ON FUNCTIONS OF FINITE BAIRE INDEX

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Abstract. It is proved that every function of finite Baire index on a separable metric space K is a D-function, i.e., a difference of bounded semi-continuous functions on K. In fact it is a strong D-function, meaning it can be approximated arbitrarily closely in D-norm, by simple D-functions. It is shown that if the n^{th} derived set of K is non-empty for all finite n, there exist D-functions on K which are not strong D-functions. Further structural results for the classes of finite index functions and strong D-functions are also given.

1. Introduction

Throughout, let K be a separable metric space. A function $f : K \to \mathbb{R}$ is called a difference of bounded semi-continuous functions if there exist bounded lower semi-continuous functions u and v on K with $f = u - v$. We denote the class of all such functions by $DBSC(K)$. We shall also refer to members of $DBSC(K)$ as *D-functions*. A classical theorem of Baire (cf. [H, p.274]) yields that $f \in DBSC(K)$ if and only if there exists a sequence (φ_j) of continuous functions on K so that

$$(1) \qquad \sup_{k\in K} \sum |\varphi_j(k)| < \infty \quad \text{and} \quad f = \sum \varphi_j \text{ point-wise.}$$

Now defining $\|f\|_D = \inf\{\sup_{k\in K} \sum |\varphi_j|(k) : (\varphi_j)$ is a sequence of continuous functions on K satisfying (1)$\}$, it easily follows that $DBSC(K)$ is a Banach algebra; and of course $DBSC(K) \subset B_1(K)$ where $B_1(K)$ denotes the (bounded) first Baire class of functions on K; i.e., the space of all bounded functions on K which are the limit of a point-wise convergent sequence of continuous functions on K.

$DBSC(K)$ appears as a natural object in functional analysis. For example, if X is a separable Banach space and K is the unit ball of X^* in the weak*-topology, then X contains a subspace isomorphic to c_0 if and only if there is an f in $X^{**} \sim X$

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with $f \mid K$ in $DBSC(K)$ (cf. [HOR], [R1]). Natural invariants for $DBSC(K)$ are used in a fundamental way in [R1], to prove that c_0 embeds in X provided X is non-reflexive and Y^* is weakly sequentially complete for all subspaces Y of X.

We investigate here a special subclass of $DBSC(K)$, which we term $SD(K)$, and show that all functions of finite Baire index belong to this class.

To motivate the definitions of these objects we first recall the following class of functions. Define $B_{1/2}(K)$ to be the set of all uniform limits of functions in $DBSC(K)$. (The terminology follows that in [HOR].) Functions in $B_{1/2}(K)$ may be characterized in terms of an intrinsic oscillation behavior, which we now give.

For $f : K \to \mathbb{R}$ a given bounded function, let Uf denote the upper semi-continuous envelope of f; $Uf(x) = \overline{\lim}_{y\to x} f(y)$ for all $x \in K$. (We use non-exclusive lim sups; thus equivalently, $Uf(x) = \inf_U \sup_{y\in U} f(y)$, the inf over all open neighborhoods of x.) Now we define $\underline{\mathrm{osc}}\, f$, the lower oscillation of f, by

$$\underline{\mathrm{osc}}\, f(x) = \overline{\lim}_{y\to x} |f(y) - f(x)| \quad \text{for all} \quad x \in K\ . \tag{2}$$

Finally, we define $\mathrm{osc}\, f$, the oscillation of f, by

$$\mathrm{osc}\, f = U\, \underline{\mathrm{osc}}\, f\ . \tag{3}$$

Now let $\varepsilon > 0$. We define the (finite) oscillation sets of f, $\mathrm{os}_j(f,\varepsilon)$, as follows. Set $\mathrm{os}_0(f,\varepsilon) = K$. Suppose $j \geq 0$ and $\mathrm{os}_j(f,\varepsilon)$ has been defined. Let $\mathrm{os}_{j+1}(f,\varepsilon) = \{x \in L : \mathrm{osc}\, f \mid L(x) \geq \varepsilon\}$, where $L = \mathrm{os}_j(f,\varepsilon)$.

We recall the following fact ([HOR]).

Proposition 1.1. *Let $f : K \to \mathbb{R}$ be a given function. The following are equivalent:*

1. $f \in B_{1/2}(K)$.
2. *For all $\varepsilon > 0$, there is an n with $\mathrm{os}_n(f,\varepsilon) = \emptyset$.*

(The proof given in [HOR] for compact metric spaces works for arbitrary separable ones; cf. also [R2].)

Remark. Actually, the sets defined in [HOR] use what we term here the upper oscillation of f, defined by $\overline{\mathrm{osc}}\, f(x) = \overline{\lim}_{y,z\to x} |f(y) - f(z)|$. It is easily seen that $\overline{\mathrm{osc}}\, f$ is upper semi-continuous and

$$\frac{1}{2}\,\overline{\mathrm{osc}}\, f \leq \mathrm{osc}\, f \leq \overline{\mathrm{osc}}\, f \tag{4}$$

Now define $K_j(f,\varepsilon)$ inductively by

$$K_0(f,\varepsilon) = K \quad \text{and} \quad K_{j+1}(f,\varepsilon) = \{x \in K_j : \overline{\text{osc}}\, f \mid K_j(x) \ge \varepsilon\}\ .$$

We then have by (4) that

$$K_j(f,2\varepsilon) \subset \text{os}_j(f,\varepsilon) \subset K_j(f,\varepsilon) \ \text{ for all } \ j\ . \tag{5}$$

Thus f satisfies 2 of 1.1 if and only if for all $\varepsilon > 0$, there is an n with $K_n(f,\varepsilon) = \emptyset$.

Proposition 1.1 suggests the following quantitative notion.

Definition 1. *Let $f : K \to \mathbb{R}$ be a given bounded function and $\varepsilon > 0$. We define $i(f,\varepsilon)$, the ε-oscillation index of f, to be* $\sup\{n : \text{os}_n(f,\varepsilon) \neq \emptyset\}$.

Thus Proposition 1.1 says that $f \in B_{1/2}(K)$ if and only if $i(f,\varepsilon) < \infty$ for all $\varepsilon > 0$.

Definition 2. *A bounded function $f : K \to \mathbb{R}$ is said to be of finite Baire index if there is an n with $\text{os}_n(f,\varepsilon) = \emptyset$ for all $\varepsilon > 0$. We then define $i(f)$, the oscillation index of f, by*

$$i(f) = \max_{\varepsilon > 0} i(f,\varepsilon)\ .$$

Evidently f is continuous if and only if $i(f) = 0$.

Remark. In [HOR], an index $\beta(f)$ is defined as $\beta(f) = \sup_{\varepsilon>0} \min\{j : K_j(f,\varepsilon) = \emptyset\}$. It follows from the remark following Proposition 1.1 that f is of finite index if and only if $\beta(f) < \infty$, and then in fact $\beta(f) = i(f) + 1$.

In [HOR], it is proved that finite index functions belong to $B_{1/4}(K)$, a class properly containing the D-functions. We obtain here that every function of finite Baire index belongs to $DBSC(K)$. In fact, we show that it belongs to the following subclass:

Definition 3. *A function $f : K \to \mathbb{R}$ is said to be a strong D-function if there exists a sequence (φ_n) of simple D-functions with $\|f - \varphi_n\|_D \to 0$. We denote the class of all strong D-functions by $SD(K)$.*

We may thus formulate one of our main results as follows:

Theorem 1.2. *Let $f : K \to \mathbb{R}$ be a function of finite Baire index. Then f belongs to $SD(K)$.*

As we show below it is easily seen that every simple D-function has finite Baire index. Thus Theorem 1.2 yields that $SD(K)$ equals the closure, in D-norm, of the functions of finite index on K. Our proof essentially proceeds from first principles. An alternate argument, using transfinite oscillations, is given in [R2].

An interesting special case of 1.2: *Let $f : [0,1] \to \mathbb{R}$ be bounded such that $\lim_{y\uparrow x} f(y)$, $\lim_{y\downarrow x} f(y)$ exist for all x. Then f is in $SD[0,1]$.* The fact that such functions are in $DBSC[0,1]$ was initially proved jointly by the first and third named authors, and precedes the work given here [C]. (It is a standard elementary result that if f has these properties, then $\mathrm{os}_1(f,\varepsilon)$ is *finite* for all $\varepsilon > 0$, hence $i(f) = 1$.)

It is evident that the simple D-functions form an algebra, hence $SD(K)$ is a Banach algebra. It is proved in [R2] that $SD(K)$ is a lattice, i.e., $|f| \in SD(K)$ if $f \in SD(K)$. We prove here that the functions of finite index form an algebra and a lattice. This follows immediately from the following result.

Theorem 1.3. *Let f, g be bounded real-valued functions on K, of finite index. Let h be any of the functions $f + g$, $f \cdot g$, $\max\{f, g\}$, $\min\{f, g\}$. Then*

$$i(h) \le i(f) + i(g) \ . \tag{6}$$

It is evident that if f is of finite index, then for any non-zero scalar λ, $i(\lambda f) = i(f)$; also it is easy to show that $i(|f|) \le i(f)$. However the assertions of Theorem 1.3 appear to lie below the surface. The quantitative result which does the job (Theorem 2.8 below), is then applied to yield a necessary condition for a function to be in $SD(K)$, which is also sufficient in the case of upper semi-continuous functions.

Theorem 1.4. *Let $f : K \to \mathbb{R}$ be a given bounded function.*

(a) If $f \in SD(K)$, then

$$\lim_{\varepsilon\to 0} \varepsilon i(f,\varepsilon) = 0 \tag{7}$$

(b) If f is semi-continuous and satisfies (7), *then $f \in SD(K)$.*

It is proved in [R2] that every SD-function is a difference of strong D-semi-continuous functions. Evidently Theorem 1.4 yields an effective criterion for distinguishing the class of strong D-semi-continuous functions. However one may construct functions, e.g., on $K = \omega^\omega + 1$, which are not D-functions but satisfy (7), or which are D-functions but not SD-functions, and still satisfy (7). An effective intrinsic criterion involving the "ω^{th} oscillation", which does distinguish SD-functions from D-functions, is given in [R2].

We conclude the article by applying Theorem 1.4(a) to show that $DBSC(K) \sim SD(K)$ is non-empty for all interesting K.

Proposition 1.5. *Assume that $K^{(j)}$, the j^{th} derived set of K, is non-empty for all $j = 1, 2, \ldots$. There exists a function f on K which is in $DBSC(K)$ but not in $SD(K)$.*

(An alternate proof of 1.5, using transfinite oscillations, is given in [R2].)

Recall that $K^{(j)}$ is defined inductively: For M a topological Hausdorff space, let M' denote the set of cluster points of M. Let $K^{(0)} = K$ and $K^{(j+1)} = (K^{(j)})'$ for all j. Now if K fails the hypotheses of 1.5 there is an integer n with $K^{(n+1)} = \emptyset$. Then every bounded function on K is of index at most n, hence belongs to $SD(K)$. It can also be shown that if K satisfies the hypotheses of 1.5, there exists an $f \in B_{1/2}(K) \sim DBSC(K)$, and also an $f \in B_1(K) \sim B_{1/2}(K)$.

Section 2.

We begin with some preliminary results.

Lemma 2.1. *Let f be a bounded non-negative lower semi-continuous function on K. Then $f \in DBSC(K)$ and $\|f\|_D = \|f\|_\infty$. Hence if f is bounded semi-continuous, $\|f\|_D \le 3\|f\|_\infty$.*

Proof. By a classical result of Baire (cf. [H]), there exists a sequence (φ_j) of continuous functions on K with $0 \le \varphi_1 \le \varphi_2 \le \cdots$ and $\varphi_j \to f$ pointwise. Setting $u_1 = \varphi_1$, $u_j = \varphi_j - \varphi_{j-1}$ for $j > 1$, we have that $u_j \ge 0$ for all j and $\sum u_j = f$ point-wise. Thus $\|f\|_D \ge \|f\|_\infty$; the reverse inequality is trivial.

To see the last statement, let e.g., f be bounded upper semi-continuous, $\lambda = \|f\|_\infty$, and note that $\lambda - f$ is non-negative lower semi-continuous. Thus $\|\lambda - f\|_D = \|\lambda - f\|_\infty \le 2\lambda$, so $\|f\|_D \le \lambda + \|\lambda - f\|_D \le 3\lambda$. □

Remark. It thus follows that if f is a D-function, then $\|f\|_D = \inf\{\|u + v\|_\infty :$ $u, v \ge 0$ are bounded lower semi-continuous with $f = u - v\}$.

Of course it follows immediately from Lemma 2.1 that if U is an open non-empty subset of K, then $\|\chi_U\|_D = 1$, for χ_U is lower semi-continuous. In this case, the sequence (φ_j) mentioned above can be easily chosen, using Urysohn's lemma. Indeed, if U is closed, this is trivial. Otherwise, let $\varepsilon_0 > 0$ be such that $\operatorname{dist}(x_0, \partial U) > \varepsilon_0$ for some $x_0 \in U$; set $F_n = \{x \in U : \operatorname{dist}(x, \partial U) \geq \frac{\varepsilon_0}{n}\}$. Then $U = \bigcup_{j=1}^{\infty} F_j$ and for all j, F_j is closed, $F_j \subset \operatorname{Int} F_{j+1}$. Now choose $[0,1]$-valued continuous functions (φ_j) on K so that for all j, $\varphi_j = 1$ on F_j and $\overline{\{x : \varphi_j(x) \neq 0\}} \subset \operatorname{Int} F_{j+1}$. Then $\varphi_j \to \chi_U$ pointwise.

Evidently it follows that if W is a closed subset of K, then $\|\chi_W\|_D \leq 2$. In fact, if W is a difference of closed sets; i.e., $W = W_1 \sim W_2$, with W_i closed for $i = 1, 2$, we again have that $\|\chi_W\|_D \leq 2$, for $\|\chi_W\|_D \leq \|\chi_{W_1}\|_D \|\chi_{\sim W_2}\|_D \leq 2 \cdot 1 = 2$.

The following result shows that the simple D-functions are precisely those functions built up from the differences of closed sets.

Proposition 2.2. *Let f be a simple real-valued function on K. The following are equivalent:*

1) *$f \in B_{1/2}(K)$;*
2) *f is of finite Baire index;*
3) *$f \in DBSC(K)$;*
4) *There exist disjoint differences of closed sets $W_1, \ldots, W_m$ and scalars $c_1, \ldots, c_m$ with*

$$f = \sum_{i=1}^{m} c_i \chi_{W_i} \ .$$

Proof. Let us suppose f is non constant, let $r_1, \ldots r_k$ be the distinct values of f, and set $\varepsilon = \min\{|r_i - r_j| : i \neq j,\ 1 \leq i, j \leq k\}$. Now if W is a non-empty subset of K, $w \in W$, and $\operatorname{osc} f \mid W(w) < \varepsilon$, then $f \mid W$ is continuous at w; in fact there is an open neighborhood U of w with $f(x) = f(w)$ for all $x \in U \cap W$.

Now suppose 1) holds, and let $n = i(f, \varepsilon)$. By Proposition 1.1, $n < \infty$. We then obtain that defining $K_0 = K$ and $K_{j+1} = \{x \in K_j : f|K_j$ is discontinuous at $x\}$, for $1 \leq j \leq n+1$, then $K_{n+1} = \emptyset$ and if $0 < \varepsilon' \leq \varepsilon$, $\operatorname{os}_j(f, \varepsilon') = K_j$ for all $1 \leq j \leq n$. Hence in fact $i(f) = i(f, \varepsilon) = n$, so 2) is proved. Of course 2) implies 1) by Proposition 1.1.

It remains only to show that 1) $\Rightarrow$ 4), for evidently 4) $\Rightarrow$ 3) $\Rightarrow$ 1). Now fixing $0 \leq j \leq n$, we have that f is continuous on $K_j \sim K_{j+1}$. Let then $\ell = \ell(j)$ and

$r_1^j, \dots, r_\ell^j$ be the distinct values of f on $K_j \sim K_{j+1}$; let $W_i^j = \{x \in K_j \sim K_{j+1} : f(x) = r_i^j\}$. Then W_i^j is a clopen subset of $K_j \sim K_{j+1}$; it follows easily that in fact W_i^j is then again a difference of closed sets in K, for all i, $1 \le i \le \ell$, and thus

$$f = \sum_{j=0}^{n} \sum_{i=1}^{\ell(j)} r_i^j \chi_{W_i^j} ,$$

proving 4). □

Remark. The above proof yields that moreover if $W \subset K$, and χ_W is a D-function, then W is a (disjoint) finite union of differences of closed sets; the converse is again immediate. This condition is incidentally equivalent to the condition that W belongs to the algebra $\mathcal{D}$ of sets generated by the closed subsets of K.

We give some more preliminary results, before passing to the proof of Theorem 1.2. For $f : K \to \mathbb{R}$, we set $\operatorname{supp} f = \{k \in K : f(k) \neq 0\}$. If $W \subset K$, we say that f is supported on W if $\operatorname{supp} f \subset W$.

Lemma 2.3. *Let U be a non-empty open subset of K, and f a bounded function on K, supported and continuous on U. Then $f \in SD(K)$ and $\|f\|_D = \|f\|_\infty$.*

Proof. Let us first show the norm identity. Note that since f is bounded, if u is a continuous function on K with $u(x) = 0$ for all $x \notin U$, then $f \cdot u$ is continuous on K. Now choose $u_1, u_2, \dots$ continuous non-negative functions on K with $\chi_U = \sum u_j$ point-wise. But then $f = \sum f \cdot u_j$ point-wise, $f \cdot u_j$ is continuous on K for all j, and $\sum |f u_j| \le \|f\|_\infty \sum u_j \le \|f\|_\infty$, so $\|f\|_D \le \|\sum |f u_j|\,\|_\infty \le \|f\|_\infty$; the reverse inequality is trivial.

To see that f is a strong D-function, assume without loss of generality that $\|f\|_\infty = 1$. Now fix n a positive integer, and for each j, $-n \le j \le n$, define K_j^n by

$$K_j^n = \left\{ x \in U : \frac{j}{n} \le f(x) < \frac{j+1}{n} \right\} . \tag{8}$$

Finally, define φ_n by

$$\varphi_n = \sum_{j=-n}^{n} \frac{j}{n} \chi_{K_j^n} . \tag{9}$$

Then evidently by the continuity of f, K_j^n is a difference of closed sets in U, and hence in K, for all j, so φ_n is a simple D-function; moreover we have

$$0 \le f - \varphi_n \le \frac{1}{n} . \tag{10}$$

Thus to show that $\|f-\varphi_n\|_D \to 0$ as $n\to\infty$, we need only show that $f-\varphi_n$ is lower semi-continuous; for then $\|f-\varphi_n\|_D \le \frac{1}{n}$ by (10) and Lemma 2.1.

Let $\psi = f-\varphi_n$, and suppose it were false that ψ is lower semi-continuous. We may then choose $x\in K$ and (x_m) a sequence in K with $x_m\to x$ so that $(\psi(x_m))$ converges and

$$\lim_{m\to\infty}\psi(x_m) < \psi(x)\ . \tag{11}$$

Evidently then $x\in U$, since $x\notin U$ implies $\psi(x)=0\le\psi(x_m)$ for all m. By passing to a subsequence, we may then assume without loss of generality that there is a j, $-n\le j\le n$, with $x_m\in K_j^n$ for all m. But since f is continuous on U, $\lim_{m\to\infty} f(x_m) = f(x)$; if also $x\in K_j^n$, then since $\psi(x_m) = f(x_m)-\frac{j}{n}$ for all m, we have that $\lim_{m\to\infty}\psi(x_n) = f(x)-\frac{j}{n} = \psi(x)$, a contradiction. If $x\notin K_j^n$, by continuity of f we must have that $f(x) = \frac{j+1}{n}$. But then $x\in K_n^{j+1}$, so $\psi(x) = 0 < \frac{j+1}{n}-\frac{j}{n} = \lim_{m\to\infty}\psi(x_m)$ again contradicting (11). $\square$

Our next preliminary result deals with extension issues. (For $W\subset K$ and $f: W\to\mathbb{R}$, $f\cdot\chi_W$ denotes the function which is zero off W and agrees with f on W.)

Lemma 2.4. *Let $W\subset K$ be a difference of closed sets and f in $DBSC(W)$. Then $f\cdot\chi_W$ is in $DBSC(K)$ and*

$$\|f\cdot\chi_W\|_{D(K)} \le 2\|f\|_{D(W)}\ ; \tag{12}$$

if W is an open set, then

$$\|f\cdot\chi_W\|_{D(K)} = \|f\|_{D(W)}\ . \tag{13}$$

Moreover if $f\in SD(W)$, then $f\chi_W\in SD(K)$.

Proof. Suppose first that W is open, and let (φ_j) in $C(K)$ be such that the φ_j's are non-negative and $\sum\varphi_j = \chi_W$ point-wise. Let $\varepsilon>0$ and choose (ψ_j) in $C(W)$ with $\sum|\psi_j| < \|f\|_{D(W)}+\varepsilon$ and $f=\sum\psi_j$ point-wise on W. Now identifying ψ_j with $\psi_j\cdot\chi_W$, $\psi_j\cdot\varphi_i$ is continuous on K for all i and j, and we have that $\sum_{i,j}|\psi_j\varphi_i| = \sum_j|\psi_j|\chi_W \le \|f\|_{D(W)}+\varepsilon$, with $\sum_{i,j}\psi_j\varphi_i = f\chi_W$. Thus $\|f\chi_W\|_{D(K)} \le \|f\|_{D(W)}+\varepsilon$ for all $\varepsilon>0$; so $\|f\chi_W\|_{D(K)}\le\|f\|_{D(W)}$. The reverse inequality is trivial, so (13) is established.

Next, suppose that W is closed, and again let $\varepsilon > 0$. As noted following Lemma 2.1, we may choose u, v non-negative lower semi-continuous on W with

$$f = u - v \quad \text{and} \quad \|u + v\|_\infty < \|f\|_{D(W)} + \varepsilon\ . \tag{14}$$

Now let $\lambda = \|u+v\|_\infty$ and let $\tilde{u} = \lambda\chi_{\sim W} + u\chi_W$, $\tilde{v} = \lambda\chi_{\sim W} + v\chi_W$. It follows easily that $\tilde{u}$ and $\tilde{v}$ are both non-negative lower semi-continuous on K and of course

$$f\chi_W = \tilde{u} - \tilde{v}\ , \qquad \|\tilde{u} + \tilde{v}\|_\infty = 2\lambda\ . \tag{15}$$

Thus by the observation following Lemma 2.1, $\|f \cdot \chi_W\|_D \le 2\lambda < 2\|f\|_{D(W)} + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, (12) is proved for closed W.

Now suppose W is a difference of closed sets. Choose U open, L closed with $W = U \cap L$. Then W is a relatively closed subset of U, so we have that $f \cdot \chi_L \mid U$ belongs to $DBSC(U)$ with $\|f \cdot \chi_L \mid U\|_{D(U)} \le 2\|f\|_{D(W)}$. But then by (13), $f \cdot \chi_W = (f \cdot \chi_L) \mid U \cdot \chi_U$ belongs to $DBSC(K)$ and $\|f \cdot \chi_W\| \le \|f \cdot \chi_L \mid U\|_{D(W)} \le 2\|f\|_{D(W)}$, proving (12).

Finally, suppose $f \in SD(W)$. Then given $\varepsilon > 0$, choose g a simple D-function on W with

$$\|g - f\|_{D(W)} < \varepsilon\ . \tag{16}$$

By Proposition 2.2, there are disjoint differences of closed sets in W, $W_1, \dots, W_k$, and scalars $c_1, \dots, c_k$ with $g = \sum_{i=1}^k c_i \chi_{W_i}$ on W. But then for all i, W_i is actually a difference of closed sets in K, and thus $g \cdot \chi_W$ is a simple D-function on K. Then by (12),

$$\|(g - f)\chi_W\| = \|g\chi_W - f\chi_W\| < 2\varepsilon\ . \tag{17}$$

Thus the final assertion of the Lemma is established. $\square$

Remark. Using the comment following Proposition 2.2, we obtain that if $W \subset K$ is in $\mathcal{D}$ (i.e., χ_W is a D-function), then for $f : W \to \mathbb{R}$ a bounded function, f is a D-function on W if and only if $f\chi_W$ is a D-function on K; moreover $f \in SD(W)$ if and only if $f\chi_W \in SD(K)$.

Before giving the proof of Theorem 1.2, we recall the following standard result.

Lemma 2.5. *Let $\varepsilon > 0$, and suppose $f : K \to \mathbb{R}$ is such that $\operatorname{osc} f \le \varepsilon$ on K. There exists $\varphi : K \to \mathbb{R}$ continuous with $|f - \varphi| \le \varepsilon$ on K.*

Proof. Let Lf be the lower semi-continuous envelope of f; $Lf(x) = \underline{\lim}_{y\to x} f(y)$ for all $x \in X$. Then we have that

$$\overline{\operatorname{osc}}\, f = Uf - Lf\ . \tag{18}$$

Since $\overline{\operatorname{osc}}\, f \le 2 \operatorname{osc} f$, $\overline{\operatorname{osc}}\, f \le 2\varepsilon$ on K. Thus we have by assumption that

$$Uf - \varepsilon \le Lf + \varepsilon\ . \tag{19}$$

By the Hahn interposition theorem (cf. [H], p.276), there exists φ continuous with

$$Uf - \varepsilon \le \varphi \le Lf + \varepsilon\ . \tag{20}$$

Since $f \le Uf$ and $Lf \le f$, φ satisfies the conclusion of the Lemma. $\square$

We now treat the proof of Theorem 1.2. It is convenient to consider a larger class; for $n \ge 0$, let $\mathcal{G}_n$ denote the family of all bounded functions $f : K \to \mathbb{R}$ so that there exists an open set U with f supported on U and $i(f \mid U) \le n$. The following quantitative result yields Theorem 1.2 immediately.

Theorem 2.6. *Let $n \ge 0$ and $f \in \mathcal{G}_n$. Then $f \in SD(K)$ and*
$\|f\|_D \le (2^{n+1} - 1)\|f\|_\infty$.

Remark. Of course it follows *a-posteriori* that if we prove the result just for functions f of index n, then it holds immediately for functions in $\mathcal{G}_n$, by Lemma 2.4. The class $\mathcal{G}_n$ is needed for our proof, however. We also note that the argument given in [R2], using transfinite oscillations, gives the optimal estimate: if $i(f) \le n$, then $\|f\|_D \le (2n + 1)\|f\|_\infty$.

We prove 2.6 by induction on n. The case $n = 0$ follows immediately from Lemma 2.3. Now let $n > 0$ and suppose 2.6 proved for "n" $= n - 1$.

Lemma 2.7. *Let $f \in \mathcal{G}_n$ and $\varepsilon > 0$. There exist functions g and h with $f = g + h$, $g \in \mathcal{G}_n$, $h \in SD(K)$, and*

$$\|h\|_D \le (2^{n+1} - 1)\|f\|_\infty \quad \text{and} \quad \|g\|_\infty \le \varepsilon \tag{21}$$

Proof. Let $\lambda_j = 2^{j+1} - 1$ for $j = 0, 1, 2, \ldots$. Let U be chosen with f supported in U and $i(f \mid U) \leq n$. Let $W = \{x \in U : \operatorname{osc} f(x) \geq \varepsilon\}$. It follows that W is a relatively closed subset of U and

$$i(f \mid W) \leq n - 1 \ . \tag{22}$$

Thus by induction hypothesis and Lemma 2.4,

$$f \cdot \chi_W \in SD(K) \quad \text{and} \quad \|f \cdot \chi_W\|_D \leq 2\lambda_{n-1}\|f\|_\infty \ . \tag{23}$$

Now by Lemma 2.5, we may choose $\varphi : U \sim W \to \mathbb{R}$, φ continuous on $U \sim W$, with

$$\|\varphi\|_\infty \leq \|f\|_\infty \quad \text{and} \quad |\varphi(x) - f(x)| \leq \varepsilon \quad \text{for all} \quad x \in U \sim W \ , \tag{24}$$

Indeed, 2.5 gives $\tilde{\varphi}$ with $\tilde{\varphi}$ continuous and $|\tilde{\varphi} - f| \leq \varepsilon$ on $U \sim W$. But simply define $\varphi(x) = \tilde{\varphi}(x)$ if $|\tilde{\varphi}(x)| \leq \|f\|_\infty$, and $\varphi(x) = \|f\|_\infty \operatorname{sgn} f(x)$ otherwise.

Let g and h be defined by

$$g = (f - \varphi)\chi_{U \sim W} \quad , \quad h = f \cdot \chi_W + \varphi \cdot \chi_{U \sim W} \ . \tag{25}$$

Now evidently $\operatorname{supp} g \subset U \sim W$; since φ is continuous on $U \sim W$, it follows that $i((f - \varphi) \mid U \sim W) \leq i(f \mid U) \leq n$; hence $g \in \mathcal{G}_n$, and by (24), $\|g\|_\infty \leq \varepsilon$.

Evidently, $f = g + h$; finally, by (23) and Lemma 2.3, $h \in SD(K)$ and

$$\|h\|_D \leq (2\lambda_{n-1} + 1)\|f\|_\infty = \lambda_n \|f\|_\infty \ . \qquad \square$$

Proof of Theorem 2.6 for n. Fix $\varepsilon > 0$. We may choose by induction sequences (h_j) and (g_j) so that for all j,

$$f = h_1 + \cdots + h_j + g_j \tag{26i}$$

$$h_j \in SD(K) \quad , \quad g_j \in \mathcal{G}_n \tag{26ii}$$

$$\|h_1\|_D \leq \lambda_n \|f\|_\infty \quad , \quad \|h_j\|_D \leq \frac{\varepsilon}{2^{j-1}} \ \text{ for } \ j > 1 \tag{26iii}$$

$$\|g_j\|_\infty \leq \frac{\varepsilon}{\lambda_n 2^j} \ . \tag{26iv}$$

Indeed, by Lemma 2.7, we may choose $h_1 \in SD(K)$ and $g_1 \in \mathcal{G}_n$ with $f = h_1 + g_1$, $\|h_1\|_D \leq \lambda_n \|f\|_\infty$, $\|g_1\|_\infty \leq$ [illegible]

Now suppose $j \geq 1$ and $h_1, \ldots, h_j$, g_j chosen satisfying (26i)–(26iv). Since $g_j \in \mathcal{G}_n$, by Lemma 2.7 we may choose $h_{j+1} \in SD(K)$ and $g_{j+1} \in \mathcal{G}_n$ with $g_j = h_{j+1} + g_{j+1}$,

$$\|h_{j+1}\|_D \leq \lambda_n \|g_j\|_\infty \quad \text{and} \quad \|g_{j+1}\|_\infty \leq \frac{\varepsilon}{\lambda_n 2^{j+1}} \ . \tag{27}$$

Then (26i)–(26iv) hold at $j+1$.

Since the D-norm is trivially larger than the sup-norm and $\|g_j\|_\infty \to 0$, it follows from (26i) and (26iii) that $\sum h_i$ converges uniformly to f. Since $DBSC(K)$ is a Banach space, $\sum \|h_j\|_D < \infty$, and $h_j \in SD(K)$ for all j, it follows that $f \in SD(K)$. Finally, we have by (26iii) that

$$\|f\|_D \leq \lambda_n \|f\|_\infty + \sum_{j=2}^\infty \frac{\varepsilon}{2^{j-1}} = \lambda_n \|f\|_\infty + \varepsilon \ . \tag{28}$$

Since $\varepsilon > 0$ is arbitrary, Theorem 2.6 is proved. $\square$

We turn now to Theorem 1.3. This follows immediately from the following result.

Theorem 2.8. *Let $f, g \in B_{1/2}(K)$, and $\varepsilon > 0$. Then the following hold.*

(a) $i(f+g, \varepsilon) \leq i(f, \frac{\varepsilon}{2}) + i(g, \frac{\varepsilon}{2})$.

(b) $i(f \cdot g, \varepsilon) \leq i(f, \frac{\varepsilon}{2G}) + i(g, \frac{\varepsilon}{2F})$ *where* $F = \|f\|_\infty$, $G = \|g\|_\infty$, *and it is assumed that* $F, G > 0$.

(c) $i(h, \varepsilon) \leq i(f, \varepsilon) + i(g, \varepsilon)$ *where* $h = f \vee g$ *or* $h = f \wedge g$.

We give the detailed proof of (a) (which is also needed later), and then indicate how (b), (c) follow by the same method.

We first note the following fact.

Lemma 2.9. *Let $W_1, \ldots, W_n$ be closed non-empty sets with $K = \bigcup_{i=1}^n W_i$ and $f : K \to \mathbb{R}$ a bounded function. Then*

$$\operatorname{osc} f = \max_{1 \leq i \leq n} (\operatorname{osc} f \mid W_i) \chi_{W_i} \ . \tag{29}$$

Proof. We first note that

$$\underline{\operatorname{osc}}\, f = \max_{1 \leq i \leq n} (\underline{\operatorname{osc}}\, f \mid W_i) \chi_{W_i} \ . \tag{30}$$

For let $x \in K$ and choose (x_m) in K with $x_m \to x$ and $\underline{\operatorname{osc}}\, f(x) = \lim_{n\to\infty} |f(x_n) - f(x)|$. We may choose i and $m_1 < m_2 < \cdots$ with $x_{m_j} \in W_i$ for all j. But then

$x \in W_i$ and so $\underline{\operatorname{osc}} f(x) \leq \underline{\operatorname{osc}} f \mid W_i(x) \leq \max_\ell(\underline{\operatorname{osc}} f \mid W_\ell)\chi_{W_\ell}(x)$. The reverse inequality is trivial, so (30) follows.

Now again let $x \in K$ and choose (x_m) in K with $x_m \to x$ and $\operatorname{osc} f(x) = \lim_{n\to\infty} \underline{\operatorname{osc}} f(x_m)$. By (30), we may again choose $m_1 < m_2 < \cdots$ and i with $\underline{\operatorname{osc}} f(x_{m_j}) = \underline{\operatorname{osc}} f \mid W_i \chi_{W_i}(x_{m_j})$ for all j. Now if $\operatorname{osc} f(x) = 0$, (29) is trivial. Otherwise, without loss of generality, $\operatorname{osc} f(x_{m_j}) > 0$ for all j; hence then $x_{m_j} \in W_i$ and so $x \in W_i$, whence $\operatorname{osc} f(x) \leq \operatorname{osc} f \mid W_i(x) \leq \max_\ell(\operatorname{osc} f \mid W_\ell)\chi_{W_\ell}(x)$. Again the reverse inequality is trivial, so (29) holds. □

Now let f, g be as in Theorem 2.8, and $\varepsilon > 0$ be given. For each $n = 1, 2, \ldots$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ with $\theta_i = 0$ or 1 for all $1 \leq i \leq n$, we define closed subsets $L(\boldsymbol{\theta})$ of K as follows:

$$L(0) = \left\{x \in K : \operatorname{osc} f(x) \geq \frac{\varepsilon}{2}\right\} \quad ; \quad L(1) = \left\{x \in K : \operatorname{osc} g(x) \geq \frac{\varepsilon}{2}\right\} . \tag{31}$$

If $n \geq 1$ and $L(\boldsymbol{\theta}) = L(\theta_1, \ldots, \theta_n)$ is defined, let

$$\begin{cases} L(\theta_1, \ldots, \theta_{n+1}) = \left\{x \in L(\boldsymbol{\theta}) : \operatorname{osc} f \mid L(\boldsymbol{\theta}) \geq \frac{\varepsilon}{2}\right\} & \text{if } \ \theta_{n+1} = 0 \\ L(\theta_1, \ldots, \theta_{n+1}) = \left\{x \in L(\boldsymbol{\theta}) : \operatorname{osc} g \mid L(\boldsymbol{\theta}) \geq \frac{\varepsilon}{2}\right\} & \text{if } \ \theta_{n+1} = 1 . \end{cases} \tag{32}$$

These sets are closed, since $\operatorname{osc} f$, $\operatorname{osc} g$ are upper semi-continuous functions. We then have for all n that

$$\operatorname{os}_n(f + g, \varepsilon) \subset \bigcup_{\boldsymbol{\theta} \in \{0,1\}^n} L(\boldsymbol{\theta}) . \tag{33}$$

We prove this by induction on n. Now for $n = 1$, since it is easily seen that $\operatorname{osc}(f+g) \leq \operatorname{osc} f + \operatorname{osc} g$, we then have that $\operatorname{osc}(f+g)(x) \geq \varepsilon$ implies $\operatorname{osc} f(x) \geq \frac{\varepsilon}{2}$ or $\operatorname{osc} g(x) \geq \frac{\varepsilon}{2}$; this gives $\operatorname{os}_1(f + g, \varepsilon) \subset L(0) \cup L(1)$. Suppose (33) is proved for n, and suppose $K_n = \operatorname{osc}_n(f + g, \varepsilon)$ and $x \in \operatorname{os}_{n+1}(f + g, \varepsilon)$. Thus $\operatorname{osc}(f + g) \mid K_n(x) \geq \varepsilon$. By the preceding lemma and (33), we may then choose $\boldsymbol{\theta} \in \{0, 1\}^n$ with $x \in K_n \cap L(\boldsymbol{\theta})$ and

$$\begin{aligned} \operatorname{osc}(f + g) \mid K_n(x) &= \operatorname{osc}(f + g) \mid K_n \cap L(\boldsymbol{\theta})(x) \\ &\leq \operatorname{osc}(f + g) \mid L(\boldsymbol{\theta})(x) \\ &\leq \operatorname{osc} f \mid L(\boldsymbol{\theta})(x) + \operatorname{osc} g \mid L(\boldsymbol{\theta})(x) . \end{aligned}$$

It follows immediately that $x \in L(\theta_1, \ldots, \theta_n, 0) \cup L(\theta_1, \ldots, \theta_n, 1)$; thus (32) holds at $n + 1$.

Next, fix n and $\boldsymbol{\theta} \in \{0,1\}^n$. Let

$$(34) \quad j = j(\boldsymbol{\theta}) = \#\{1 \le i \le n : \theta_i = 0\} \quad , \quad k = k(\boldsymbol{\theta}) = \#\{1 \le i \le n : \theta_i = 1\} \ .$$

Then we claim

$$(35) \qquad L(\boldsymbol{\theta}) \subset \mathrm{os}_j\left(f, \frac{\varepsilon}{2}\right) \cap \mathrm{os}_k\left(g, \frac{\varepsilon}{2}\right) \ .$$

Again we prove this by induction on n. The case $n = 1$ is trivial, by the definitions of $L(0)$ and $L(1)$. Now suppose (35) is proved for n, and $(\theta_1, \dots, \theta_{n+1})$ is given; let $j = j(\theta_1, \dots, \theta_n)$ and $k = k(\theta_1, \dots, \theta_n)$. Now if $\theta_{n+1} = 0$, then $j(\theta_1, \dots, \theta_{n+1}) = j + 1$ and $k(\theta_1, \dots, \theta_{n+1}) = k$; then by (35), $L(\theta_1, \dots, \theta_{n+1}) \subset L(\theta_1, \dots, \theta_n) \subset \mathrm{os}_k(g, \frac{\varepsilon}{2})$ and by definition and (35),

$$\begin{aligned} L(\theta_1, \dots, \theta_{n+1}) &\subset \left\{x \in \mathrm{os}_j\left(f, \frac{\varepsilon}{2}\right) : \mathrm{osc}\, f \mid \mathrm{os}_j\left(f, \frac{\varepsilon}{2}\right)(x) \ge \frac{\varepsilon}{2}\right\} \\ &= \mathrm{os}_{j+1}\left(f, \frac{\varepsilon}{2}\right) \ . \end{aligned}$$

Of course if $\theta_{n+1} = 1$, we obtain by the same reasoning that $L(\theta_1, \dots, \theta_{n+1}) \subset \mathrm{os}_j(f, \frac{\varepsilon}{2}) \cap \mathrm{os}_{k+1}(g, \frac{\varepsilon}{2})$ and $j = j(\theta_1, \dots, \theta_{n+1})$, $k + 1 = k(\theta_1, \dots, \theta_{n+1})$; thus (35) is proved for $n+1$, and so established for all n by induction.

Now suppose, for a given n, that $\mathrm{os}_n(f+g, \varepsilon) \ne \emptyset$. Then by (33), there is a $\boldsymbol{\theta} \in \{0,1\}^n$ with $L(\boldsymbol{\theta}) \ne \emptyset$. Thus letting j and k be as in (34), we have by (35) that $\mathrm{os}_j(f, \frac{\varepsilon}{2}) \ne \emptyset$ and $\mathrm{os}_k(g, \frac{\varepsilon}{2}) \ne \emptyset$. But then $n = j + k \le i(f, \frac{\varepsilon}{2}) + i(g, \frac{\varepsilon}{2})$. Theorem 2.8(a) is thus established.

To see 2.8(b), note for any y and $x \in K$ that

$$(36) \qquad |f(y)g(y) - f(x)g(x)| \le G|f(y) - f(x)| + F|g(y) - g(x)| \ .$$

Hence we have that fixing $x \in K$, then $\underline{\mathrm{osc}}\, fg(x) \le G\, \underline{\mathrm{osc}}\, f(x) + F\, \underline{\mathrm{osc}}\, g(x)$, whence

$$(37) \qquad \mathrm{osc}\, fg(x) \le G \,\mathrm{osc}\, f(x) + F \,\mathrm{osc}\, g(x) \ .$$

Thus $\mathrm{osc}\, fg(x) \ge \varepsilon$ implies $\mathrm{osc}\, f(x) \ge \frac{\varepsilon}{2G}$ or $\mathrm{osc}\, g(x) \ge \frac{\varepsilon}{2F}$. We now prove (b) by defining the sets $L(\boldsymbol{\theta})$ by $L(0) = \mathrm{os}_1(f, \frac{\varepsilon}{2G})$, $L(1) = \mathrm{os}_1(g, \frac{\varepsilon}{2F})$, and for $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{n+1})$, $L(\theta_1, \dots, \theta_{n+1}) = \{x \in L(\boldsymbol{\theta}) : \mathrm{osc}\, f \mid L(\boldsymbol{\theta}) \ge \frac{\varepsilon}{2G}\}$ if $\theta_{n+1} = 0$, and $L(\theta_1, \dots, \theta_{n+1}) = \{x \in L(\boldsymbol{\theta}) : \mathrm{osc}\, g \mid L(\boldsymbol{\theta}) \ge \frac{\varepsilon}{2F}\}$ if $\theta_{n+1} = 1$. Then we proceed exactly as in case (a). Finally, for case (c), we note that if h is as in (c) and $x \in K$, then

$$\operatorname{osc} h(x) \geq \varepsilon \ \text{ implies } \ \operatorname{osc} f(x) \geq \varepsilon \text{ or } \operatorname{osc} g \geq \varepsilon \ . \tag{38}$$

Suppose this were false. Then we can choose $0 < \varepsilon' < \varepsilon$ and U an open neighborhood of x with

$$\operatorname{osc} f(u) < \varepsilon' \ \text{ and } \ \operatorname{osc} g(u) < \varepsilon' \ \text{ for all } \ u \in U \ . \tag{39}$$

Now fix $u \in U$; we can then choose V an open neighborhood of u with $V \subset U$ and

$$|f(v) - f(u)| < \varepsilon' \ \text{ and } \ |g(v) - g(u)| < \varepsilon' \ \text{ for all } \ v \in V \ . \tag{40}$$

Suppose e.g., $h = f \vee g$ and $v \in V$ with $(f \vee g)(v) = f(v)$, $(f \vee g)(u) = g(u)$. But then by (40) and the above,

$$f(v) \geq g(v) > g(u) - \varepsilon' \ \text{ so } \ f(v) - g(u) > -\varepsilon' \tag{41}$$

and

$$f(v) < f(u) + \varepsilon' \leq g(u) + \varepsilon' \ \text{ so } \ f(v) - g(u) < \varepsilon' \ . \tag{42}$$

It thus follows from (40)–(42) that

$$|h(v) - h(u)| < \varepsilon' \ . \tag{43}$$

If e.g., $f \vee g(v) = f(v)$ and $f \vee g(u) = f(u)$, (43) follows immediately from (40), so (43) holds for all $v \in V$. Thus we obtain $\underline{\operatorname{osc}}\, h(u) \leq \varepsilon'$; but since $u \in U$ is arbitrary, we also have $\operatorname{osc} h(x) \leq \varepsilon'$, a contradiction. The proof for $h = f \wedge g$ is the same.

Evidently (38) yields that $\operatorname{os}_1(h, \varepsilon) \subset \operatorname{os}_1(f, \varepsilon) \cup \operatorname{os}_1(g, \varepsilon)$; we then proceed as in case (a), except that the sets $L(\theta_1, \dots, \theta_n)$ are defined by replacing "ε" by "$\frac{\varepsilon}{2}$" in (31), (32). $\square$

We next treat Theorem 1.4. We first recall the following fact.

Lemma 2.9. *Let $f \in D(K)$. Then $\varepsilon i(f, \varepsilon) \leq 4\|f\|_D$.*

This follows immediately from the definitions, the fact that $\operatorname{os}_j(f, \varepsilon) \subset K_j(f, \varepsilon)$ for all j, and Lemma 2.4 of [HOR]. (A direct proof of 2.9 is given in [R2] yielding the refinement that $\varepsilon i(f, \varepsilon) \leq \|f\|_D$.)

Proof of Theorem 1.4. Suppose first that $f \in SD(K)$, $\eta > 0$, and choose g a simple D-function with $\|f - g\|_D \le \eta$. It then follows by Lemma 2.9 that

$$\varepsilon i(f - g, \varepsilon) \le 4\eta \quad \text{for all} \quad \varepsilon > 0 \ . \tag{44}$$

Now since g is a simple D-function, g has finite index (by Proposition 2.2); say $N = i(g)$. Then by Theorem 2.8(a) and (44), for any $\varepsilon > 0$,

$$\begin{aligned} \varepsilon i(f, \varepsilon) &\le \varepsilon i\Big(f - g, \frac{\varepsilon}{2}\Big) + \varepsilon i\Big(g, \frac{\varepsilon}{2}\Big) \\ &\le 8\eta + \varepsilon N \ . \end{aligned}$$

Hence $\overline{\lim}_{\varepsilon \to 0} \varepsilon i(f, \varepsilon) \le 8\eta$. Since $\eta > 0$ is arbitrary, (7) is proved.

Finally, to prove (b) of Theorem 1.4, suppose without loss of generality that f is upper semi-continuous and satisfies (7), let $\eta > 0$, and choose $0 < \varepsilon < \eta$ with

$$\varepsilon i(f, \varepsilon) < \eta \ . \tag{45}$$

Let then $n = i(f, \varepsilon)$ and set $K^j = \text{os}_j(f, \varepsilon)$ for all j. Thus $K^n \ne \emptyset$, $K^{n+1} = \emptyset$, and for $0 \le j \le n$, $\text{osc}(f \mid K^j \sim K^{j+1}) < \varepsilon$. Thus for all j, we may choose by Lemma 2.5 a continuous function φ_j on $K^j \sim K^{j+1}$ with

$$|\varphi_j - f| \le \varepsilon \quad \text{on} \quad K^j \sim K^{j+1} \ . \tag{46}$$

Now set $g = \sum_{j=0}^n \varphi_j \chi_{K^j \sim K^{j+1}}$. By Lemmas 2.3 and 2.4, $g \in SD(K)$. Now fixing j and letting $W = K^j \sim K^{j+1}$, then $(f - g) \mid W$ is upper semi-continuous, hence by Lemma 2.1 and (46),

$$\|f - g\|_{D(W)} \le 3\|f - g\|_\infty \le 3\varepsilon \ . \tag{47}$$

Then by Lemma 2.4,

$$\|(f - g)\chi_W\|_{D(K)} \le 6\varepsilon \ . \tag{48}$$

Hence

$$\begin{aligned} \|f - g\|_D &= \sum_{j=0}^n \|(f - g)\chi_{K^j \sim K^{j+1}}\|_D \\ &\le \sum_{j=0}^n \|(f - g)\chi_{K^j \sim K^{j+1}}\|_D \\ &\le 6n\varepsilon + 6\varepsilon \\ &\le 7\eta \quad \text{by (45)} \end{aligned}$$

Since $\eta > 0$ is arbitrary and $SD(K)$ is closed in $DBSC(K)$, we obtain that $f \in SD(K)$, thus completing the proof of Theorem 1.4. $\square$

Remark. Define $B^0_{1/2}(K)$ to be the family of all bounded functions $f : K \to \mathbb{R}$ which satisfy (7). Evidently we have (by the preceding result) that $SD(K) \subset B^0_{1/2}(K) \subset B_{1/2}(K)$. We have moreover that $B^0_{1/2}(K)$ is an algebra and a lattice, by Theorem 2.8. As noted in the introduction, it can be shown that there are non-D-functions in $B^0_{1/2}(K)$, and also $(DBSC(K) \sim SD(K)) \cap B^0_{1/2}(K) \neq \emptyset$ (for suitable K). It can be seen that $B^0_{1/2}(K)$ is a complete linear topological space under the quasi-norm $\|f\| = \sup_{\varepsilon>0} \varepsilon i(f,\varepsilon) + \|f\|_\infty$.

We finally consider Proposition 1.5. The construction uses some preliminary results.

Lemma 2.10. *Let $n \geq 1$ and $K = K_0 \supset K_1 \supset \cdots \supset K_n$ be closed non-empty sets with K_i nowhere dense relative to K_{i-1} for all $1 \leq i \leq n$. Also let $K_{n+1} = \emptyset$. Let $E = \bigcup_{0 \leq i \leq [n/2]} K_{2i} \sim K_{2i+1}$. Then*

$$i(\chi_E) = i(\chi_E, \varepsilon) = n \quad \text{for all} \quad 0 < \varepsilon \leq 1 \ . \tag{49}$$

Moreover $\|\chi_E\|_D \leq n+1$.

Proof. Fix $0 < \varepsilon \leq 1$. We prove by induction on j that

$$\mathrm{os}_j(\chi_E, \varepsilon) = K_j \quad \text{for all} \quad 0 \leq j \leq n \ . \tag{50}$$

Then since χ_E is constant on K_n, $\mathrm{os}_{n+1}(\chi_E,\varepsilon) = \emptyset$, yielding (49).

Now χ_E is constant on $K_0 \sim K_1$, an open set; since K_1 is nowhere dense in K, given $x \in K_1$, there exists a sequence (x_m) in $K_0 \sim K_1$ with $x_m \to x$. But then $(\mathrm{osc}\,\chi_E)(x) \geq \lim_{m\to\infty}(\chi_E(x_m) - \chi_E(x)) = 1$, hence (50) is proved for $j = 0$.

Suppose now (50) is proved for $0 \leq j < n$. Again if $x \in K_{j+1}$, since K_{j+1} is nowhere dense in K_j, choose a sequence (x_m) in K_j with $x_m \to x$. Now by definition of E, $|\chi_E(x_m) - \chi_E(x)| = 1$ for all m. Thus $\mathrm{osc}\,\chi_E \mid K_j(x) \geq 1$, which proves that $K_{j+1} \subset \mathrm{os}_{j+1}(\chi_E, \varepsilon)$. But χ_E is constant on $K_j \sim K_{j+1}$, whence $K_{j+1} \supset \mathrm{osc}_{j+1}(\chi_E,\varepsilon)$. Thus (50) holds.

To see the final inequality in 2.10, we have that [illegible]

$\|\chi_{K_{2i}\sim K_{2i+1}}\|_D \le 2$ for all $1 \le i \le [n/2]$ (by Lemma 2.4); hence

$$\|\chi_E\|_D \le \sum_{i=0}^{[n/2]} \|\chi_{K_{2i}\sim K_{2i+1}}\|_D$$
$$\le 1 + 2[n/2] \le n+1 \qquad \square$$

Remark. Actually the final inequality in 2.10 follows from (49). In fact it is proved in [R2] that if $E \subset K$ is such that $i(\chi_E) = n$, then $\|\chi_E\|_D = n$ or $n+1$ (and both possibilities can occur).

Lemma 2.11. (a) *Let $n \ge 1$ and suppose $K^{(n)} \ne \emptyset$. There exist non-empty closed sets $K_1, \dots, K_n$ satisfying the hypotheses of Lemma 2.10.*

(b) *Suppose $K^{(n)} \ne \emptyset$ for all $n = 1, 2, \dots$. There exist disjoint open subsets $U_1, U_2, \dots$ of K with $U_n^{(n)} \ne \emptyset$ for all n.*

Proof.

(a) If K is perfect, it can be seen that there exists a closed perfect nowhere dense subset L of K; we then easily obtain the desired sets (K_j) with K_j a perfect nowhere dense result of K_{j-1}. Evidently the same reasoning holds if K has a perfect non-empty subset. Otherwise, simply let $K_j = K^{(j)}$, $1 \le j \le n$. Alternatively, we may just observe that the hypotheses imply K has a closed subset homeomorphic to $\omega^n + 1$.

(b) First note that if $x \in K^{(n)}$, then

$$x \in U^{(n)} \text{ for all open neighborhoods } U \text{ of } x\ . \tag{51}$$

Next, note that the hypotheses imply that $K^{(n)}$ is infinite for all n. We may thus choose distinct points $x_1, x_2, \dots$, with $x_n \in K^{(n)}$ for all n. Now it follows that if U is an open set containing infinitely many of the x_j's, there exists an n and an open neighborhood V of x_n with $\bar{V} \subset U$ so that $U \sim \bar{V}$ contains infinitely many of the x_j's. We may then choose $k_1 < k_2 < \cdots$ and $U_1, U_2, \dots$ open sets with $\bar{U}_i \cap \bar{U}_j = \emptyset$ for all $i \ne j$ and $x_{k_n} \in U_n$ for all n. (51) then yields that (b) holds. $\square$

We finally observe the following simple "localization" property for D-functions.

Lemma 2.12. *Let $U_1, U_2, \dots$ be disjoint non-empty open subsets of K, $U = \bigcup_{j=1}^{\infty} U_j$, $\lambda < \infty$, and $f : K \to \mathbb{R}$ a function supported on U with $\|f \mid U_j\|_D \le \lambda$*

Proof. Let $\varepsilon > 0$. For each j, choose a sequence of continuous functions on K, $(\varphi_i^j)_{i=1}^\infty$, with $0 \le \varphi_i^j \le 1$ for all i and $\chi_{U_j} = \sum_{i=1}^\infty \varphi_i^j$ pointwise. Also, choose $(h_i^j)_{i=1}^\infty$ continuous functions on U_j, with $\sum |h_i^j| \le \lambda + \varepsilon$ and $f \mid U_j = \sum h_i^j$ pointwise. Now let

$$f_{jk\ell} = \varphi_k^j h_\ell^j \chi_{U_j} \quad \text{for all} \quad j,k,\ell\ . \tag{52}$$

Then $f_{jk\ell}$ is continuous on K since h_ℓ^j is bounded continuous on K and supported on U_j, and

$$\sum_{j,k,\ell} |\varphi_k^j h_\ell^j \chi_{U_j}| = \sum_j \sum_\ell |h_\ell^j| \chi_{U_j} \le \lambda + \varepsilon\ ,$$
$$\sum_j \sum_\ell \sum_k \varphi_k^j h_\ell^j \chi_{U_j} = \sum_j \sum_\ell h_\ell^j \chi_{U_j} = \sum_j f \chi_{U_j} = f\ .$$

Thus $\|f\|_D \le \lambda + \varepsilon$; since $\varepsilon > 0$ is arbitrary, the result follows. □

We are now prepared for the

Proof of Proposition 1.5.

By Lemmas 2.10 and 2.11, we may choose disjoint non-empty open subsets $U_1, U_2, \ldots$ of K, and for each n a subset E_n of U_n so that

$$i(\chi_{E_n}) = n = i(\chi_{E_n}, \varepsilon) \quad \text{for all} \quad 0 < \varepsilon \le 1\ . \tag{53}$$

and

$$\|\chi_{E_n}\|_{D(U_n)} \le n + 1\ . \tag{54}$$

Now let $f = \sum_{n=1}^\infty \chi_{E_n}/n$ pointwise. Thus by Lemma 2.12 and (54), $f \in DBSC(K)$ (with $\|f\|_D \le 2$). However fixing n and letting $\varepsilon = \frac{1}{n}$, then by (53), $i(\chi_{E_n}, 1) = n$ $(= i(\frac{1}{n}\chi_{E_n}, \frac{1}{n}))$ and so

$$\varepsilon i(f, \varepsilon) \ge \frac{1}{n} i\Big(f \mid U_n, \frac{1}{n}\Big) = 1\ . \tag{55}$$

Thus f fails (7), so $f \notin SD(K)$ by Theorem 1.4. □

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