A_\infty(K)$. A classical example of this type of result is: X is reflexive if and only if $X^{**} \subset C(K)$.

A second example is the following theorem. ($B_1(K)$ is the class of bounded Baire-1 functions on K and $DBSC(K)$ is the subclass of differences of bounded semicontinuous functions on K. The precise definitions appear below in §1.) We write $Y \hookrightarrow X$ if Y is isomorphic to a subspace of X.

Theorem A. *Let X be a separable Banach space and let $K = Ba(X^*)$ with the weak* topology.*

a) [35] $\ell_1 \hookrightarrow X$ *iff* $X^{**} \setminus B_1(K) \neq \emptyset$.

b) [7] $c_0 \hookrightarrow X$ *iff* $[X^{**} \cap DBSC(K)] \setminus C(K) \neq \emptyset$.

Theorem A provides the motivation for this paper: What can be said about X if $X^{**} \cap [B_1(K) \setminus DBSC(K)] \neq \emptyset$? To study this problem we consider various subclasses of

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$B_1(K)$ for an arbitrary compact metric space K. J. Bourgain has also used this approach and some of our results and techniques overlap with those of [8,9,10]. In a different direction, generalizations of $B_1(K)$ to spaces where K is not compact metric with ensuing applications to Banach space theory have been developed in [22].

In §1 we consider two subclasses of $B_1(K)$ denoted $B_{1/4}(K)$ and $B_{1/2}(K)$ satisfying

$$C(K) \subseteq DBSC(K) \subseteq B_{1/4}(K) \subseteq B_{1/2}(K) \subseteq B_1(K) \ . \tag{0.1}$$

Our interest in these classes stems from Theorem B (which we prove in §3).

Theorem B. *Let K be a compact metric space and let (f_n) be a uniformly bounded sequence in $C(K)$ which converges pointwise to $F \in B_1(K)$.*

- *a) If $F \notin B_{1/2}(K)$, then (f_n) has a subsequence whose spreading model is equivalent to the unit vector basis of ℓ_1.*
- *b) If $F \in B_{1/4}(K) \setminus C(K)$, there exists (g_n), a convex block subsequence of (f_n), whose spreading model is equivalent to the summing basis for c_0.*

Theorem B may be regarded as a local version of Theorem A (see Corollary 3.10). In fact the proof is really a localization of the proof of Theorem A. In Theorem 3.7 we show that the converse to a) holds and thus we obtain a characterization of $B_1(K) \setminus B_{1/2}(K)$ in terms of ℓ_1 spreading models. We do not know if the condition in b) characterizes $B_{1/4}(K)$ (see Problem 8.1).

Given that our main objective is to deduce information about the subspaces of X from the nature of $F \in X^{**} \cap B_1(K)$, it is useful to introduce the following definition.

Let $\mathcal{C}$ be a class of separable infinite-dimensional Banach spaces and let $F \in B_1(K)$. F is said to *govern* $\mathcal{C}$ if whenever $(f_n) \subseteq C(K)$ is a uniformly bounded sequence converging pointwise to F, then there exists a $Y \in \mathcal{C}$ which embeds into $[(f_n)]$, the closed linear span of (f_n). We also say that F *strictly governs* $\mathcal{C}$ if whenever $(f_n) \subseteq C(K)$ is a uniformly bounded sequence converging pointwise to F, there exists a convex block subsequence (g_n) of (f_n) and a $Y \in \mathcal{C}$ with $[(g_n)]$ isomorphic to Y.

Theorem A (b) can be more precisely formulated as: if $F \in DBSC(K) \setminus C(K)$, then F governs $\{c_0\}$. (In fact Corollary 3.5 below yields that $F \in B_1(K) \setminus C(K)$ strictly governs $\{c_0\}$ if and only if $F \in DBSC(K)$.) In §4 we prove that the same result holds if $F \in DSC(K) \setminus C(K)$. (A more general result, with a different proof, has been obtained by Elton [13].) We also note in §4 that there are functions that govern $\{c_0\}$ but are not in $DSC(K)$.

In §6 we give a characterization of $B_{1/4}(K)$ (Theorem 6.1) and use it to give an example of an $F \in B_{1/4}(K) \setminus C(K)$ which does not govern $\{c_0\}$. Thus Theorem B (b) is best possible.

In §7 we note that there exists a K and an $F \in B_{1/2}(K)$ which governs $\{\ell_1\}$. We also give an example of an $F \in B_{1/2}(K)$ which governs $\mathcal{C} = \{X : X$ is separable and X^* is nonseparable$\}$ but does not govern $\{\ell_1\}$.

§1 contains the definitions of the classes $DBSC(K)$, $DSC(K)$, $B_{1/2}(K)$ and $B_{1/4}(K)$. At the end of §1 we briefly recall the notion of spreading model. In §2 we recall some ordinal indices which are used to study $B_1(K)$. A detailed study of such indices can be found in [25]. Our use of these indices and many of the results of this paper have been motivated by [8,9,10]. Proposition 2.3 precisely characterizes $B_{1/2}(K)$ in terms of our index.

In §5 we show that the inclusions in (0.1) are, in general, proper. We first deduce this from a Banach space perspective. Subsequently, we consider the case where K is countable. Proposition 5.3 specifies precisely how large K must be in order for each separate inclusion in (0.1) to be proper.

In §8 we summarize some problems raised throughout this paper and raise some new questions regarding $B_{1/4}(K)$.

We are hopeful that our approach will shed some light on the central problem: if X is infinite dimensional, does X contain an infinite dimensional reflexive subspace or an isomorph of c_0 or ℓ_1? A different attack has been mounted on this problem in the last few years by Ghoussoub and Maurey. The interested reader should also consult their papers (*e.g.*, [18,19,20,21]). Another fruitful approach has been via the theory of types ([26], [24], [38]). We wish to thank S. Dilworth and R. Neidinger for useful suggestions.

1. Definitions.

In this section we give the basic definitions of the Baire-1 subclasses in which we are interested. Let K be a compact metric space. $B_1(K)$ shall denote the class of bounded Baire-1 functions on K, *i.e.*, the pointwise limits of (uniformly bounded) pointwise converging sequences $(f_n) \subseteq C(K)$. $DBSC(K) = \{F : K \to \mathbb{R} \mid$ there exists $(f_n)_{n=0}^\infty \subseteq C(K)$ and $C < \infty$ such that $f_0 \equiv 0$, (f_n) converges pointwise to F and

$$\sum_{n=0}^{\infty} |f_{n+1}(k) - f_n(k)| \le C \quad \text{for all} \quad k \in K\Big\} \ . \tag{1.1}$$

If $F \in DBSC(K)$ we set $|F|_D = \inf\{C \mid$ there exists $(f_n)_{n=0}^\infty \subseteq C(K)$ converging pointwise to F satisfying (1.1) with $f_0 \equiv 0\}$. $DBSC(K)$ is thus precisely those F's which are the "difference of bounded semicontinuous functions on K." Indeed if (f_n) satisfies (1.1), then $F = F_1 - F_2$ where $F_1(k) = \sum_{n=0}^\infty (f_{n+1} - f_n)^+(k)$ and $F_2(k) = \sum_{n=0}^\infty (f_{n+1} - f_n)^-(k)$ are both (lower) semicontinuous. The converse is equally trivial.

It is easy to prove that $(DBSC(K), |\cdot|_D)$ is a Banach space by using the series criterion for completeness. The fact that $\|F\|_\infty \le |F|_D$ but the two norms are in general not equivalent on $DBSC(K)$, leads naturally to the following two definitions.

$$\begin{aligned} B_{1/2}(K) = \Big\{F \in B_1(K) \mid \ &\text{there exists a sequence} \\ &(F_n) \subseteq DBSC(K) \ \text{ converging uniformly to } F\Big\} \text{ and} \\ B_{1/4}(K) = \Big\{F \in B_1(K) \mid \ &\text{there exists } (F_n) \\ &\text{converging uniformly to } F \text{ with } \sup_n |F_n|_D < \infty\Big\} \ . \end{aligned}$$

It can be shown that $DBSC(K)$ is a Banach algebra under pointwise multiplication, and hence $B_{1/2}(K)$ can be identified with $C(\Omega)$, where Ω is the "structure space" or "maximal ideal space" of Ω. Thus $B_{1/4}(K)$ also has a natural interpretation in the general context of commutative Banach algebras.

There is a natural norm on $B_{1/4}(K)$ given by

$$|F|_{1/4} = \inf\Big\{C : \ \text{there exists } (F_n) \text{ converging uniformly with } \sup_n |F_n|_D \le C\Big\}$$

Furthermore $(B_{1/4}(K), |\cdot|_{1/4})$ is a Banach space. One way to see this is to use the following elementary

Lemma 1.1. *Let (M, d_1) be a complete metric space and let d_2 be a metric on M with $d_1(x,y) \leq d_2(x,y)$ for all $x, y \in M$. If all d_2-closed balls in M are also d_1-closed, then (M, d_2) is complete.*

The hypotheses of the lemma apply to $M = \{F : |F|_{1/4} \leq 1\}$ and d_1, d_2 given, respectively, by $\|\cdot\|_\infty$ and $|\cdot|_{1/4}$.

Remark 1.2. While we shall confine our attention to $B_{1/2}$ and $B_{1/4}$, one could of course continue the game, defining

$$\begin{aligned} B_{1/8}(K) = \big\{F \in B_1(K) \mid \ &\text{there exists } (F_n) \subseteq DBSC(K) \\ &\text{with } \ |F_n - F|_{1/4} \to 0\big\} \ \text{ and} \\ B_{1/16}(K) = \big\{F \in B_1(K) \mid \ &\text{there exists } F_n \\ &\text{with } \ \sup_n |F_n|_D < \infty \ \text{ and } \ |F_n - F|_{1/4} \to 0\big\} \ . \end{aligned}$$

This could be continued obtaining

$$DBSC(K) \subseteq \cdots \subseteq B_{1/2^{2n}}(K) \subseteq B_{1/2^{2n-1}}(K) \subseteq \cdots \subseteq B_{1/2}(K)$$

with $B_{1/2^{2n}}(K)$ having a norm $|\cdot|_{1/2^{2n}}$ which, using Lemma 1.1, is easily seen to be complete.

There is another class of Baire-1 functions that shall interest us, the differences of (not necessarily bounded) semi-continuous functions on K.

$$\begin{aligned} DSC(K) = \big\{F : K \to \mathbb{R} \mid \ &\text{there exists a uniformly bounded sequence} \\ &(f_n)_{n=0}^\infty \subseteq C(K) \ \text{ converging pointwise to } F \text{ with} \\ &\sum_{n=0}^\infty |f_{n+1}(k) - f(k)| < \infty \ \text{ for } \ k \in K\big\} \ . \end{aligned}$$

An interesting subclass of $DSC(K)$, is $PS(K)$, the pointwise limits of *pointwise stabilizing* (pointwise ultimately constant) sequences.

$$\begin{aligned} PS(K) = \big\{F \in B_1(K) \mid \ &\text{there exists a uniformly bounded sequence} \\ &(f_n) \subseteq C(K) \ \text{ with the property that for all } k \in K \text{ there exists} \\ &m \in \mathbb{N} \ \text{ such that } \ f_n(k) = F(k) \ \text{ for } \ n \geq m\big\} \ . \end{aligned}$$

Remark 1.3. We discuss $PS(K)$ in Proposition 4.9. Both of these classes were considered in [10], and as noted there, if an indicator function $\mathbf{1}_A \in B_1(K)$, then $\mathbf{1}_A \in PS(K)$. Indeed A must be both F_σ and G_δ (cf. Proposition 2.1 below) and so we can write $A = \bigcup_n F_n = \bigcap_n G_n$ where $F_1 \subseteq F_2 \subseteq \cdots$ are closed sets and $G_1 \supseteq G_2 \supseteq \cdots$ are open sets. Then by the Tietze extension theorem, for each n choose $f_n \in Ba(C(K))$ with f_n identically 1 on F_n and identically 0 on $K \setminus G_n$. Thus for all $k \in K$, $(f_n(k))_n$ is ultimately $\mathbf{1}_A(k)$.

The *summing basis* (s_n) for (an isomorph of) c_0 is characterized by

$$\|\sum a_n s_n\| = \sup_k |\sum_{i=1}^k a_i| \ .$$

Let (x_n) be a seminormalized basic sequence. A basic sequence (e_n) is said to be a *spreading model* of (x_n) if for all $k \in \mathbb{N}$ and all $\varepsilon > 0$ there exist N so that if $N < n_1 < n_2 < \cdots < n_k$ and $(a_i)_1^k \subseteq \mathbb{R}$ with $\sup_i |a_i| \le 1$, then

$$\Big| \|\sum_{i=1}^k a_i x_{n_i}\| - \|\sum_{i=1}^k a_i e_i\| \Big| < \varepsilon \ .$$

For further information on spreading models see [4].

We recall that if $(f_n) \subseteq Ba(C(K))$ converges pointwise to $F \in B_1(K) \setminus C(K)$ then there exists a $C = C(F)$ such that (f_n) has a basic subsequence (f_n') with basis constant C which C-dominates (s_n). Thus $C\|\sum a_n f_n'\| \ge \|\sum a_n s_n\|$, for all $(a_n) \subseteq \mathbb{R}$ (see *e.g.*, [31]). Furthermore (f_n') can be taken to have a spreading model [4]. The constant C depends only on $\sup\{\mathrm{osc}(F,k) \mid k \in K\}$ (see §2 for the definition of $\mathrm{osc}(F,k)$).

Finally we recall that a sequence (g_n) in a Banach space is a *convex block subsequence* of (f_n) if $g_n = \sum_{i=p_n+1}^{p_{n+1}} a_i f_i$ where (p_n) is an increasing sequence of integers, $(a_i) \subseteq \mathbb{R}^+$ and for each n, $\sum_{i=p_n+1}^{p_{n+1}} a_i = 1$.

2. Ordinal Indices for $B_1(K)$.

Let (K,d) be a compact metric space and let $F : K \to \mathbb{R}$ be a bounded function. The Baire characterization theorem [3] states that $F \in B_1(K)$ iff for all closed nonempty $L \subseteq K$, $F\big|_L$ has a point of continuity (relative to the compact space (L,d)). This leads naturally to an ordinal index for Baire-1 functions which we now describe.

For a closed set $L \subseteq K$ and $\ell \in L$ let the *oscillation of* $F\big|_L$ at ℓ be given by $\mathrm{osc}_L(F,\ell) = \lim_{\varepsilon\downarrow 0} \sup\{f(\ell_1) - f(\ell_2) \mid \ell_i \in L \text{ and } d(\ell_i,\ell) < \varepsilon \text{ for } i = 1,2\}$. We define the *oscillation of* F *over* L by $\mathrm{osc}_L F = \sup\{F(\ell_1) - F(\ell_2) \mid \ell_1,\ell_2 \in L\}$.

For $\delta > 0$, let $K_0(F,\delta) = K$ and if $\alpha < \omega_1$ let

$$K_{\alpha+1}(F,\delta) = \big\{k \in K_\alpha(F,\delta) \mid \mathrm{osc}_{K_\alpha(F,\delta)}(F,k) \geq \delta\big\} \ .$$

For limit ordinals α, set

$$K_\alpha(F,\delta) = \bigcap_{\beta<\alpha} K_\beta(F,\delta) \ .$$

Note that $K_\alpha(F,\delta)$ is always closed and $K_\alpha(F,\delta) \supseteq K_\beta(F,\delta)$ if $\alpha < \beta$. The index $\beta(F,\delta)$ is given by

$$\beta(F,\delta) = \inf\big\{\alpha < \omega_1 \mid K_\alpha(F,\delta) = \emptyset\big\}$$

provided $K_\alpha(F,\delta) = \emptyset$ for some $\alpha < \omega_1$ and $\beta(F,\delta) = \omega_1$ otherwise. Since K is separable, the transfinite sequence $(K_\alpha(F,\delta))_{\alpha<\omega_1}$ must stabilize: there exists $\beta < \omega_1$ so that $K_\alpha(F,\delta) = K_\beta(F,\delta)$ for $\beta \geq \alpha$.

The Baire characterization theorem yields that $\beta(F,\delta) < \omega_1$ for all $\delta > 0$ iff $F \in B_1(K)$. In fact we have the following proposition. In its statement $\mathcal{A}$ denotes the algebra of *ambiguous* subsets of K. Thus $A \in \mathcal{A}$ iff A is both F_σ and G_δ. Also we write $[F \leq a]$ for the set $\{k \in K \mid F(k) \leq a\}$.

Proposition 2.1. *Let $F : K \to \mathbb{R}$ be a bounded function on the compact metric space K. The following are equivalent.*

1) *$F \in B_1(K)$.*
2) *$\beta(F,\delta) < \omega_1$ for all $\delta > 0$.*
3) *For a and b real, $[F \leq a]$ and $[F \geq b]$ are both G_δ subsets of K.*

4) For U an open subset of $\mathbb{R}$, $F^{-1}(U)$ is an F_σ subset of K.

5) For $a < b$, $[F \le a]$ and $[F \ge b]$ may be separated by disjoint sets in $\mathcal{A}$. Equivalently, there exists $A \in \mathcal{A}$ with $[F \le a] \subseteq A$ and $A \cap [F \ge b] = \emptyset$.

6) F is the uniform limit of a sequence of $\mathcal{A}$-simple functions ($\mathcal{A}$-measurable functions with finite range).

7) F is the uniform limit of a sequence $(g_n) \subseteq DSC(K)$.

8) F is the uniform limit of a sequence $(g_n) \subseteq PS(K)$.

The proof is standard and can be compiled from [23]. We are more interested in an analogous characterization of $B_{1/2}(K)$. Before stating that proposition we need a few more definitions.

$\mathcal{D}$ shall denote the algebra of all finite unions of differences of closed subsets of K. $\mathcal{D}$ is easily seen to be a subalgebra of $\mathcal{A}$.

One of the statements in our next proposition involves another ordinal index for Baire-1 functions, $\alpha(F; a, b)$, which as we shall see is closely related to our index. For $a < b$, let $K_0(F; a, b) = K$ and for any ordinal α, let

$$\begin{aligned} K_{\alpha+1}(F; a, b) = \{k \in K_\alpha(F; a, b) \mid \quad &\text{for all } \ \varepsilon > 0 \text{ and } i = 1, 2, \\ &\text{there exist } k_i \in K_\alpha(F; a, b) \ \text{ with } \ d(k_i, k) \le \varepsilon \ , \\ &F(k_1) \ge b \ \text{ and } \ F(k_2) \le a\} \ . \end{aligned}$$

Equivalently, $K_{\alpha+1} = \overline{K_\alpha \cap [F \le a]} \cap \overline{K_\alpha \cap [F \ge b]}$. At limit ordinals α we set

$$K_\alpha(F; a, b) = \bigcap_{\beta < \alpha} K_\beta(F; a, b) \ .$$

As before these sets are closed and decreasing. We let $\alpha(F; a, b) = \inf\{\gamma < \omega_1 \mid K_\gamma(F; a, b) = \emptyset\}$ if $K_\gamma(F; a, b) = \emptyset$ for some $\gamma < \omega_1$ and let $\alpha(F; a, b) = \omega_1$ otherwise.

Remark 2.2. The index $\alpha(F; a, b)$ is only very slightly different from the index $L(F, a, b)$ considered by Bourgain [8]. $L(F; a, b) = \inf\{\eta < \omega_1 \mid$ there exists a transfinite increasing sequence of open sets $(G_\alpha)_{\alpha \le \eta}$ with $G_0 = \emptyset$, $G_\eta = K$, $G_{\alpha+1} \setminus G_\alpha$ is disjoint from either $[F \le a]$ or $[F \ge b]$ for all $\alpha < \eta$ and $G_\gamma = \bigcup_{\alpha < \gamma} G_\alpha$ if $\gamma \le \eta$ is a limit ordinal$\}$. In fact one can show that if $\alpha(F; a, b) = \eta + n$ where η is a limit ordinal and $n \in \mathbb{N}$, then

$L(F,a,b) \in \{\eta+2n, \eta+2n-1\}$. In Proposition 2.3 we shall show that $\alpha(F;a,b) < \omega$ for all $a < b$ iff $\beta(F,\delta) < \omega$ for all $\delta > 0$. We note that a more general result has subsequently been obtained in [25]. Indeed if we define $\beta(F) = \sup\{\beta(F;\delta) \mid \delta > 0\}$ and $\alpha(F) = \sup\{\alpha(F;a,b) \mid a < b \text{ rational}\}$ then Kechris and Louveau have shown that $\beta(F) \le \omega^\xi$ iff $\alpha(F) \le \omega^\xi$.

Also we note that the following result follows from [8]. Let X be a separable Banach space not containing ℓ_1. Let $K = Ba(X^*)$ in its weak* topology. Then

$$\sup\bigl\{\beta(x^{**}|_K) : x^{**} \in X^{**}\bigr\} < \omega_1 \ .$$

Proposition 2.3. *Let $F : K \to \mathbb{R}$ be a bounded function on the compact metric space K. The following are equivalent*

1) *$F \in B_{1/2}(K)$.*
2) *F is the uniform limit of $\mathcal{D}$-simple functions on K.*
3) *For $a < b$, $[F \le a]$ and $[F \ge b]$ may be separated by disjoint sets in $\mathcal{D}$.*
4) *$\beta(F) \le \omega$.*
5) *$\alpha(F;a,b) < \omega$ for all $a < b$.*

Proof.

4) $\Rightarrow$ 5). This follows from the elementary observation that for all ordinals α and reals $a < b$, $K_\alpha(F;a,b) \subseteq K_\alpha(F,b-a)$, and the fact that 4) holds if and only if $\beta(F,\delta) < \omega$ for all $\delta > 0$.

5) $\Rightarrow$ 3). Let $K_i = K_i(F;a,b)$. Thus $K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n = \emptyset$ where $n = \alpha(F;a,b)$. Let

$$D = \bigcup_{i=1}^n \overline{(F \le a \cap K_{i-1})} \setminus \overline{([F \ge b] \cap K_{i-1})} \in \mathcal{D}\ .$$

Since $K_i = \overline{([F \le a] \cap K_{i-1})} \cap \overline{([F \ge b] \cap K_{i-1})}$,

$$\begin{aligned} D &= \bigcup_{i=1}^{n} \big(\overline{[F \le a] \cap K_{i-1}} \setminus K_i \big) \\ &\supseteq \bigcup_{i=1}^{n} \Big[\big([F \le a] \cap K_{i-1} \big) \setminus K_i \Big] \\ &= \bigcup_{i=1}^{n} \big([F \le a] \cap (K_{i-1} \setminus K_i) \big) = [F \le a] \ . \end{aligned}$$

Since K_{i-1} is closed,

$$\begin{aligned} D &\subseteq \bigcup_{i=1}^{n} \big(K_{i-1} \setminus \overline{[F \ge b] \cap K_{i-1}} \ \big) \\ &\subseteq \bigcup_{i=1}^{n} \Big[K_{i-1} \setminus \big([F \ge b] \cap K_{i-1} \big) \Big] \\ &= \bigcup_{i=1}^{n} \big(K_{i-1} \setminus [F \ge b] \big) = K \setminus [F \ge b] \ . \end{aligned}$$

3) $\Rightarrow$ 2). This is a standard exercise in real analysis.

2) $\Rightarrow$ 1). Since every $\mathcal{D}$-simple function can be expressed in the form $\sum_{i=1}^{k} a_i \mathbf{1}_{L_i}$ where the L_i's are closed sets and $DBSC(K)$ is a linear space it suffices to recall that $\mathbf{1}_L \in DBSC(K)$ whenever L is closed. In fact $\mathbf{1}_L$ is upper semicontinuous.

1) $\Rightarrow$ 4). Let F be the uniform limit of $(F_n) \subseteq DBSC(K)$. For $\delta > 0$ and n sufficiently large, $\beta(F, 2\delta) \le \beta(F_n, \delta)$ and thus is suffices to prove that for $G \in DBSC(K)$, $\beta(G, \delta) < \omega$ for $\delta > 0$. This is immediate from the following

Lemma 2.4. *If $m \in \mathbb{N}$, $\delta > 0$ and $G : K \to \mathbb{R}$ is such that $K_m(G, \delta) \ne \emptyset$, then $|G|_D \ge m\delta/4$.*

Proof. Let $(g_n) \subseteq C(K)$ converge pointwise to G. It suffices to show that there exist integers $n_1 < n_2 < \cdots < n_{m+1}$ and $k \in K$ such that $|g_{n_{i+1}}(k) - g_{n_i}(k)| > \delta/4$ for $1 \le i \le m$.

Let $n_1 = 1$, $k_0 \in K_m(G, \delta)$ and let U_0 be a neighborhood of k_0 for which $\operatorname{osc}_{U_0} g_{n_1} < \delta/8$. Choose k_0^1 and k_0^2 in $U_0 \cap K_{m-1}(G, \delta)$ with $G(k_0^1) - G(k_0^2) > 3\delta/4$. Then choose

$n_2 > n_1$ such that $g_{n_2}(k_0^1) - g_{n_2}(k_0^2) > 3\delta/4$. Thus there is a nonempty neighborhood $U_1 \subset U_0$ of either k_0^1 or k_0^2 such that for $k \in U_1$, $|g_{n_2}(k) - g_{n_1}(k)| > \delta/4$.

Similarly we can find a neighborhood $U_2 \subseteq U_1$ of a point in $K_{m-1}(G,\delta)$ and $n_3 > n_2$ so that for $k \in U_2$, $|g_{n_3}(k) - g_{n_2}(k)| > \delta/4$, etc. ∎

Remarks 2.5. 1. Of course by using a bit more care one can show that $|G|_D \ge m\delta/2$ whenever $K_m(G,\delta) \ne \emptyset$.

2. Following [25] we say that for $F \in B_1(K)$, $F \in B_1^\xi(K)$ iff $\beta(F) \le \omega^\xi$. Thus $B_{1/2}(K) \equiv B_1^1(K)$ by Proposition 2.3, a result also observed in [25].

3. We do not yet have an index characterization of $B_{1/4}(K)$, however we have a necessary condition (which may be sufficient). To describe this we first must generalize our index above. Let $F : K \to \mathbb{R}$ and let $(\delta_i)_{i=1}^\infty$ be positive numbers. Set $K_0(F,(\delta_i)) = K$ and for $0 \le i$ set

$$K_{i+1}\big(F,(\delta_j)\big) = \big\{k \in K_i\big(F,(\delta_j)\big) \mid \mathrm{osc}_{K_i(F,(\delta_j))}(F,k) \ge \delta_{i+1}\big\} \ .$$

Proposition 2.6. *Let $F \in B_{1/4}(K)$. Then there exists an $M < \infty$ so that if $K_n(F,(\delta_i)) \ne \emptyset$, then $\sum_{i=1}^n \delta_i \le M$.*

Proof. Let F be the uniform limit of (G_n) with $|G_n|_D \le C < \infty$ for all n. Suppose that $K_n(F,(\delta_i)) \ne \emptyset$ for some sequence $(\delta_i)_{i=1}^\infty \subseteq \mathbb{R}^+$. Since $K_n(F,(\delta_i)) \subseteq K_n(G_m,(\delta_i/2))$ for large m, the latter set is non-empty as well. The proof of Lemma 2.4 yields

$$\text{(2.1)} \qquad \begin{cases} \text{If } G : K \to \mathbb{R} \text{ and } (\delta_i)_{i=1}^\infty \subseteq \mathbb{R}^+ \text{ is such that } K_n(G,(\delta_i)) \ne \emptyset, \\ \text{then } |G|_D \ge 4^{-1} \sum\limits_{i=1}^n \delta_i. \end{cases}$$

Thus by (2.1) we have, for large m, $C \ge |G_m|_D \ge 4^{-1} \sum_{i=1}^n \delta_i$ and so $\sum_{i=1}^n \delta_i \le 4C$. ∎

We shall explore in greater detail in §3 and §8 some questions related to the problem of an index characterization of Baire-1/4. The following proposition gives a sufficient index criterion for a function to be Baire-1/4. It also shows (via Proposition 2.3) that if $F \in B_{1/2}(K) \setminus B_{1/4}(K)$, then $\beta(F) = \omega$.

Proposition 2.7. *Let $F \in B_1(K)$. If $\beta(F) < \omega$, then $F \in B_{1/4}(K)$.*

Proof. Without loss of generality let $F : K \to [0,1]$ with $\beta(F) \le n$. Thus $\alpha(F;a,b) \le n$ for all $a < b$. It follows from the proof of 5) $\Rightarrow$ 3) in Proposition 2.3 that for all $0 < a < b < 1$ there exists a $D \in \mathcal{D}$ with $|\mathbf{1}_D|_D \le 2n$, $[F \le a] \subseteq D$ and $[F \ge b] \cap D = \emptyset$. Thus for all $m < \infty$ there exist sets $D_1 \supseteq D_2 \supseteq \cdots \supseteq D_m$ in $\mathcal{D}$ with $[F \ge i/m] \subseteq D_i$, $[F \le (i-1)/m] \cap D_i = \emptyset$ and $|\mathbf{1}_{D_i}|_D \le 2n$ for $i \le m$. In particular if $G = \sum_{i=1}^m m^{-1}\mathbf{1}_{D_i}$, then $\|F - G\|_\infty \le m^{-1}$ and $|G|_D \le 2n$. ∎

The following proposition is related to work of A. Sersouri [39]. It is of interest to us because it shows that a separable Banach space X can have functions of large index in X^{**} and yet be quite nice. In fact it shows there are Baire-1 functions of arbitrarily large index which strictly govern the class of quasireflexive (order 1) Banach spaces. Our proof was motivated by discussions with A. Pełczyński.

Proposition 2.8. *For all $\gamma < \omega_1$ there exists a quasireflexive (of order 1) Banach space Q_γ such that $Q_\gamma^{**} = Q_\gamma \oplus \langle F_\gamma \rangle$ where $\beta(F_\gamma) > \gamma$.*

(The index $\beta(F_\gamma)$ is computed with respect to $Ba(Q_\gamma^*)$.)

Remark 2.9. In §6 we shall show the existence of a quasireflexive space whose new functional (in the second dual) is Baire-1/4.

Proof of Proposition 2.8. We use interpolation, namely the method of [12]. (This has also been used in [19] in a slightly different manner to produce a quasireflexive space from a weak* convergent sequence.)

To begin let $\gamma < \omega_1$ be any ordinal and choose a compact metric space K containing an ambiguous set A_γ with $\alpha(\mathbf{1}_{A_\gamma}; \frac{1}{4}, \frac{3}{4}) > \gamma$. (For example $\mathbf{1}_{A_\alpha}$ could be taken to be one of the functions F_δ described in §5 with $\delta > \omega^\gamma +$.) Choose a sequence $(f_n) \subseteq Ba(C(K))$ converging pointwise to $\mathbf{1}_{A_\gamma}$ such that $(\mathbf{1}_{A_\gamma}, f_1, f_2, \ldots)$ is basic in $C(K)^{**}$. Let W be the closed convex hull of $\{\pm f_n\}_{n=1}^\infty$ in $C(K)$. Let Q_γ be the Banach space obtained from $W \subseteq Ba(C(K))$ by [DFJP]-interpolation. Thus for all $n \in \mathbb{N}$, $\|\cdot\|_n$ is the gauge of $U_n = 2^n W + 2^{-n} Ba(C(K))$, and $Q_\gamma = \{x \in C(K) : \| x \| \equiv (\sum_n \|x\|_n^2)^{1/2} < \infty\}$. Following the notation of [12], we let $C = Ba(Q_\gamma) = \{x \in C(K) : \| x \| \le 1\}$ and let

$j : Q_\gamma \to C(K)$ be the natural semiembedding.

We first observe that Q_γ is quasireflexive of order 1. Indeed it is easy to check that $\widetilde{W}$, the weak* closure of W in $C(K)^{**}$ is just

$$\widetilde{W} = \left\{ \sum_{i=1}^{\infty} a_i f_i + a_\infty \mathbf{1}_{A_\gamma} : |a_\infty| + \sum_{i=1}^{\infty} |a_i| \le 1 \right\} .$$

Furthermore $\widetilde{C} \subseteq [\widetilde{W}]$ ([12], Lemma 1(v)) which has the basis $(\mathbf{1}_{A_\gamma}, f_1, f_2, \ldots)$. Now $j^{**} : Q_\gamma^{**} \to C(K)^{**}$ is one-to-one and $(j^{**})^{-1}(C(K)) = Q_\gamma$ (Lemma 1(iii)). Thus if $F_\gamma \in Q_\gamma^{**}$ satisfies $j^{**}F_\gamma = \mathbf{1}_{A_\gamma}$, then $Q_\gamma^{**} = Q_\gamma \oplus \langle F_\gamma \rangle$. Of course F_γ must be the weak* limit of $(j^{-1}(f_n))_n$ in Q_γ^{**}.

It remains to show that $\beta(F_\gamma) \ge \gamma$. We shall prove

$$\overline{\alpha}(F_\gamma; \tfrac{1}{4}, \tfrac{3}{4}) \ge \alpha(\mathbf{1}_{A_\gamma}; \tfrac{1}{4}, \tfrac{3}{4}) \tag{2.2}$$

where $\overline{\beta}$ is the index computed with respect to $F_\gamma \in B_1(3Ba(Q_\gamma^*))$. Since $\beta(F_\gamma) \ge \alpha(F_\gamma; \frac{1}{12}, \frac{1}{4}) \ge \overline{\alpha}(F_\gamma; \frac{1}{4}, \frac{3}{4})$, the result follows.

Since $\|j\| \le 3$, if $K_0 = 3Ba(Q_\gamma^*)$ and $H_0 = Ba(C(K)^*)$, then $j^*H_0 \subseteq K_0$. More generally if $K_{\beta+1} = \{y^* \in K_\beta \mid$ for all non-empty relative weak* neighborhoods U of y^* in K_β there exists $y_1^*, y_2^* \in U$ with $F_\gamma(y_1^*) \ge \frac{3}{4}$ and $F_\gamma(y_2^*) \le \frac{1}{4}\}$ and $H_{\beta+1}$ is defined similarly in terms of $\mathbf{1}_{A_\gamma}$, then $j^*H_{\beta+1} \subseteq K_{\beta+1}$ for all β, since j^* is ω^*-continuous and $F_\gamma(j^*x^*) = (j^{**}F_\gamma)x^* = \mathbf{1}_{A_\gamma}(x^*)$. This proves (2.2). ∎

3. Theorem B.

For the proof of Theorem (B) (a) we need a lemma. Recall that a collection of pairs of subsets of K, $(A_i, B_i)_{i=1}^n$, is said to be (*Boolean*) *independent* if for all $I \subseteq \{1,\ldots,n\}$, $\bigcap_{i\in I} A_i \cap \bigcap_{i\notin I} B_i \neq \emptyset$.

Lemma 3.1. *Let $F : K \to \mathbb{R}$ be the pointwise limit of $(f_n) \subseteq C(K)$. If $K_m(F;a,b) \neq \emptyset$ for some $m \in \mathbb{N}$ and $a < b$, then for $a < a' < b' < b$ there exists a subsequence (f_n') of (f_n) so that if $n_1 < \cdots < n_m$, then $(A_{n_i}', B_{n_i}')_{i=1}^m$ are independent where $A_{n_i}' = [f_{n_i}' \le a']$ and $B_{n_i}' = [f_{n_i}' \ge b']$.*

Proof. The proof is similar to that of Lemma 2.4 and is actually a local version of the proof of the main result of [35] (see [8] for a more general discussion of the consequences of $K_\beta(F;a,b) \neq \emptyset$).

We first show how to choose a finite subsequence $(f_{n_i})_{n=1}^m$ of (f_n) so that $(A_{n_i}, B_{n_i})_{i=1}^m$ is independent, where $A_{n_i} = [f_{n_i} \le a']$ and $B_{n_i} = [f_{n_i} \ge b']$. Let $k_\phi \in K_m(F;a,b)$. Thus there exist k_0 and k_1 in $K_{m-1}(F;a,b)$ with $F(k_0) \le a$ and $F(k_1) \ge b$. Choose n_1 and neighborhoods U_0 and U_1 of k_0 and k_1, respectively, so that $f_{n_1} < a'$ on U_0 and $f_{n_1} > b'$ on U_1. Let $k_{\varepsilon_1,\varepsilon_2} \in U_{\varepsilon_1} \cap K_{m-2}(F;a,b)$ for $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ with $F(k_{\varepsilon_1,0}) \le a$ and $F(k_{\varepsilon_1,1}) \ge b$ for $\varepsilon_1 \in \{0,1\}$. Choose $n_2 > n_1$ and neighborhoods $U_{\varepsilon_1,\varepsilon_2} \subseteq U_{\varepsilon_1}$ of $k_{\varepsilon_1,\varepsilon_2}$ so that $f_{n_2} < a'$ on $U_{\varepsilon_1,0}$ and $f_{n_2} > b'$ on $U_{\varepsilon_1,1}$ (for $\varepsilon_1, \varepsilon_2 \in \{0,1\}$). Continue up to f_{n_m}. The sets $(A_{n_i}, B_{n_i})_1^m$ are then independent since for $I \subseteq \{1,\ldots,m\}$, $\bigcap_{i\in I} A_{n_i} \cap \bigcap_{i\notin I} B_{n_i} \supseteq U_{\varepsilon_1,\ldots,\varepsilon_m} \neq \emptyset$ where $\varepsilon_i = 0$ if $i \in I$ and $\varepsilon_i = 1$ if $i \notin I$.

Now the existence of an infinite subsequence (f_n') satisfying the conclusion of 3.1 follows immediately from Ramsey's theorem. Indeed, by the latter, there exists (f_n') a subsequence of (f_n) so that (f_n') satisfies the conclusion, or such that for *all* $n_1 < \cdots < n_m$, $(A_{n_i}', B_{n_i}')_{i=1}^m$ *is not* independent. But we have proved that the second alternative is impossible.

Proof of Theorem B(a). (f_n) is a bounded sequence in $C(K)$ converging pointwise to $F \notin B_{1/2}(K)$. By Proposition 2.3 there exists $a < b$ so that $K_m(F;a,b) \neq \emptyset$ for all $m \in \mathbb{N}$. By passing to a subsequence we may assume (f_n) has a spreading model. Furthermore by Lemma 3.1, passing to subsequences and diagonalization we may assume that for some

$a < a' < b' < b$, $(A_{n_i}, B_{n_i})_{i=1}^m$ is independent whenever $m \le n_1 < n_2 < \cdots < n_m$ and $A_{n_i} = [f_{n_i} \le a']$, $B_{n_i} = [f_{n_i} \ge b']$. By Proposition 4 of [36] it follows that there exists $C < \infty$ so that $(f_{n_i})_{i=1}^m$ is C-equivalent to the unit vector basis of ℓ_1^m whenever $m \le n_1 < \cdots < n_m$. ∎

The proof of Theorem B(b) will require a more precise version of Theorem A(b) and the following elementary lemma (which follows easily from the Hahn-Banach theorem). If C is a subset of a Banach space X, $\widetilde{C}$ denotes the w^*-closure of C in X^{**}.

Lemma 3.3. *Let C and D be convex subsets of X. Then $md(C, D) = md(\widetilde{C}, \widetilde{D})$. By $md(C, D)$ we mean the minimum distance,*

$$\inf\big\{\|c - d\| \mid c \in C \ , \ d \in D\big\} \ .$$

The variant of Theorem A(b) which we need is

Lemma 3.4. *Let $F : K \to \mathbb{R}$ be bounded and let $(f_n) \subseteq C(K)$ converge pointwise to F with $\sum_{n=0}^\infty |f_{n+1}(k) - f_n(k)| \le M$ for all $k \in K$ ($f_0 \equiv 0$). Suppose $\operatorname{osc}(F, k) > \delta$ for some $\delta > 0$. Then there exists a subsequence (f_n') of (f_n) which is $C = C(M, \delta)$ equivalent to the summing basis.*

Let $F \in B_1(K) \setminus C(K)$. It is evident that if F strictly governs $\{c_0\}$, then $F \in DBSC(K)$. The next result shows that the converse is true.

Corollary 3.5. *Let $F \in DBSC(K)$ and let (f_n), M and δ be as in the hypothesis of Lemma 3.4. Let $(g_n) \subseteq C(K)$ converge pointwise to F with $\sup_n \|g_n\|_\infty < \infty$. Then there exists (h_n), a convex block subsequence of (g_n), which is $C(M, \delta)$-equivalent to the summing basis.*

The proof is straightforward from Lemmas 3.3 and 3.4.

Proof of Theorem B(b). Let $F \in B_{1/4}(K) \setminus C(K)$ and let $(f_n) \subseteq C(K)$ be a bounded sequence converging pointwise to F. Choose $(F_n) \subseteq DBSC(K)$ which converges uniformly to F so that $\sup_n |F_n|_D < M < \infty$. For each $n \in \mathbb{N}$, choose $(f_i^n)_{i=0}^\infty \subseteq C(K)$, $f_0^n \equiv 0$, which converges pointwise to F_n and satisfies $\sum_{i=0}^\infty |f_{i+1}^n(k) - f_i^n(k)| \le M$ for $k \in K$.

Since $F \notin C(K)$ we may assume there exists $\delta > 0$ so that for all n, $\text{osc}_K(F_n, k) > \delta > 0$ for some $k \in K$. Thus, by Lemma 3.4, we may suppose for all n, $(f_i^n)_{i=1}^\infty$ is $C = C(M, \delta)$-equivalent to the summing basis. We may also assume $\|F_n - F_{n+1}\|_\infty < \varepsilon_n$ where $\varepsilon_n \downarrow 0$ and for all $n \in \mathbb{N}$, $\sum_{i=n+1}^\infty \varepsilon_i < \varepsilon_n$.

By induction and Lemma 3.3 we may replace each sequence $(f_i^n)_{i=1}^\infty$ by a convex block subsequence $(g_i^n)_{i=1}^\infty$ such that for $n > 1$,

$$(*) \qquad \begin{cases} \text{there exists a convex block subsequence } (h_i^n)_{i=1}^\infty \text{ of } (g_i^{n-1})_{i=1}^\infty \\ \text{with } \|g_i^n - h_i^n\|_\infty < \varepsilon_{n-1} \text{ for } i \in \mathbb{N}. \end{cases}$$

Let $(g_n^n)_{n=1}^\infty$ be the diagonal sequence. Clearly (g_n^n) converges pointwise to F. Also by $(*)$ for $n > k$, $md(g_n^n, co(g_j^k)_{j=1}^\infty) < \sum_{j=k}^n \varepsilon_j < \varepsilon_{k-1}$. In fact for k fixed, there exists a convex block subsequence $(d_n^k)_{n>k}$ of $(g_j^k)_{j=1}^\infty$ with $\|g_n^n - d_n^k\|_\infty < \varepsilon_{k-1}$ for $n > k$. Thus for any k, $(g_n^n)_{n>k}$ is an ε_{k-1}-perturbation of a sequence $(d_n^k)_{n>k}$ which is C'-equivalent to the summing basis where C' depends solely on C.

By Lemma 3.3 applied to $co(f_n)$ and $co(g_n^n)$, there are convex block subsequences (g_n) of (f_n) and $(\bar{g}_n)$ of (g_n^n) with $\|g_n - \bar{g}_n\|_\infty \to 0$. Since $(\bar{g}_n)_{n>i}$ is an ε_{i-1}-perturbation of a sequence which is C'-equivalent to the summing basis, $(\bar{g}_n)$ and hence (g_n) has a subsequence which has spreading model equivalent to the summing basis. ∎

Remark 3.6. The constant of equivalence of the spreading model of (g_n) with the summing basis depends solely upon $\sup_{k\in K} \text{osc}_K(F, k)$ and $|F|_{1/4}$.

Our next theorem is a converse to Theorem B(a).

Theorem 3.7. *Let $F \in B_1(K)$. Assume that whenever $(f_n) \subseteq C(K)$ is a uniformly bounded sequence converging pointwise to F, then any spreading model of (f_n) is equivalent to the unit vector basis of ℓ_1. Then $F \notin B_{1/2}(K)$.*

Lemma 3.8. *Let $F \in B_{1/2}(K) \setminus C(K)$, $\|F\|_\infty \leq 1$. Then there exists $(f_n) \subseteq C(K)$ converging pointwise to F with spreading model (e_n) and a function $M : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$\| \sum a_n e_n \| \leq M(\varepsilon) \| \sum a_n s_n \| + \varepsilon \sum |a_n| \tag{3.1}$$

for all $(a_n) \subseteq \mathbb{R}$ *and* $\varepsilon > 0$.

Proof. Let $(g_n) \subseteq Ba(C(K))$ converge pointwise to F and let $\varepsilon_n \downarrow 0$. By the proof of Theorem B(b) we can choose (f_n), a convex block subsequence of (g_n) such that for all m, $(f_n)_{n=m}^{\infty}$ is an ε_m-perturbation of a sequence which is $M(\varepsilon_m, F)$-equivalent to the summing basis. ∎

Proof of Theorem 3.7. This is immediate from Lemma 3.8, since if (e_n) satisfies (3.1), then

$$\lim_{n\to\infty} \frac{1}{n}\Big\|\sum_{i=1}^{n}(-1)^i e_i\Big\| = 0 \ .$$

In particular (e_i) is not equivalent to the unit vector basis of ℓ_1. ∎

The proof of Theorem 3.7 combined with Theorem B(a) yields the following result. *Let* $F \in B_1(K)$. *Then* $F \notin B_{1/2}(K)$ *if and only if there exists* $(f_n) \subseteq C(K)$, *a uniformly bounded sequence converging pointwise to* F, *so that if* (g_n) *is a convex block subsequence of* (f_n), *then some subsequence of* (g_n) *has the unit vector basis of* ℓ^1 *as a spreading model.*

We do not know if the converse to Theorem B(b) is valid.

Problem 3.9. Let $F \in B_1(K)$ and $C < \infty$ be such that whenever (f_n) is a uniformly bounded sequence in $C(K)$ converging pointwise to F, then there exists (g_n), a convex block subsequence of (f_n) with spreading model C-equivalent to the summing basis. Is $F \in B_{1/4}(K)$?

We now turn to the Banach space implications of Theorem B. Let K be compact metric and let X be a closed subspace of $C(K)$. For example, K could be $Ba(X^*)$ but we do not require this. X^{**} is naturally isometric to $X^{\perp\perp} \subseteq C(K)^{**}$. In this setting it can be shown (see [35]) that if $B_1(X) = \{x^{**} \in X^{**} :$ there exists $(x_n) \subseteq X$ with (x_n) converging weak* in X^{**} to $x^{**}\}$, then $B_1(X) \subseteq B_1(C(K))$ and $B_1(C(K))$ is naturally identified with $B_1(K)$.

Corollary 3.10. *Let* K *be compact metric and let* X *be a closed subspace of* $C(K)$.

a) If $X^{**} \cap [B_1(K)\backslash B_{1/2}(K)] \neq \emptyset$, *then* X *contains a basic sequence with spreading model equivalent to the unit vector basis of* ℓ_1.

b) If $[X^{**} \cap B_{1/4}(K)] \setminus X \neq \emptyset$ then X contains a basic sequence with spreading model equivalent to the summing basis.

Remark 3.11. This corollary has immediate purely local consequences. Thus if X and K are as above and X does not contain ℓ_n^∞'s uniformly, then $X^{**} \cap B_{1/4}(K) \subset X$. Moreover if X is B-convex, *i.e.*, does not contain ℓ_n^1's uniformly, then $X^{**} \setminus X \subset B_{1/2}(K) \setminus B_{1/4}(K)$.

4. $DSC(K)$.

Theorem 4.1. *Let K be compact metric and let $F \in DSC(K) \setminus C(K)$. Then F governs $\{c_0\}$.*

Remark 4.2. If X is a separable Banach space, $K = Ba(X^*)$ in its weak* topology and $F \in X^{**}$, then if $F \in DSC(K)$, $F \in DBSC(K)$ (and hence for such functions Theorem 4.1 follows from Theorem A). To see this assertion, first choose (f_n) uniformly bounded in $C(K)$ so that $f_n \to F$ pointwise and $\sum_{n=1}^\infty |f_{n+1}(k) - f_n(k)| < \infty$ for all $k \in K$. Now since $F \in B_1(X)$, we may choose (g_j) a convex block subsequence of (f_j) and (x_j) a sequence in X with $\|g_j - x_j\| < 2^{-j}$ for all j. But then it follows that $x_j \to F$ pointwise and moreover $\sum_{j=1}^\infty |x_{j+1}(k) - x_j(k)| < \infty$ for all $k \in K$. Thus by the uniform boundedness principle,

$$\sup_{k\in K} \sum_{j=1}^\infty |x_{j+1}(k) - x_j(k)| < \infty ,$$

so $F \in DBSC(K)$.

Theorem 4.1 follows from the stronger result of Elton [13] which was motivated by work of Fonf [16].

Theorem. [13]. *Let X be a Banach space and let $\mathcal{E}$ be the set of extreme points of $Ba(X^*)$. Let (x_i) be a normalized basic sequence in X such that $\sum_{i=1}^\infty |x^*(x_i)| < \infty$ for all $x^* \in \mathcal{E}$. Then $c_0 \hookrightarrow [(x_i)]$.*

Theorem 4.1. can be phrased in this way provided $\mathcal{E}$ is replaced by $\overline{\mathcal{E}}$. However we wish to present a separate proof of our weaker result which seems to be of interest in its own right. The main step is given by the following lemma. If Y is a subspace of $C(K)$, $U \subseteq K$ and $r > 0$, we say U *r-norms* Y if $\|y|_U\|_\infty \geq r\|y\|$ for all $y \in Y$.

Lemma 4.3. *Let L be a compact metric space and let (f_i) be a normalized basic sequence in $C(L)$. If $c_0 \not\hookrightarrow [(f_i)]$, then there exists a nonempty compact set $K \subseteq L$ and a normalized block basis (g_i) of (f_i) so that*

$$\left\{\begin{array}{l}\text{for any nonempty relatively open subset } U \text{ of } K \text{ there are an} \\ r > 0 \text{ and an } n_0 \in \mathbb{N} \text{ such that } U \text{ } r\text{-norms } [(g_n)_{n=n_0}^{\infty}].\end{array}\right. \tag{4.1}$$

Remark 4.4. It can be deduced from [36] that $[(x_n)]$ contains an isomorph of ℓ_1 iff there exists a compact set $K \subseteq L$ such that (4.1) holds for some fixed $r > 0$ independent of U.

Proof of Lemma 4.3. Let $(U_m)_{m=1}^{\infty}$ be a base of open sets for L. We inductively construct for each m a normalized block basis $(f_i^m)_{i=1}^{\infty}$ of (f_i) and a certain subsequence M of $\mathbb{N}$.

Let $(f_i^0) = (f_i)$ and suppose $(f_i^m)_{i=1}^{\infty}$ has been chosen. There are two possibilities.

(i) There is a normalized block basis (g_i) of $(f_i^m)_{i=1}^{\infty}$ with $\|g_i|_{U_m}\|_{\infty} \to 0$ as $i \to \infty$.

(ii) There exists no such sequence.

If (i) holds, choose $(f_i^{m+1})_{i=1}^{\infty}$ to be a normalized block basis of $(f_i^m)_{i=1}^{\infty}$ with

$$\|f|_{U_m}\|_{\infty} < 2^{-k}\|f\|_{\infty} \text{ for all } f \in \big[(f_i^{m+1})_{i=k}^{\infty}\big] \tag{4.2}$$

and put m in M. If (ii) holds let $(f_i^{m+1})_{i=1}^{\infty} = (f_i^m)_{i=1}^{\infty}$ and put m in $\mathbb{N}\setminus M$. Let $K = L \setminus \bigcup_{m\in M} U_m$ and for all $n \in M$ let $g_n = f_{n+1}^{n+1}$. We may assume M is infinite or else the conclusion of the lemma is satisfied with $K = L$ and $g_i = f_i^m$ ($m = \max M$ or 0 if $M = \emptyset$).

First we check that $K \neq \emptyset$. If $K = \emptyset$, then $L \subseteq \bigcup_{n\in M} U_n$. By compactness there exists $n_1 \in M$ so that $L \subseteq \bigcup_{n\in M\ ,\ n\le n_1} U_n$. But then since $\|g_{n_1}|_{U_n}\|_{\infty} < 2^{-(n_1+1)}$ for $n \in M$ with $n \le n_1$, we have $\|g_{n_1}\|_{\infty} < 1$, a contradiction.

We claim that K and (g_n) satisfy (4.1). If not there exist (h_n), a normalized block basis of (g_n) and a U_m such that $K \cap U_m \neq \emptyset$ and so $m \notin M$ yet $\|h_i|_{K\cap\overline{U}_m}\| < 2^{-i}$ for all i. Indeed there must exist $m' \in M$ with $K \cap U_{m'} \neq \emptyset$ and (h_i), a normalized block basis of (g_n), with $\|h_i|_{K\cap U_{m'}}\| < 2^{-i}$. Then choose $m \in \mathbb{N}$ so that $\overline{U}_m \subseteq U_{m'}$ and $K \cap U_m \neq \emptyset$. Let $j_0 = m$ and if j_i is defined choose $j_{i+1} > j_i$ so that

$$\overline{U}_m \cap [h_{j_i} \ge 2^{-i}] \subseteq \bigcup_{\substack{n\in M \\ n\le j_{i+1}}} U_n \ .$$

This can be done since $\overline{U}_m \cap [h_{j_i} \geq 2^{-i}] \subseteq \overline{U}_m \cap [h_{j_i} \geq 2^{-j_i}] \subseteq L \setminus K = \bigcup_{n\in M} U_n$. This completes the definition of $j_1, j_2, \ldots$. Now for $t \in U_m$, $|h_{j_i}(t)| \geq 2^{-i}$ for at most one i. Indeed let i_0 be the first integer such that $|h_{j_{i_0}}(t)| \geq 2^{-i_0}$ (if such an i_0 exists). Then $t \in \bigcup_{n\in M\ ,\ n\leq j_{i_0+1}} U_n$ and for $i > i_0$, h_{j_i} is a normalized element in $[(g_j)_{j\geq j_i,\, j\in M}] = [(f_{j+1}^{j+1})_{j\geq j_i,\, j\in M}] \subseteq [(f_p^{j_i+1})_{p\geq j_i+1}]$. Thus if $t \in U_n$ with $n \leq j_{i_0+1}, n \in M$, then $h_{j_i} \in [(f_p^{n+1})_{p\geq j_i+1}]$ and so by (4.2), $|h_{j_i}(t)| \leq \|h_{j_i}\big|_{U_n}\| < 2^{-j_i} \leq 2^{-i}$.

Thus $\sum_{i=1}^\infty |h_{j_i}(t)| \leq 2$ for all $t \in \overline{U}_m$. Since $\overline{U}_m$ norms $[h_{j_i}]$, it follows from [7] that $c_0 \hookrightarrow [h_{j_i}]$, a contradiction. ∎

Proof of Theorem 4.1. Let (f_n) be a bounded sequence in $C(K)$ converging pointwise to F. By Lemma 3.3 and passing to a convex block subsequence of (f_n), if necessary, we may suppose that $\sum_{n=1}^\infty |f_{n+1}(k) - f_n(k)| < \infty$ for all $k \in K$. Also since $F \notin C(K)$, by passing to a subsequence $(f'_n) \subseteq (f_n)$ we may assume that $(h_n) \equiv (f'_{2n} - f'_{2n+1})$ is a seminormalized basic sequence satisfying $\sum_{n=1}^\infty |h_n(k)| < \infty$ for all $k \in K$. If $c_0 \not\hookrightarrow [(h_n)]$, then by Lemma 4.3 there exist (g_n), a normalized block basis of (h_n), and a closed nonempty set $K_0 \subseteq K$ satisfying (4.1) (with K replaced by K_0).

For $m \in \mathbb{N}$ set $K_m = \{k \in K_0 : \sum_{n=1}^\infty |g_n(k)| \leq m\}$. Since (g_n) is a normalized block basis of (h_n), $\sum_{n=1}^\infty |g_n(k)| < \infty$ for all $k \in K$ and thus $\bigcup_{m=1}^\infty K_m = K_0$. By the Baire category theorem there exists m_0 so that K_{m_0} has nonempty interior U (relative to K_0). Choose n_0 and $r > 0$ so that U r-norms $[(g_n)_{n\geq n_0}]$. Since $\sum |g_n| \leq m_0$ on U, (g_n) is equivalent to the unit vector basis of c_0 [7], a contradiction. ∎

A natural problem is to classify those functions $F \in B_1(K)$ which govern $\{c_0\}$. We do not know how to do this, but it is easy to see that this class is strictly larger than $DSC(K)$.

Example 4.5. Let L be a countable compact metric space, large enough so that there exists an $F \in B_1(L) \setminus DBSC(L)$ (see Proposition 5.3). Choose a bounded sequence $(f_n) \subseteq C(L)$ which converges pointwise to F and let $X = [(f_n)]$. $C(L)$ is c_0-saturated (every infinite dimensional subspace of $C(L)$ contains c_0 isomorphically) and thus X is c_0-saturated. Thus F governs $\{c_0\}$ by Lemma 3.3. Let $K = Ba(X^*)$. $F \notin DSC(K)$ or otherwise (Remark 4.2) $F \in DBSC(K)$ and hence $F \in DBSC(L)$. Using this example, it can be shown that if

K is any uncountable compact metric space, there exists an $F \in B_1(K) \setminus DSC(K)$ which governs $\{c_0\}$.

Question 4.6. Let $F \in B_1(K)$. If F governs $\{c_0\}$ does there exist a bounded sequence $(f_n) \subseteq C(K)$ converging pointwise to F and a w^*-closed set $L \subseteq Ba[(f_n)]^*$ such that L norms $[(f_n)]$ and $F\big|_L \in DSC(L)$? (Could L be taken to be countable?)

Question 4.7. Let $F \in B_1(K)$. Suppose there exists $(f_n) \subseteq C(K)$, a bounded sequence converging pointwise to F and satisfying $\sum_{n=1}^\infty |f_{n+1}(k) - f_n(k)| < \infty$ for all k in some residual set (complement of a first category set). Does F govern $\{c_0\}$?

We should also mention the following result of Bourgain which gives some global information about the class $DSC(K)$.

Proposition 4.8. [10] *Let $F \in DSC(K) \setminus C(K)$ and let (f_n) be a bounded sequence in $C(K)$ converging pointwise to F with $\sum |f_{n+1}(k) - f(k)| < \infty$ for all $k \in K$. Then there exists a subsequence (f_n') of (f_n) with $[(f_n')]^*$ separable.*

It follows that if $F \in DSC(K) \setminus C(K)$, then F strictly governs the class $\mathcal{C}$ of infinite dimensional Banach spaces with separable duals. However we don't know that if F governs $\{c_0\}$, then F strictly governs $\mathcal{C}$. (A negative answer, of course, would give a negative answer to 4.6.)

We give a somewhat different proof than that of [10].

Proof. We may assume that $\|f_n\| = 1$ for all n. As mentioned in the introduction there exists a subsequence (f_n') of (f_n) which is basic and C_1-dominates the summing basis for some $C_1 < \infty$. It follows that $(h_n)_1^\infty$ is seminormalized basic where $h_1 = f_1'$ and $h_n = f_n' - f_{n-1}'$ for $n > 1$. [Indeed let $(a_i)_1^m$ be given and let $1 \le n < m$ with $\|\sum_1^n a_i h_i\| = 1$. $\sum_{i=1}^n a_i h_i = (a_1 - a_2) f_1' + \cdots + (a_{n-1} - a_n) f_{n-1}' + a_n f_n' \equiv f + a_n f_n'$. If $\|f\| \ge \frac{1}{2}$, then $\|\sum_1^m a_i h_i\| \ge C_2^{-1} \|f\| \ge 2^{-1} C_2^{-1}$ where C_2 is the basis constant of (f_n'). Otherwise

$|a_n| \geq \frac{1}{2}$ and so

$$\Big\| \sum_{i=1}^{m} a_i h_i \Big\| = \Big\| \sum_{i=1}^{m-1} (a_i - a_{i+1}) f_i' + a_m f_m' \Big\|$$

$$\geq (C_2+1)^{-1} \Big\| \sum_{i=n}^{m-1} (a_i - a_{i+1}) f_i' + a_m f_m' \Big\|$$

$$\geq (C_2+1)^{-1} C_1^{-1} \Big\| \sum_{i=n}^{m-1} (a_i - a_{i+1}) s_i + a_m s_m \Big\|$$

$$\geq (C_2+1)^{-1} C_1^{-1} \Big| \sum_{i=n}^{m-1} (a_i - a_{i+1}) + a_m \Big|$$

$$= (C_2+1)^{-1} C_1^{-1} |a_n| \geq 2^{-1} (C_2+1)^{-1} C_1^{-1} \ .]$$

Also for $k \in K$, $\sum_{n=1}^{\infty} |h_n(k)| < \infty$. Thus (h_n) is shrinking. Indeed if (h_n) has basis constant C and $g_n = \sum_{i=p_n+1}^{p_{n+1}} a_i h_i$ is a normalized block basis, then for $k \in K$

$$|g_n(k)| \leq \left(\max_{p_n+1 \leq i \leq p_{n+1}} |a_i| \right) \sum_{i=p_n+1}^{p_{n+1}} |h_i(k)|$$

$$\leq (C+1) \min_i \|h_i\|^{-1} \sum_{i=p_n+1}^{p_{n+1}} |h_i(k)|$$

which goes to 0 as $n \to \infty$. ∎

The following proposition characterizes the subclass $PS(K)$ of $DSC(K)$ which was defined in §1.

Proposition 4.9. *Let $F \in B_1(K)$. The following are equivalent.*

a) *$F \in PS(K)$.*
b) *For all closed $L \subseteq K$, $F\big|_L$ is continuous on a relatively open dense subset of L.*
c) *There exists $\eta < w_1$ and a family $(K_\alpha)_{\alpha \leq \eta}$ of closed subsets of K with $K_0 = K$, $K_\eta = \emptyset$, $K_\gamma = \bigcap_{\alpha<\gamma} K_\alpha$ if γ is a limit ordinal and $K_\alpha \supseteq K_\beta$ if $\alpha < \beta$, such that $F\big|_{K_\alpha \setminus K_{\alpha+1}}$ is continuous for all α.*
d) *There exists a sequence (K_n) of closed subsets of K with $K_n \subseteq K_{n+1}$ for all n such that $K = \bigcup_n K_n$ and $F\big|_{K_n}$ is continuous for all n.*

Remark 4.10. Property (c) suggests the following index for $PS(K)$:

$$I(F) = \inf\big\{\eta < w_1 : \exists\ (K_\alpha)_{\alpha\le\eta} \ \text{ satisfying (c)}\big\}\ .$$

Proof of 4.9. d) $\Rightarrow$ a): Let (K_n) be as in d) and for $n \in \mathbb{N}$ let $f_n \in C(K_n)$ be given by $f_n = F\big|_{K_n}$. By the Tietze extension theorem there exists an extension of f_n, $\widetilde{f}_n \in C(K)$, with $\|\widetilde{f}_n\|_\infty \le \|F\|_\infty$. Clearly $(\widetilde{f}_n)$ is pointwise stabilizing and has limit F.

a) $\Rightarrow$ b): For $n \in \mathbb{N}$ set

$$L_n = \big\{k \in L : f_m(k) = F(k) \ \text{ for } \ m \ge n\big\}$$

where $(f_n) \subseteq C(K)$, $\|f_n\| \le \|F\|$ and (f_n) is pointwise stabilizing with limit F. Let $G = \bigcup_n \operatorname{int}(L_n)$. Thus G is open in L. Also by the Baire Category theorem, G is dense in L.

b) $\Rightarrow$ c): Let $K_0 = K$ and let $K_1 = \sim G_0$ where G_0 is a dense open subset of K and F is continuous on G_0. Now if K_α is defined choose G_α, a dense open subset of K_α, so that $F\big|_{K_\alpha}$ is continuous on G_α and set $K_{\alpha+1} = K_\alpha \setminus G_\alpha$. At limit ordinals γ, set $K_\gamma = \bigcap_{\alpha<\gamma} K_\alpha$. Since K is a separable metric space, $K_\eta = \emptyset$ for some $\eta < w_1$.

c) $\Rightarrow$ d): Let $(K_\alpha)_{\alpha\le\eta}$ be as in c). Let $\mathcal{E}_n \downarrow 0$ and for each n set $K_{\alpha,n} = \{k \in K_\alpha : d(k, K_{\alpha+1}) \ge \mathcal{E}_n\}$ where d is the metric on K. Let $K_n = \bigcup_{\alpha<\eta} K_{\alpha,n}$. We note that K_n is closed. Indeed let $(k_i) \subseteq K_n$ converge to k. Then there exists $\alpha < \eta$ so that $k \in K_\alpha$ but $k \notin K_{\alpha+1}$. We claim that $k_i \in K_{\alpha,n}$ for sufficiently large i and thus $k \in K_{\alpha,n}$ since $K_{\alpha,n}$ is closed. To see this note first that if $k_i \notin K_\alpha$, then $d(k_i, k) \ge \mathcal{E}_n$. Thus for large i, $k_i \in K_\alpha$ and (since $k \notin K_{\alpha+1}$) $k_i \notin K_{\alpha+1}$. Hence $k_i \in K_{\alpha,n}$ for large i (since the $K_{\alpha,n}$'s are disjoint in n).

Finally $F\big|_{K_n}$ is continuous, for if $(k_i) \subseteq K_n$ and (k_i) converges to $k \in K_{\alpha,n}$, then by the above argument $k_i \in K_{\alpha,n}$ for large i and $F\big|_{K_{\alpha,n}}$ is continuous. ∎

We end this section with an improvement of Proposition 4.8 in a special case.

Proposition 4.11. *Let K be a compact metric space and let F be a simple Baire-1 function on K. Then there exists $(f_n) \subseteq C(K)$ converging pointwise to F such that $[(f_n)]$ embeds into $C(L)$ for some countable compact space L.*

Proof. First we consider the case where K is totally disconnected. Choose $\mathcal{E}_0 > 0$ so that if $F(k_1) \neq F(k_2)$, then $|F(k_1) - F(k_2)| > \mathcal{E}_0$. Let $K_\alpha = K_\alpha(F, \mathcal{E}_0)$ for $\alpha \leq \eta$ with $K_\eta = \emptyset$. By our choice of $\mathcal{E}_0$, $F\big|_{K_\alpha \setminus K_{\alpha+1}}$ is continuous (with respect to K_α) for all $\alpha < \eta$.

Choose a countable partition (D_j) of K into closed sets with the following properties.

a) $\operatorname{diam} D_j \to 0$

b) for each j, D_j is a relatively clopen subset of $K_\alpha \setminus K_{\alpha+1}$ for some $\alpha < \eta$ such that $F\big|_{D_j}$ is constant.

This can be done as follows. For each α choose a finite partition of relatively clopen subsets of $K_\alpha \setminus K_{\alpha+1}$ such that F is constant on each set of the partition. Each such set is relatively open in K_α and thus may be in turn partitioned into a countable number of relatively clopen subsets of K_α. List all the sets thus obtained for all $\alpha < \eta$ as $(C_i)_{i=1}^\infty$. Each C_i is closed in K and thus may in turn be partitioned into a finite number of closed subsets of diameter not exceeding $1/i$. We list all these sets as $(D_j)_{j=1}^\infty$.

Let $L = K/\{D_j\}$ be the quotient space of K. Since each D_j is closed and $\operatorname{diam} D_j \to 0$, L is compact metric. For $n \in \mathbb{N}$ choose $\hat{f}_n \in C(L)$ with $\|\hat{f}_n\|_\infty \leq \|F\|_\infty$ and $\hat{f}_n(D_j)$ equal to the constant value of $F\big|_{D_j}$ for $j \leq n$. Let $\phi : K \to L$ denote the quotient map and let $f_n = \hat{f}_n \circ \phi$. Clearly $f_n \in C(K)$, $\|f_n\| \leq \|F\|$ and (f_n) converges pointwise to F. Also $[(f_n)]$ is isometric to $[(\hat{f}_n)] \subseteq C(L)$.

For the general case let $\phi : \Delta \to K$ be a continuous surjection and let F be a simple Baire-1 function on K. By the first part of the proof there exist $(f_n) \subseteq C(\Delta)$ converging pointwise to $F \circ \phi$ and a countable compact metric space L such that $[(f_n)] \hookrightarrow C(L)$. Let (g_n) be a bounded sequence in $C(K)$ converging pointwise to F. By Lemma 3.3 there exist convex block subsequences (h_n) and (d_n) of (g_n) and (f_n), respectively, such that $\sum \|g_n \circ \phi - d_n\| < \infty$. Thus $[(g_n)] \cong [(g_n \circ \phi)] \hookrightarrow C(L)$. ∎

Question 4.12. Does Proposition 4.11 remain true if we only assume $F \in PS(K)$ or even $F \in DSC(K)$? Note that if F satisfies the conclusion of 4.11, F strictly governs the class of c_0-saturated spaces, while it is not clear that DSC functions have this property.

5. The Baire-1 Solar System.

In this section we shall examine the relationships between the various classes of Baire-1 functions which we have defined. We begin with a result which follows easily from the Banach space theory — that developed above and some examples presented in later sections.

Proposition 5.1. *Let K be an uncountable compact metric space. Then*

$$C(K) \subsetneqq DBSC(K) \subsetneqq B_{1/4}(K) \subsetneqq B_{1/2}(K) \subsetneqq B_1(K)\ . \tag{5.1}$$

Proof. Since $C(K)$ and $C(K')$ are isomorphic whenever K and K' are both uncountable compact metric spaces [29], it suffices to separately consider each of the inclusions in (5.1). Thus if we show $C(K') \neq DBSC(K')$ for some uncountable compact metric space K', then $C(K) \neq DBSC(K)$ as well. Indeed if $j : C(K) \to C(K')$ is an onto isomorphism, then $\widetilde{\jmath} = j^{**}|_{B_1(K)} : B_1(K) \to B_1(K')$. is an onto isomorphism satisfying $\widetilde{\jmath}(DBSC(K)) = DBSC(K')$, $\widetilde{\jmath}(B_{1/4}(K)) = B_{1/4}(K')$ and $\widetilde{\jmath}(B_{1/2}(K)) = B_{1/2}(K')$.

For the first inclusion, $C(K) \subsetneqq DBSC(K)$, let $X = c_0$. Then $K = (Ba(X^*), w^*)$ is uncountable compact metric and, as is well known, $X^{**} \subseteq DBSC(K)$. In particular if $F \in X^{**} \setminus X$, then $F \in DBSC(K) \setminus C(K)$.

The fact that $B_{1/4}(K) \supsetneqq DBSC(K)$ follows from Theorem A(b) and our example in §6 where we produce a nonreflexive separable Banach space X not containing c_0 such that $X^{**} \subseteq B_{1/4}(K)$, where $K = Ba(X^*)$.

For the next inclusion let $X = J$, the James space. J is not reflexive and has no spreading model isomorphic to c_0 or ℓ_1 [1]. Thus if $K = (Ba(J^*), w^*)$, then $X^{**} \setminus X \subseteq B_{1/2}(K) \setminus B_{1/4}(K)$ by virtue of Theorem B.

For the last inclusion let Y be the quasi-reflexive space of order 1 (see the proof of Proposition 6.3) whose dual is $J(e_i)$, where (e_i) is the unit vector basis of Tsirelson's space. It is proved in [32] that the only spreading models of Y are isomorphic to ℓ_1. Thus by Theorem 3.7, if $Y^{**} = Y \oplus \langle F\rangle$ and $K = Ba(Y^*)$, then $F \notin B_{1/2}(K)$. An alternative method would be to consider the quasi-reflexive spaces Q_γ constructed in Proposition 2.8. ∎

Remark 5.2. How does the class $DSC(K)$ relate to the classes in (5.1)? Of course we always have $DBSC(K) \subseteq DSC(K)$ and in fact for K an uncountable compact metric space we have the following diagram.

Thus $DSC(K)$ is an asteroid in the Baire-1 solar system. Indeed our proof of Proposition 5.1 along with Theorem 4.1 yields that $B_{1/4}(K) \setminus DSC(K) \neq \emptyset$, $B_{1/2}(K) \setminus [DSC(K) \cup B_{1/4}(K)] \neq \emptyset$ and $B_1(K) \setminus [DSC(K) \cup B_{1/2}(K)] \neq \emptyset$. The fact that $DSC(K) \cap B_1(K) \setminus B_{1/2}(K)$, $DSC(K) \cap B_{1/2}(K) \setminus B_{1/4}(K)$ and $DSC(K) \cap B_{1/4}(K) \setminus DBSC(K)$ are all nonempty follows from Proposition 5.3 below.

We now turn to the case where K is a *countable* compact metric space. In this setting we have, of course, $DSC(K) = B_1(K)$. However if K is large enough, the classes in (5.1) are still distinct. Since every countable compact metric space is homeomorphic to some countable ordinal, given the order topology [30], we confine ourselves to this setting.

Proposition 5.3.

a) If $K = \omega^{\omega^2}+$, then $B_{1/4}(K) \setminus DBSC(K) \neq \emptyset$.

b) If $K = \omega^{\omega}+$, then $B_{1/2}(K) \setminus B_{1/4}(K) \neq \emptyset$.

c) If $K = \omega^{\omega}+$, then $B_1(K) \setminus B_{1/2}(K) \neq \emptyset$.

d) If $K = \omega^{+}$, then $DBSC(K) \setminus C(K) \neq \emptyset$.

Before proving this proposition we need some terminology. Recall that an indicator function $\mathbf{1}_A$ is Baire-1 iff A is ambiguous (simultaneously F_σ and G_δ). Thus if $A \subseteq K$ where K is countable compact metric, then $\mathbf{1}_A \in B_1(K)$. We begin with a discussion of such functions.

Let δ be a countable compact ordinal space (in its order topology). Recursively we define $I_0 = \emptyset$, $I_1 = \{x \in \delta : x \text{ is an isolated point of } \delta\}$, and for $\alpha > 1$, $I_\alpha =$

$\{x \in \delta \setminus \bigcup_{\beta<\alpha} I_\beta : x$ is an isolated point of $\delta \setminus \bigcup_{\beta<\alpha} I_\beta\}$. The I_α's are just the relative complements of the usual derived sets.

Let us say an ordinal is *even* if it is of the form $\gamma + 2n$ for some $n \in \mathbb{N}$ where $\gamma = 0$ or γ is a limit ordinal. Let $F_\delta = \mathbf{1}_{A_\delta}$ where $A_\delta = \bigcup_{\alpha \text{ even}} I_\alpha$. We have

°1) $\|F_{\omega^n+}\|_\infty = 1$ and $|F_{\omega^n+}|_D = n$.

°2) $|F_\delta|_D = \infty$ if $\delta \geq \omega^\omega+$.

°1) implies °2) trivially. To see °1), one first notes that $K_n(F_{\omega^n+}, 1) \neq \emptyset$. Indeed, $K_\alpha(F_\delta, 1)$ is just the α^{th} derived set of δ. Hence $|F_{\omega^n+}|_D \geq n$ by the proof of Lemma 2.4. We leave the reverse inequality to the reader.

Definition. We say that a function $F : \omega^n+ \to \mathbb{R}$ is of *type* 0 if $F = n^{-1}F_{\omega^n+}$. The domain of F, ω^n+, is called a *space of type* 0.

Thus if F is a function of type 0 with domain ω^n+, $|F|_D = 1$ and $\|F\|_\infty = n^{-1}$.

More generally for $n \in \mathbb{N}$ we have the

Definition. A class of real valued functions $\mathcal{F}_n$ defined on countable compact metric spaces is said to be of *type* n if

a) For $F \in \mathcal{F}_n$, $|F|_D \geq n$.

b) For $F \in \mathcal{F}_n$, F is the uniform limit of (F_m) with $\sup_m |F_m|_D \leq 1$.

c) For each $\varepsilon > 0$, there is an $F \in \mathcal{F}_n$ with $\|F\|_\infty < \varepsilon$.

The domain of $F \in \mathcal{F}_n$ is called a *space of type* n.

Lemma 5.4. *For $n \in \mathbb{N} \cup \{0\}$ there exists a class $\mathcal{F}_n$ of functions of type n.*

Proof. We have seen that $\mathcal{F}_0$ exists. Suppose $\mathcal{F}_n$ exists. To obtain functions $F \in \mathcal{F}_{n+1}$ we begin with a function $G \in \mathcal{F}_0$ defined on a set K. Let $(t_i)_{i=1}^\infty$ be a list of the isolated points of K. We enlarge K as follows. To each t_i we adjoin a sequence of disjoint clopen sets $K_1^i, K_2^i, \ldots$ clustering only at t_i. Each of the K_j^i's is a space of type n supporting a function F_j^i of type n with $\|F_j^i\|_\infty \leq (i + j + m)^{-1}$. Here $m \in \mathbb{N}$ is arbitrary but fixed. K_{n+1}, the new space of type $n + 1$, is this enlarged space. Set

$$F(t) = \begin{cases} G(t)\ , & t \in K \\ F_j^i(t)\ , & t \in K_{ij}\ . \end{cases}$$

Let $\mathcal{F}_{n+1}$ be the set of all such F's thusly obtained. We must check that $\mathcal{F}_{n+1}$ satisfies a) and b) with n replaced by $n+1$ (c) is immediate). b) holds since F is the uniform limit of (F_k) where

$$F_k(t) = \begin{cases} G(t) \ , & t \in K \\ F_j^i(t) \ , & t \in K_j^i \text{ with } i+j \le k \\ 0 \ , & \text{otherwise} \end{cases}$$

and each F_k is the uniform limit of $(F_{k,n})_{n=1}^\infty$ where $|F_{k,n}|_D \le 1$ for all n.

Finally we check a). Let $(f_m)_0^\infty \subseteq C(K_{n+1})$, $f_0 \equiv 0$, converge pointwise to F. Since $F\big|_K = G$ and $|G|_D = 1$, for $\varepsilon > 0$ there exist $t_{i_0} \in K$ and $k \in \mathbb{N}$ with $\sum_{i=0}^{k-1} |f_{i+1}(t_{i_0}) - f_i(t_{i_0})| > 1 - \varepsilon$. Moreover by the nature of G we may assume $|G(t_{i_0})| < \varepsilon$. Since the $K_j^{i_0}$'s cluster at t_{i_0} and each f_i is continuous there exists $j_0 \in \mathbb{N}$ so that for $t \in K_{j_0}^{i_0}$, $\sum_{i=0}^{k-1} |f_{i+1}(t) - f_i(t)| > 1 - \varepsilon$ and $|f_k(t)| < \varepsilon$. But on $K_{j_0}^{i_0}$, (f_m) converges pointwise to $F_{j_0}^{i_0}$ and $|F_{j_0}^{i_0}|_D \ge n$. Thus there exists $t \in K_{j_0}^{i_0}$ with

$$|f_{k+1}(t)| + \sum_{i>k} |f_{i+1}(t) - f_i(t)| > n - \varepsilon \ .$$

It follows that

$$\sum_{i=0}^{\infty} |f_{i+1}(t) - f_i(t)| > n + 1 - 3\varepsilon$$

which proves a). ∎

Remark 5.4. Our proof yields that the spaces of type-n can be constructed within $\omega^{\omega \cdot (n+1)}+$.

Proof of Proposition 5.3. a) Let $K = \omega^{\omega^2}+$ and choose (by Remark 5.4) a sequence $(K_n)_{n=0}^\infty$ of disjoint clopen subspaces of K with K_n of type-n. Let F_n be a function of type-n supported on K_n with $\|F_n\|_\infty \to 0$ and let F be the sum of the F_n's. Clearly $|F|_D = \infty$ since $|F_n|_D \ge n$. Yet F is the uniform limit of a sequence of functions with $|\cdot|_D$ not exceeding 1.

b) Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of disjoint clopen subspaces of type-0 of $\omega^\omega + = K$ such that K_n supports a function F_n, which is a multiple of a function of type-0, with $\|F_n\|_\infty \le n^{-1}$ and $|F_n|_D \ge n$. Define

$$F(t) = \begin{cases} F_n(t) & \text{if } t \in K_n \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $F \in B_{1/2}(K) \setminus B_{1/4}(K)$.

c) The type-0 function $F_{\omega^\omega +}$ is not Baire-1/2.

d) $F_{\omega +}$ is $DBSC$. ∎

It is easy to check that the results of Proposition 5.3 are best possible.

6. A Characterization of $B_{1/4}(K)$ and an Example.

In this section we give an example which shows that functions of class Baire-1/4 need not govern $\{c_0\}$. Thus Theorem B(b) is best possible. Before giving the example we give a sufficient (and necessary) criterion for a function to be Baire-1/4.

Theorem 6.1. *Let K be a compact metric space and let $F \in B_1(K)$. Then $F \in B_{1/4}(K)$ iff there exists a $C < \infty$ such that for all $\varepsilon > 0$ there exists a sequence $(S_n)_{n=0}^\infty \subseteq C(K)$, $S_0 \equiv 0$, with $S_n(k) \to F(k)$ for all $k \in K$ and such that for all subsequences (n_i) of $\{0\} \cup \mathbb{N}$ and $k \in K$,*

$$\sum_{j \in B((n_i),k)} |S_{n_{j+1}}(k) - S_{n_j}(k)| \le C \ . \tag{6.1}$$

Here $B((n_i),k) = \{j : |S_{n_{j+1}}(k) - S_{n_j}(k)| \ge \varepsilon\}$.

Proof. First assume $F \in B_{1/4}(K)$, let $\varepsilon > 0$ and let $\varepsilon_n \downarrow 0$. By the proof of Theorem B(b) there exists $(f_n)_{n=0}^\infty \subseteq C(K)$, $f_0 \equiv 0$, converging pointwise to F with the following property. For each $m \in \mathbb{N}$, there exists $(h_j^m)_{j=0}^\infty \subseteq C(K)$ with $h_0^m \equiv 0$ and

$$\sum_{j=0}^\infty |h_{j+1}^m(k) - h_j^m(k)| \le M \equiv 2|F|_{1/4} \ , \quad \text{for} \quad k \in K \ . \tag{6.2}$$

Furthermore $\|h_j^m - f_j\|_\infty \le \varepsilon_m$ for $j \ge m$.

Let $\varepsilon > 0$ and fix m with $4\varepsilon_m < \varepsilon$. Let $(S_n)_{n=0}^\infty = (0, f_m, f_{m+1}, \ldots)$, and let (n_i) be a subsequence of $\{0\} \cup \mathbb{N}$ and let $k \in K$ be fixed. Then

$$\sum_{j \in B((n_i),k)} |S_{n_{j+1}}(k) - S_{n_j}(k)| \le \sum_{j=0}^\infty |h_{j+1}^m(k) - h_j^m(k)| + 2\varepsilon_m \# B((n_i),k) \ . \tag{6.3}$$

Since $|f_p(k) - f_q(k)| \ge \varepsilon$ implies for $p > q \ge m$ or $q = 0$ that $|h_p^m(k) - h_q^m(k)| \ge \varepsilon - 2\varepsilon_m > \varepsilon/2$, (6.2) yields that $\# B((n_i),k) \le 2M/\varepsilon$. Thus (6.3) yields (6.1) with $C = 2M = 4|F|_{1/4}$.

For the converse, let $C > \varepsilon > 0$ and let $(S_n)_0^\infty \subseteq C(K)$, $S_0 \equiv 0$, converge pointwise to F and satisfy (6.1) for any subsequence (n_i) of $\{0,1,2,\ldots\}$ and any $k \in K$. For $k \in K$ we linearly extend the sequence $(S_n(k))_{n=0}^\infty$ to $(S_r(k))_{r\ge 0}$. Precisely, if $r = \lambda n + (1-\lambda)(n+1)$ we set $S_r(k) = \lambda S_n(k) + (1-\lambda) S_{n+1}(k)$. Since the S_n's are continuous, $S_r \in C(K)$ as well. Furthermore, if $0 \le r_1 < r_2 < r_3 < \cdots$, $k \in K$ and $B = B((r_i),k) = \{j : |S_{r_{j+1}}(k) - S_{r_j}(k)| \ge \varepsilon\}$, then

$$\sum_{j\in B} |S_{r_{j+1}}(k) - S_{r_j}(k)| \le 3C \ . \tag{6.4}$$

Indeed if $J_n = \{j \in B : n \le r_j < r_{j+1} \le n+1\} \ne \emptyset$, then $\varepsilon \le \sum_{j\in J_n} |S_{r_{j+1}}(k) - S_{r_j}(k)| \le |S_{n+1}(k) - S_n(k)|$. If $j \in B \setminus \bigcup_n J_n$, there exists integers ℓ_j and m_j with $\ell_j - 1 \le r_j < \ell_j \le m_j < r_{j+1} \le m_{j+1}$. Thus by linearity for some choice of $p_j \in \{\ell_j - 1, \ell_j\}$ and $q_j \in \{m_j, m_j+1\}$ we have $\varepsilon \le |S_{r_{j+1}}(k) - S_{r_j}(k)| \le |S_{q_j}(k) - S_{p_j}(k)|$. Thus

$$\begin{aligned}\sum_{j\in B} |S_{r_{j+1}}(k) - S_{r_j}(k)| &\le \sum_{\{n:J_n\ne\emptyset\}} |S_{n+1}(k) - S_n(k)| \\ &\quad + \sum_{2j\in B\setminus\bigcup_n J_n} |S_{q_{2j}}(k) - S_{p_{2j}}(k)| + \sum_{2j+1\in B\setminus\bigcup_n J_n} |S_{q_{2j+1}}(k) - S_{p_{2j+1}}(k)| \le 3C \ .\end{aligned}$$

We shall construct a sequence $(f_n)_{n=0}^\infty \subseteq C(K)$, $f_0 \equiv 0$, such that for $k \in K$,

$$\sum_{n=0}^\infty |f_{n+1}(k) - f_n(k)| \le 4C \quad \text{and} \tag{6.5}$$

$$\text{if } H \text{ is the pointwise limit of } (f_n) \text{ then } \|H - F\|_\infty \le 5\varepsilon \ . \tag{6.6}$$

This will complete the proof.

Each f_n shall be an average of functions S_t where $t : K \to [0,\infty)$ is continuous and $S_t(k) \equiv S_{t(k)}(k)$ for $k \in K$. Let $f_0 = S_0 \equiv 0$. Let $\alpha_1^1 : [0,\infty) \to [0,1]$ be identically 0 on $[0,\varepsilon]$, identically 1 on $[3\varepsilon/2,\infty)$ and linear on $[\varepsilon, 3\varepsilon/2]$. Let $\alpha_2^1 : [0,\infty) \to [0,1]$ be identically 0 on $[0,3\varepsilon/2]$, identically 1 on $[2\varepsilon,\infty)$ and linear on $[3\varepsilon/2,\varepsilon]$. For $i = 1,2$ let $t_i(k) = \alpha_i^1(|S_1(k)|)$. Let $f_1 = 2^{-1}(S_{t_1} + S_{t_2})$. We next define continuous functions $t_{i,j}$ for $i = 1,2$ and $j = 1,2,3,4$ by $t_{i,j}(k) = t_i(k) + \alpha_j^2(|S_2(k) - S_{t_i}(k)|)(2 - t_i(k))$. Here $\alpha_j^2 : [0,\infty) \to [0,1]$ is identically 0 on $[0,(4+j-1)\varepsilon/4]$, identically 1 on $[(4+j)\varepsilon/4,\infty)$ and linear on $[(4+j-1)\varepsilon/4, (4+j)\varepsilon/4]$. Set $f_2 = 8^{-1}\sum_{i=1}^2 \sum_{j=1}^4 S_{t_{i,j}}$.

In general if $f_n = 2^{-1}2^{-2}\cdots 2^{-n}\sum S_{t_{i_1,\ldots,i_n}}$, where the indices of summation range over $\{(i_1,\ldots,i_n) : 1 \le i_j \le 2^j\}$. We define $t_{i_1,\ldots,i_n}$ for $1 \le i_{n+1} \le 2^{n+1}$ by

$$\begin{aligned} t_{i_1,\ldots,i_{n+1}}(k) &= t_{i_1,\ldots,i_n}(k) \\ &\quad + \alpha^{n+1}_{i_{n+1}}\big(|S_{n+1}(k) - S_{t_{i_1,\ldots,i_n}}(k)|\big)\big(n+1-t_{i_1,\ldots,i_n}(k)\big)\ . \end{aligned}$$

The functions α^{n+1}_j for $1 \le j \le 2^{n+1}$ are defined as before to be identically 0 on $[0,\varepsilon+(j-1)\varepsilon 2^{-n-1}]$, identically 1 on $[\varepsilon+j\varepsilon 2^{-n-1}]$ and linear on $[\varepsilon+(j-1)\varepsilon 2^{-n-1}, \varepsilon+j\varepsilon 2^{-n-1}]$.

The point of the construction is this. For $k \in K$ and $(i_1,\ldots,i_n)$ fixed, $|S_{t_{i_1,\ldots,i_{n+1}}}(k) - S_{t_{i_1,\ldots,i_n}}(k)|$ is either 0 or a number exceeding ε for all but perhaps one choice of i_{n+1}. [This is because the nonconstant parts of the α^{n+1}_j's are disjointly supported.] Also except for at most one value of i_{n+1}, if $|S_{t_{i_1,\ldots,i_{n+1}}}(k) - S_{t_{i_1,\ldots,i_n}}(k)| \ge \varepsilon$ then $S_{t_{i_1,\ldots,i_{n+1}}}(k) = S_{n+1}(k)$.

We next check (6.5). Fix $k \in K$ and $m \in \mathbb{N}$. A simple calculation using the triangle inequality shows that

$$\sum_{n=0}^{m} |f_{n+1}(k) - f_n(k)| \le \text{AVE}\sum_{n=0}^{m} |S_{t_{i_1,\ldots,i_{n+1}}}(k) - S_{t_{i_1,\ldots,i_n}}(k)| \tag{6.7}$$

where the average is taken over $\{(i_1,\ldots,i_m) : 1 \le i_j \le 2^j \text{ for all } j\}$. If we fix $(i_1,\ldots,i_m)$ and let

$$B = \big\{n \le m \mid |S_{t_{i_1,\ldots,i_{n+1}}}(k) - S_{t_{i_1,\ldots,i_n}}(k)| \ge \varepsilon\big\}$$

then

$$\sum_{n\in B} |S_{t_{i_1,\ldots,i_{n+1}}}(k) - S_{t_{i_1,\ldots,i_n}}(k)| \le 3C$$

by (6.4).

Now for $1 \le n \le m$ fixed, the percentage of terms in the "AVE" of (6.7) for which $0 < |S_{t_{i_1,\ldots,i_{n+1}}}(k) - S_{t_{i_1,\ldots,i_n}}(k)| < \varepsilon$ is at most 2^{-n-1}. It follows that

$$\text{AVE}\ \sum_{n=0}^{m} |S_{t_{i_1,\ldots,i_{n+1}}}(k) - S_{t_{i_1,\ldots,i_n}}(k)| \le 3C + 2^{-1}\varepsilon + \cdots + 2^{-m-1}\varepsilon$$

and (6.5) follows from this since $\varepsilon < C$.

(6.5) implies (f_n) is pointwise convergent to some function H. For fixed $k \in K$ choose $m \in \mathbb{N}$ so that $2^{-m}C < \varepsilon$, $|S_m(k) - F(k)| < \varepsilon$ and $|f_m(k) - H(k)| < \varepsilon$. We claim that $|f_m(k) - S_m(k)| < 3\varepsilon$, which proves (6.6). Indeed

$$f_m(k) = \text{AVE}\ S_{t_{i_1,\ldots,i_m}}(k) \quad\text{and}\quad C \ge |S_{t_{i_1,\ldots,i_m}}(k) - S_m(k)| \ge 2\varepsilon$$

for at most $2^{-m}\#\{(i_1,\ldots,i_m) : i_j \le 2^j\}$ choices of $(i_1,\ldots,i_m)$. Thus $|f_m(k) - S_m(k)| \le 2\varepsilon + 2^{-m}C < 3\varepsilon$. ∎

Remark 6.2. Let $F \in B_1(K)$. Our proof shows that $F \in B_{1/4}(K)$ iff there exists $C < \infty$ and $(S_n)_{n=0}^\infty \subseteq C(K)$, $S_0 \equiv 0$, converging pointwise to F such that for all $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that if (n_i) is any subsequence of $\{0, m, m+1, \ldots\}$ then (6.1) holds.

Proposition 6.3. *There exists a compact metric space K and $F \in B_{1/4}(K)$ which does not govern $\{c_0\}$.*

Proof. Let (e_i) be the unit vector basis of the Tsirelson space T constructed in [17] (see also [11]) and let $X = J(e_i)$ be its "Jamesification" as described in [6]. For completeness we recall the definition of X. Let c_{oo} be the linear space of all finitely supported functions $x : \mathbb{N} \to \mathbb{R}$ and for $n \in \mathbb{N}$ define $S_n : c_{oo} \to \mathbb{R}$ by $S_n(x) = \sum_{i=1}^n x(i)$. Let $S_0 \equiv 0$. For $x \in c_{oo}$ let

$$\|x\| = \sup\left\{ \Big\| \sum_{i=1}^m (S_{n_i} - S_{p_i - 1})(x) e_{p_i} \Big\|_T \;\Big|\; 1 \le p_1 \le n_1 < p_2 \le n_2 < \cdots < p_m \le n_m \right\} .$$

Let X be the completion of $(c_{oo}, \|\cdot\|)$.

As shown in [6], the unit vectors (u_i) form a boundedly complete normalized basis for X. Thus $X = Y^*$ (where $Y = [(u_i^*)] \subseteq X^*$). Furthermore it was shown that Y is quasi-reflexive and Y^{**} has a basis given by $\{S, u_1^*, u_2^*, \ldots\}$, where

$$S\Big(\sum a_i u_i\Big) = \sum_1^\infty a_i \ .$$

Of course (u_i^*) are the biorthogonal functionals to (u_i) and S is the weak* limit in Y^{**} of (S_n).

Let $K = Ba(X) = Ba(Y^*)$ in the weak* topology (of Y^*). Since Y does not contain c_0, our example will be complete if we can prove that $S \in B_{1/4}(K)$. By Theorem 6.1 it suffices to prove that if $\varepsilon > 0$ then for $m \in \mathbb{N}$ with $m > 2/\varepsilon$, if $x \in Ba(X)$ and (n_i) is a subsequence of $\{m, m+1, m+2, \ldots\}$, then

$$\sum_{j \in B} |S_{n_{j+1}}(x) - S_{n_j}(x)| \le 2$$

where

$$B = \left\{ j : |S_{n_{j+1}}(x) - S_{n_j}(x)| \geq \varepsilon \right\} \ .$$

We first note that $\#B < m$. Indeed if $\#B \geq m$, then by the properties of T,

$$\begin{aligned} 1 \geq \|x\| &\geq \Big\| \sum_{j \in B} \big(S_{n_{j+1}}(x) - S_{n_j}(x)\big) e_{n_j} \Big\|_T \\ &\geq 2^{-1} m\varepsilon \ , \end{aligned}$$

a contradiction. The last inequality is due to the fact that $\| \sum_A a_i e_i \|_T \geq 2^{-1} \sum_A |a_i|$ provided $\min A \leq \#A$.

Thus $m \leq \min B \leq \#B$ and so

$$\begin{aligned} \sum_{j \in B} |S_{n_{j+1}}(x) - S_{n_j}(x)| \leq 2 \Big\| \sum_{j \in B} \big(S_{n_{j+1}}(x) - S_{n_j}(x)\big) e_{n_j} \Big\| \\ \leq 2\|x\| \leq 2 \ . \end{aligned}$$

■

Remark 6.4. Our proof of Proposition 6.3 shows that there exists a quasi-reflexive (of order one) Banach space Y such that if $K = Ba(Y^*)$ then $Y^{**} \setminus Y \subseteq B_{1/4}(K)$. In particular, it follows that there exists an $F \in B_{1/4}(K) \setminus C(K)$ which strictly governs the class of quasi-reflexive Banach spaces.

7. Some Bad Baire-1/2 Functions.

In this section we show that functions of class Baire-1/2 need not be that nice.

Proposition 7.1. *There exists a compact metric space K and $F \in B_{1/2}(K)$ which governs $\{\ell_1\}$.*

Remark 7.2. The first example of an $F \in B_1(K)$ which governs $\{\ell_1\}$ was due to Bourgain [9,10]. His ingenious construction forms the motivation behind our next example (Proposition 7.3). Another example of such an F appears in [2]. While the example of [2] can be shown to be Baire-1/2, we prefer to present a very slight modification.

Proof. Let (e_n) be the unit vector basis of a Lorentz sequence space $d_{w,1}$ (see *e.g.*, [27]). Let $J(e_i)$ be the Jamesification of (e_n) (see [6]) and let (u_i) be the unit vector basis of $J(e_i)$. Thus

$$\Big\| \sum_{i=1}^{k} a_i u_i \Big\| = \sup \left\{ \Big\| \sum_{i=1}^{p} \Big(\sum_{j=n_i}^{m_i} a_j \Big) e_i \Big\|_{d_{w,1}} \ \Big|\ 1 \le n_1 \le m_1 < n_2 \le m_2 < \cdots < n_p \le m_p \right\} .$$

(u_i) is a normalized spreading basis for $J(e_i)$ which is not equivalent to the unit vector basis of ℓ_1 and thus by [36], (u_i) is weak Cauchy. Furthermore by standard block basis arguments one can show that $J(e_i)$ is hereditarily ℓ_1. Also if F is defined by $u_i \to F$ weak* then $F \in B_{1/2}(K)$ where $K = Ba(J(e_i)^*)$. But this is immediate by Theorem B(a) since (u_i), being its own spreading model, does not have ℓ_1 as a spreading model. The fact that F governs ℓ_1 follows from Lemma 3.3. Indeed if (f_n) is a bounded sequence in $C(K)$ converging pointwise to F, then some convex block subsequence of (f_n) is a basic sequence equivalent to some convex block subsequence of (u_i). Since $[(u_i)]$ is hereditarily ℓ_1, $\ell_1 \hookrightarrow [(f_n)]$. ■

Proposition 7.3. *There exists a compact metric space K and $F \in B_{1/2}(K)$ such that F does not govern $\{\ell_1\}$ yet F strictly governs $\{X : X$ is separable and X^* is not separable$\}$.*

Remark 7.4. In [33] a function $F \in B_1(K) \setminus B_{1/2}(K)$ was constructed satisfying the conclusion of Proposition 7.3. The construction we now present will be a modification of that example.

Proof of Proposition 7.3. We begin by defining a Banach space Y. (The space Y was first defined in [34]) Let $\mathcal{D} = \{\phi\} \cup \bigcup_n \{0,1\}^n$ be the dyadic tree with its natural order (see Remark 4.2) and let $(K_\alpha)_{\alpha\in\mathcal{D}}$ be the natural clopen base for the Cantor set Δ. For $f \in C(K_\alpha)$ we let $\widetilde{f} \in C(\Delta)$ be given by $\widetilde{f}(t) = f(t)$ for $t \in K_\alpha$ and $\widetilde{f}(t) = 0$ otherwise. Let

$$Y = \Bigg\{ (f_\alpha)_{\alpha\in\mathcal{D}} \mid f_\alpha \in C(K_\alpha) \text{ for all } \alpha \in \mathcal{D} \text{ and}$$

$$\|(f_\alpha)\|_Y \equiv \sup\Bigg\{ \Bigg(\sum_{k=1}^{\ell} \Big\| \sum_{\alpha\in S_k} \widetilde{f}_\alpha \Big\|_\infty^2 \Bigg)^{1/2} : (S_k)_{k=1}^{\ell} \text{ are disjoint segments in } \mathcal{D} \Bigg\} < \infty \Bigg\} .$$

Y is a Banach space under the given norm.

We shall construct a weak Cauchy sequence $(g_n) \subseteq Y$ with weak* limit F such that

$$\ell_1 \not\hookrightarrow [(g_n)] \ , \tag{7.1}$$

$$\begin{cases} [(h_n)]^* \text{ is nonseparable for every convex block} \\ \qquad \text{subsequence } (h_n) \text{ of } (g_n) \text{ and} \end{cases} \tag{7.2}$$

$$\begin{cases} \text{there exists a weak* closed set } K \subseteq Ba(Y^*) \text{ such that} \\ \qquad K \text{ norms } [(g_n)] \text{ and } F\big|_K \in B_{1/2}(K) \ . \end{cases} \tag{7.3}$$

The proposition follows immediately from (7.1)–(7.3). Indeed to see that F governs $\{X : X$ is separable and X^* is nonseparable$\}$, let (f_n) be a bounded sequence in $C(K)$ converging pointwise to F. By Lemma 3.3 there exist convex block subsequences (d_n) and (h_n) of (f_n) and (g_n), respectively, such that $\|d_n - h_n\|_{C(K)} \to 0$. Since $[(h_n)]^*$ is nonseparable, so is $[(d_n)]^*$.

Our construction of (g_n) depends upon the following (which in turn follows from our discussion of functions of type-0 in §5): for $n \in \mathbb{N}$ there exists $F_n \in B_1(\Delta)$ such that

$$\|F_n\|_\infty = 1 \text{ and} \tag{7.4}$$

$$\begin{cases} |F_n|_D = n \ . \text{ Moreover if } (h_i) \subseteq C(\Delta) \text{ converges pointwise to } F_n \text{ then} \\ \text{there exists } k \in \Delta, \text{ integers } \ell_1 < \ell_2 < \cdots \ell_{n+1} \text{ and } \varepsilon_i = \pm 1 \ (1 \le i \le n) \\ \text{such that } \sum_{i=1}^{n} \varepsilon_i (h_{\ell_{i+1}} - h_{\ell_i})(k) > n-1. \end{cases} \tag{7.5}$$

Actually our F_n's are indicator functions whose domains are countable compact metric spaces K. Of course one can embed K into Δ and the corresponding extended indicator functions have the desired properties (7.4) and (7.5).

We use "$<_L$" for the natural linear order on $\mathcal{D}$. Thus $\phi < 0 < 1 < 00 < 01 < 10 < 11 < 000 < \cdots$. For each $\alpha \in \mathcal{D}$ choose $n_\alpha \in \mathbb{N}$ and $c_\alpha \in \mathbb{R}^+$ satisfying the following seven properties:

i) $\sum_{\beta\in\mathcal{D}} c_\beta \le 1$.

ii) $c_\alpha^{-1} n_\alpha^{-1} \sum_{\beta<\alpha} n_\beta < 1/10$.

iii) $2c_\alpha^{-1} \sum_{\beta>\alpha} c_\beta < 1/10$.

iv) $1 - n_\alpha^{-1} > 9/10$.

v) $2c_{\alpha_0} c_\alpha^{-1} < 1/10$ if $\alpha <_L \alpha_0$.

vi) $c_{\alpha_0} c_\alpha^{-1} n_{\alpha_0} n_\alpha^{-1} < 1/10$ if $\alpha_0 <_L \alpha$.

vii) $\sum_{\beta\in\mathcal{D}} b_\beta^2 < \infty$ where $b_\beta = \sum_{\gamma \ge_L \beta} c_\gamma$.

Of course we could trim this list somewhat, but we prefer to list the properties in the form in which they are used. The c_α's and n_α's can be chosen as follows. Let $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ be a listing of $\mathcal{D}$ in the linear order. Let $c_{\alpha_j} = (22)^{-j}$. It is quickly checked that properties i), iii), v) and vii) hold. We then choose n_{α_j} inductively to be an increasing sequence of positive integers with $n_{\alpha_1} = 11$ (so that iv) holds). If n_{α_j} is picked, choose $n_{\alpha_{j+1}}$ to satisfy ii) and vi) for $\alpha = \alpha_{j+1}$. For each $\alpha \in \mathcal{D}$, let $F_{n_\alpha} \in B_1(K_\alpha)$ satisfy (7.4) and (7.5) (with Δ replaced by K_α and n replaced by n_α).

For each $\alpha \in \mathcal{D}$ choose $(f_\alpha^n)_{n=1}^\infty \subseteq C(K_\alpha)$, $f_\alpha^n \ge 0$ and $\|f_\alpha^n\| = 1$, so that $(f_\alpha^n)_{n=1}^\infty$ converges pointwise to F_{n_α} and is equivalent to (s_n) with

$$(7.6) \qquad |f_\alpha^1(k)| + \sum_{n=1}^\infty |f_\alpha^{n+1}(k) - f_\alpha^n(k)| \le n_\alpha \quad \text{for all} \quad k \in K_\alpha \ .$$

Let $g_n = (c_\alpha f_\alpha^n)_{\alpha\in\mathcal{D}}$. Clearly $g_n \in Y$ since $\|g_n\| \le \sum_{\alpha\in\mathcal{D}} c_\alpha \le 1$ by i). Furthermore $\ell_1 \not\hookrightarrow [(g_n)]$ by the following lemma and the fact that for all α, $\ell_1 \not\hookrightarrow [f_\alpha^n : n \in \mathbb{N}]$.

Lemma 7.5. *For all $\alpha \in \mathcal{D}$, let Y_α be a closed subspace of $C(K_\alpha)$ which does not contain ℓ_1. Let*

$$\widetilde{Y}_\alpha = \left\{(h_\beta)_{\beta\in\mathcal{D}} \in Y : h_\alpha \in Y_\alpha \ \text{ and } \ h_\beta \equiv 0 \ \text{ if } \ \alpha \ne \beta\right\} \ .$$

Let Z be the closed linear span of $\{\widetilde{Y}_\alpha : \alpha \in \mathcal{D}\}$. Then Z does not contain ℓ_1.

Proof. It is shown in [34] that Y does not contain a sequence $(h_n)_{n=1}^\infty = ((h_\alpha^n)_{\alpha\in\mathcal{D}})_{n=1}^\infty$ which is both equivalent to the unit vector basis of ℓ_1 and has the following property: for all $\alpha_0 \in \mathcal{D}$ there exists $m_0 \in \mathbb{N}$ so that for $m \geq m_0$ and $\alpha \leq_L \alpha_0$, $h_\alpha^m \equiv 0$.

But if Z contains ℓ_1, then Y must contain such a sequence (h_n). This follows easily from the fact that if $(f_n)_{n=1}^\infty$ is an ℓ_1-basis in Z, then for all $\varepsilon > 0$ and $\alpha_0 \in \mathcal{D}$, there exists a normalized block basis $(d_n)_{n=1}^\infty = ((d_\alpha^n)_{\alpha\in\mathcal{D}})_{n=1}^\infty$ of (f_n) with $\|d_{\alpha_0}^n\|_{C(K_{\alpha_0})} < \varepsilon$ for all n. ∎

Thus by [36] we may pass to a subsequence of (g_n) which is weak Cauchy. By relabeling we assume that (g_n) itself is weak Cauchy and converges weak* to $F \in Y^{**}$.

We next verify (7.2). Let (h_n) be a convex block subsequence of (g_n). For $k \in \Delta$ and $h = (h_\alpha)_{\alpha\in\mathcal{D}} \in Y$, define $\delta_k(h) = \sum_{\alpha\in\gamma_k} \widetilde{h}_\alpha(k)$ where $\gamma_k = \{\alpha \in \mathcal{D} : k \in K_\alpha\}$. Clearly δ_k is a normalized element of Y^*. We shall show that

$$(7.7) \qquad \begin{cases} \text{for all } \alpha \in \mathcal{D} \text{ there exists } k_\alpha \in K_\alpha \text{ and } h = (h_\beta) \in Ba[(h_n)] \\ \text{such that } \delta_{k_\alpha}(h) > 7/10 \text{ and } \delta_k(h) < 3/10 \text{ if } k \in \Delta \setminus K_\alpha. \end{cases}$$

As in [33] this implies $[(h_n)]^*$ is nonseparable. Indeed by (7.7) we can choose $(h^\alpha)_{\alpha\in\mathcal{D}} \subseteq Ba[(h_n)]$ and a collection of basic clopen sets $(K'_\alpha)_{\alpha\in\mathcal{D}}$ in Δ such that for all $\alpha \in \mathcal{D}$,

a) $K'_{\alpha,0} \cap K'_{\alpha,1} = \emptyset$,

b) $K'_{\alpha,\varepsilon} \subseteq K'_\alpha$ for $\varepsilon = 0, 1$ and

c) $\delta_k(h_\alpha) > 7/10$ for $k \in K'_\alpha$ and
$\delta_k(h_\alpha) < 3/10$ for $k \notin K'_\alpha$.

For each branch (a maximal subset linearly ordered by $<$) γ in $\mathcal{D}$ choose $k_\gamma \in \bigcap_{\alpha\in\gamma} K'_\alpha$. By a) and b) k_γ is well defined and $k_\gamma \neq k_{\gamma'}$ if $\gamma \neq \gamma'$. By c), $\|(\delta_{k_\gamma} - \delta_{k_{\gamma'}})|_{[(h_n)]}\| > 2/5$ if $\gamma \neq \gamma'$.

We return to the proof of (7.7). Fix $\alpha \in \mathcal{D}$ and set $h_n = (h_\beta^n)_{\beta\in\mathcal{D}}$. Since $(h_\alpha^n)_{n=1}^\infty$ is a convex block subsequence of $(c_\alpha f_\alpha^n)_{n=1}^\infty$, $(h_\alpha^n)_{n=1}^\infty$ converges pointwise to $c_\alpha F_{n_\alpha}$. Thus by (7.5) and (7.6) we may assume (by passing to a subsequence and relabeling, if necessary) that there exist $\varepsilon_i = \pm 1$ $(1 \leq i \leq n_\alpha)$ and $k_\alpha \in K_\alpha$ such that

$$n_\alpha \geq \sum_{i=1}^{n_\alpha} c_\alpha^{-1} \varepsilon_i (h_\alpha^{i+1} - h_\alpha^i)(k_\alpha) > n_\alpha - 1 \ .$$

Let $h = n_\alpha^{-1} c_\alpha^{-1} \sum_{i=1}^{n_\alpha} \varepsilon_i (h_{i+1} - h_i) \equiv (h_\beta)_{\beta\in\mathcal{D}}$. Thus $1 \geq h_\alpha(k_\alpha) > 1 - n_\alpha^{-1} > 9/10$ by iv).

Furthermore by applying (7.6) to each $\beta < \alpha$ we have from ii)

$$\begin{aligned}\sum_{\beta<\alpha} \tilde{h}_\beta(k_\alpha) &\leq \sum_{\beta<\alpha} n_\alpha^{-1} c_\alpha^{-1} c_\beta n_\beta \\ &\leq c_\alpha^{-1} n_\alpha^{-1} \sum_{\beta<\alpha} n_\beta < 1/10 \ .\end{aligned}$$

By the triangle inequality and the definition of h,

$$\begin{aligned}\sum_{\beta>\alpha} \tilde{h}_\beta(k_\alpha) &\leq c_\alpha^{-1} n_\alpha^{-1} \sum_{\beta>\alpha} 2c_\beta n_\alpha \\ &= 2c_\alpha^{-1} \sum_{\beta>\alpha} c_\beta < 1/10 \ \text{ (by iii)).}\end{aligned}$$

Thus $\delta_{k_\alpha}(h) > 9/10 - 2/10 = 7/10$ which proves the first part of (7.7).

Let $k \in \Delta \setminus K_\alpha$ be fixed. There exists a unique $\alpha_0 \in \mathcal{D}$ ($\alpha_0 \neq \alpha$) with the same length as α_0, $|\alpha| = |\alpha_0|$, such that $k \in K_{\alpha_0}$. The calculations above yield $\sum_{\beta<\alpha_0} \tilde{h}_\beta(k) + \sum_{\beta>\alpha_0} \tilde{h}_\beta(k) < 2/10$. If $\alpha_0 <_L \alpha$ then by (7.6)

$$\begin{aligned}0 \leq h_{\alpha_0}(k) &= n_\alpha^{-1} c_\alpha^{-1} \sum_{i=1}^{n_\alpha} \varepsilon_i (h_{\alpha_0}^{i+1} - h_{\alpha_0}^i)(k) \\ &\leq n_\alpha^{-1} c_\alpha^{-1} c_{\alpha_0} n_{\alpha_0} \leq 1/10 \ \text{ (by vi)).}\end{aligned}$$

If $\alpha <_L \alpha_0$ then we have (from the equality above) that

$$0 \leq h_{\alpha_0}(k) \leq n_\alpha^{-1} c_\alpha^{-1} c_{\alpha_0} 2n_\alpha = 2c_{\alpha_0} c_\alpha^{-1} < 1/10$$

by v)). It follows that $\delta_k(h) < 3/10$ which completes the proof of (7.7).

Finally, we verify (7.3). Let $S = [\alpha, \beta] \equiv \{\gamma \in \mathcal{D} \mid \alpha \leq \gamma \leq \beta\}$ be a *finite segment* in $\mathcal{D}$. For $k \in K_\beta$ and $f \in Y$ we set $\delta_{S,k}(f) = \sum_{\gamma\in S} \tilde{f}_\gamma(k)$. $\delta_{S,k}(f)$ is defined similarly if $S = [\alpha, \infty) \equiv \{\gamma \in \mathcal{D} : \alpha \leq \gamma\}$ is an infinite segment and $k \in \bigcap_{\beta\in S} K_\beta$. Define

$$\begin{aligned}K = \Bigg\{ &\sum_{i=1}^\infty a_i \delta_{S_i,k_i} : (a_i)_1^\infty \in Ba(\ell_2) \ , \ (S_i)_1^\infty \ \text{ are disjoint segments and} \\ &k_i \in \bigcap_{\beta\in S_i} K_\beta \ \text{ for every } \ i \Bigg\} \ .\end{aligned}$$

From the definition of the norm in Y it is clear that $K \subseteq Ba(Y^*)$. Furthermore it is easy to check that K is weak* closed and K 1-norms Y.

It remains to show that $F\big|_K \in B_{1/2}(K)$. For $m, n \in \mathbb{N}$ let $g(n,m) \in Y$ be given by $g(n,m) = (g(n,m)_\beta)_{\beta \in \mathcal{D}}$ where

$$g(n,m)_\beta = \begin{cases} g_\beta^n & \text{if } |\beta| \le m \\ 0 & \text{otherwise.} \end{cases}$$

Let $y^* = \sum_{i=1}^\infty a_i \delta_{S_i,k_i} \in K$. Then for m fixed,

$$\sum_{n=1}^\infty \Big| [g(n+1,m) - g(n,m)](y^*) \Big| = \sum_{n=1}^\infty \Big| \sum_{i=1}^\infty a_i \sum_{\substack{\gamma \in S_i \\ |\gamma| \le m}} [\tilde{g}_\gamma^{n+1}(k_i) - \tilde{g}_\gamma^n(k_i)] \Big|$$

$$\le \sum_{i=1}^\infty |a_i| \sum_{\substack{\gamma \in S_i \\ |\gamma| \le m}} \sum_{n=1}^\infty |\tilde{g}_\gamma^{n+1}(k_i) - \tilde{g}_\gamma^{n+1}(k_i)|$$

$$\le \sum_{i=1}^\infty |a_i| \sum_{\substack{\gamma \in S_i \\ |\gamma| \le m}} c_\gamma n_\gamma \quad \text{(by (7.6))}$$

$$\le \sum_{|\gamma| \le m} c_\gamma n_\gamma < \infty \ .$$

In particular $(g(n,m))_{n=1}^\infty$ converges pointwise on K to a function $G_m \in DBSC(K)$.

All that remains is to show that $\|G_m - F\big|_K\|_{C(K)} \to 0$ as $m \to \infty$. Let $m \in \mathbb{N}$ be fixed and let $y^* = \sum_{i=1}^\infty a_i \delta_{S_i,k_i} \in K$. Then

$$|G_m(y^*) - F(y^*)| = \Big| \sum_{i=1}^\infty a_i \sum_{\substack{\gamma \in S_i \\ |\gamma| > m}} c_\gamma \tilde{F}_{n_\gamma}(k_i) \Big|$$

$$\le \sum_{i=1}^\infty |a_i| \Bigg(\sum_{\substack{\gamma \in S_i \\ |\gamma| > m}} c_\gamma \Bigg) \ .$$

For each i set

$$b_i^m = \sum_{\substack{\gamma \in S_i \\ |\gamma| > m}} c_\gamma \ .$$

Thus

$$|G_m(y^*) - F(y^*)| \le \Bigg(\sum_{i=1}^\infty (b_i^m)^2 \Bigg)^{1/2}$$

by Hölder's inequality. The latter goes to 0 as $m \to \infty$ by vii). ■

8. Problems.

We have previously raised two problems concerning $B_{1/4}(K)$.

Problem 8.1. Let $F \in B_1(K)$ and $C < \infty$ be such that if $(f_n) \subseteq C(K)$ is a bounded sequence converging pointwise to F, then there exists (g_n), a convex block subsequence of (f_n), with spreading model C-equivalent to the summing basis. Is $F \in B_{1/4}(K)$?

Problem 8.2. Let $F \in B_1(K)$ and assume there exists a $C < \infty$ such that if $(\varepsilon_i) \subseteq \mathbb{R}^+$ and $K_n(F, (\varepsilon_i)) \neq \emptyset$, then $\sum_1^n \varepsilon_i \leq C$. Is $F \in B_{1/4}(K)$?

These problems lead naturally to the following definitions. Let $F \in B_1(K)$.

$$|F|_I = \max\left\{ \sup\left\{ \sum_{i=1}^m \delta_i : K_m\big(F, (\delta_i)\big) \neq \emptyset \right\} , \ \|F\|_\infty \right\} .$$

$$|F|_{I'} = \max\Big\{ \sup\big\{ m\delta : K_m(F, \delta) \neq \emptyset \big\} , \ \|F\|_\infty \Big\} .$$

$$|F|_S = \inf\Big\{ C : \text{ there exist } (f_n) \subseteq C(K) \text{ converging pointwise to } F \text{ with for all } (a_i)_1^k \subseteq \mathbb{R} , \ \lim_{\substack{n_1 \to \infty \\ n_1 < \cdots < n_k}} \Big\| \sum_{i=1}^k a_i f_{n_i} \Big\| \leq C \Big\| \sum_1^k a_i s_i \Big\| \Big\} .$$

Remark 8.3. We do not know if $|F|_I$ or $|F|_{I'}$ are norms. It is clear that $|F|_S$ is a norm and also that

$$\|F\|_\infty \leq |F|_{I'} \leq |F|_I \leq |F|_S \leq |F|_{1/4} \leq |F|_D$$

($|F|_S \leq |F|_{1/4}$ follows from the proof of Theorem B.) Furthermore, using the series criterion for completeness, it is easy to show that $(\{F \in B_1(K) : |F|_S < \infty\}, |\cdot|_S)$ is a Banach space.

Problem 8.4. Are $|\cdot|_I$ and $|\cdot|_S$ equivalent? Are $|\cdot|_S$ and $|\cdot|_{1/4}$ equivalent?

The solution of Problem 8.4 would of course solve Problems 8.1 and 8.2. Furthermore an affirmative answer to Problem 8.2 would yield an affirmative answer to Problems 8.1 and 8.4.

Proposition 8.5. *$|\cdot|_I$ and $|\cdot|_{I'}$ are not (in general) equivalent.*

Proof. Define $F : [0,1]^\omega \to \mathbb{R}$ as follows:

If $t_0 \neq 0$ let

$$F(t_0, t_1, \ldots) = \sin t_0^{-1} \ .$$

If $t_0 = t_1 = \cdots = t_r = 0 \neq t_{r+1}$, set

$$F(t_0, t_1, \ldots) = \frac{1}{r+2} \sin t_r^{-1} \ .$$

It's easy to see that $\operatorname{osc}(F; (0, t_1, t_2, \ldots)) = 2$ for all $t_1, t_2, \ldots \in [0,1]$ and so

$$K_1(F, \varepsilon) = \{0\} \times [0,1]^{\omega \setminus \{0\}}$$

whenever $0 < \delta < 2$. Similar calculations show that if $r = [\![\frac{2}{\varepsilon}]\!]$ then

$$K_r(F, \varepsilon) = \{0\}^r \times [0,1]^{\omega \setminus r}$$

and $K_{r+1}(F, \varepsilon) = \emptyset$. Thus $K_m(F, \varepsilon) \neq \emptyset$ implies $m\varepsilon \leq 2$. On the other hand, for $m \geq 1$,

$$K_m \left(F, \left(2, 1, \frac{2}{3}, \cdots, \frac{2}{m} \right) \right) = \{0\}^m \times [0,1]^{\omega \setminus m} \ . \qquad \blacksquare$$

We conclude by mentioning some further problems for study, some of which have been raised above.

Problem 8.6. Classify (or give useful sufficient conditions) for a function $F \in B_1(K)$ to govern $\{X : X^* \text{ is separable and } \dim X = \infty\}$. In particular is $F \in B_{1/4}(K) \setminus C(K)$ a sufficient condition?

Problem 8.7. Classify those $F \in B_1(K)$ which govern $\{\ell_1\}$, which govern $\{c_0\}$, which govern $\{X : X \text{ is reflexive}\}$ or which govern $\{X : X \text{ is quasi-reflexive}\}$.

We note that if X is a Polish Banach space (*i.e.*, $Ba(X)$ is Polish in the weak topology) then Edgar and Wheeler [14] have shown that X is hereditarily reflexive (see also [37] and [18]). Bellenot [5] and Finet [15] have independently extended this result by showing that whenever X is Polish, if $x^{**} \in X^{**} \setminus X$ then $x^{**}|_{Ba(X^*)}$ strictly governs the class of quasi-reflexive spaces of order 1.

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