ON WIDE-(s) SEQUENCES AND THEIR APPLICATIONS TO CERTAIN CLASSES OF OPERATORS

H. Rosenthal

Department of Mathematics
The University of Texas at Austin
Austin, TX 78712-1082

August 5, 1996

Abstract. A basic sequence in a Banach space is called wide-(s) if it is bounded and dominates the summing basis. (Wide-(s) sequences were originally introduced by I. Singer, who termed them P^*-sequences). These sequences and their quantified versions, termed λ-wide-(s) sequences, are used to characterize various classes of operators between Banach spaces, such as the weakly compact, Tauberian, and super-Tauberian operators, as well as a new intermediate class introduced here, the strongly Tauberian operators. This is a nonlocalizable class which nevertheless forms an open semigroup and is closed under natural operations such as taking double adjoints. It is proved for example that an operator is non-weakly compact iff for every $\varepsilon > 0$, it maps some $(1+\varepsilon)$-wide-(s)-sequence to a wide-(s) sequence. This yields the quantitative triangular arrays result characterizing reflexivity, due to R.C. James. It is shown that an operator is non-Tauberian (resp. non-strongly Tauberian) iff for every $\varepsilon > 0$, it maps some $(1+\varepsilon)$-wide-(s) sequence into a norm-convergent sequence (resp. a sequence whose image has diameter less than ε). This is applied to obtain a direct "finite" characterization of super-Tauberian operators, as well as the following characterization, which strengthens a recent result of M. González and A. Martínez-Abejón: An operator is non-super-Tauberian iff there are for every $\varepsilon > 0$, finite $(1+\varepsilon)$-wide-(s) sequences of arbitrary length whose images have norm at most ε.

§1. Introduction.

A semi-normalized basic sequence (b_j) in a Banach space is called wide-(s) if it dominates the summing basis; i.e., $\sum c_j$ converges whenever $\sum c_j b_j$ converges. Our main objective here is to show that wide-(s) sequences provide a unified approach for dealing with certain important classes of operators between Banach spaces. For example, we show

---

1991 *Mathematics Subject Classification*. B46, B4615.

Research partially supported by NSF Grant DMS-9500874 and TARP Grant ARP-275.

in Proposition 2 that an operator between Banach spaces is non-weakly compact iff it maps some wide-(s) sequence into a wide-(s) sequence. Similarly, we show in Theorem 5 that an operator is non-Tauberian iff it maps some wide-(s) sequence into a norm convergent sequence. In Corollary 6 we obtain that an operator is Tauberian iff it maps some subsequence of a given wide-(s) sequence, into a wide-(s) sequence. (Recall that $T : X \to Y$ is Tauberian if $T^{**}(X^{**} \sim X) \subset Y^{**} \sim Y$.) Theorem 5 may also be deduced from the results due to R. Neidinger and the author in [NR], and standard facts. However we give here a self-contained treatment.

After circulating the first version of this paper, we learned that wide-(s) sequences were originally introduced by I. Singer, who termed them "P^*-sequences" [S1]. Some of the pleasant permanence properties discovered by Singer, are as follows: a semi-normalized sequence (x_j) is wide-(s) iff its difference sequence $(x_{j+1} - x_j)$ is a basic sequence. Moreover if (x_j) is a semi-normalized basis for a Banach space X and (x_j^*) is its sequence of biorthogonal functionals, then (x_j) is wide-(s) iff $(\sum_{j=1}^n x_j^*)_{n=1}^\infty$ is a wide-(s) sequence in X^* (cf. Theorem 9.2, page 311 of [S2]; as noted in our earlier version, these equivalences also follow from certain arguments in [R3]). Of course convex block bases of wide-(s) sequences are also wide-(s) (see Proposition 3 below).

We prove in Corollary 17 that suitable perturbations of wide-(s) sequences have wide-s subsequences. In fact our argument yields that given (x_j) a wide-(s) sequence in a Banach space X, there exists an $\varepsilon > 0$ so that if (y_j) is any bounded perturbation of (x_j) so that all w^*-cluster points of $(x_j - y_j)$ in X^{**} have norm at most ε, then (y_j) has a wide-(s) subsequence. We also obtain that ε may be chosen depending only on λ the wide-(s) constant of (x_j), which we introduce in Definition 3. It follows immediately from this definition that if (b_j) is a λ-wide-(s) sequence in X, then $\|b_j\| \le \lambda$ for all j and there exists a sequence (f_j) in X^* with $\|f_j\| \le \lambda$ for all j so that

$$\begin{aligned} f_i(b_j) &= 1 \ \text{ all } \ 1 \le i \le j < \infty \\ f_i(b_j) &= 0 \ \text{ all } \ 1 \le j < i < \infty \ . \end{aligned}$$

(We call sequences (b_j) satisfying this condition *triangular*, because the matrix $(f_i(b_j))$ is obviously upper triangular consisting of 1's above and on the natural diagonal, zeros below.) In Theorem 11 we prove that every non-reflexive Banach space has a $(1+\varepsilon)$-wide-(s) sequence for every $\varepsilon > 0$; this immediately yields the remarkable quantitative information on triangular arrays in non-reflexive spaces discovered by R.C. James [J1], [J2] and D.P. Milman–V.D. Milman [MM] (cf. the remark following the statement of Theorem 11).

Our proof of this result involves the rather technical result Theorem 12, and some standard facts on basic-sequence selections, formulated in [R3] and repeated here for completeness as Lemmas 13 and 14. These facts are used to deduce the rather surprising result that given $\varepsilon > 0$ and any bounded non-relatively compact sequence (x_j) in a Banach space X, there is a subsequence (x'_j) of (x_j) and an x in X with (b_j) a $(2+\varepsilon)$-basic sequence, where $b_j = x'_j - x$ for all j. Moreover, if (x_j) is non-relatively weakly compact, there is a $c > 0$ and a convex block basis (u_j) of (b_j) so that (cu_j) is $(1+\varepsilon)$-wide-(s) (Corollary 15). (For an interesting recent result on uniformity in the biorthogonal constant of subsequence of uniformly separated sequences see [HKPTZ].) Corollary 15 yields also a quantitative refinement of both Theorem 11 and Proposition 2a: given $\varepsilon > 0$, a non-weakly compact operator maps some $(1+\varepsilon)$-wide-(s) sequence into a wide-(s) sequence (Theorem 11$'$, given in the second remark following the proof of Corollary 15).

Our definition of λ-wide-(s) sequences applies to finite sequences as well. This leads to localizations of our results. For example, Proposition 19 yields that an operator is non-super weakly compact iff there is a $\lambda \geq 1$ so that for all n, it maps some λ-wide-(s) sequence of length n into a λ-wide-(s) sequence. Applying a quantitative refinement of Theorem 5 (called Theorem 5$'$ and formulated in the first remark following the proof of Corollary 15), we obtain that a Banach space X is non-super reflexive iff it contains $(1+\varepsilon)$-wide-(s) sequences of arbitrarily large length, for every $\varepsilon > 0$. In Proposition 20, we localize Theorem 5 to the setting of super Tauberian operators. Our results here are motivated by recent work of M. González and A. Martínez-Abejón [GM]. In fact, Proposition 20 may be deduced from results in [GM] and our localization result, Proposition 18, whose proof does not require ultraproduct techniques. However using ultraproducts, we obtain certain of the results in [GM] by localizing our Theorem 5$'$ and Proposition 3. Thus we obtain an alternative route to the result obtained in [GM]: if T is a given operator, then if T is non-super Tauberian, and T_U is an ultrapower of T, then $\ker T_U$ is non-reflexive. We then obtain the following strengthening of Proposition 12 of [GM]: *An operator T between Banach spaces is non-super-Tauberian iff for all $\varepsilon > 0$, all n, T maps some $(1+\varepsilon)$-wide-(s) sequence of length n into a sequence whose elements have norm at most ε.* (See the remark following the proof of Proposition 20.) Our Proposition 20 also yields the immediate Corollary that the super-Tauberian operators in $\mathcal{L}(X,Y)$ are an open set. (This result is due to D.G. Tacon [T]; the deduction of this result in [GM] motivates our formulation of Proposition 20.)

Next we introduce a class of operators intermediate between Tauberian and super Tauberian; the strongly Tauberian operators. These are operators $T \in \mathcal{L}(X,Y)$ whose natural induced map $\tilde{T} : X^{**}/X \to Y^{**}/Y$ is an isomorphism. It is immediate that the strongly Tauberian operators are an open subset of $\mathcal{L}(X,Y)$ and of course have the semigroup property. We give several equivalences in Theorem 21, obtaining for example, that T is strongly Tauberian iff T^{**} is strongly Tauberian iff T maps some subsequence of a given λ-wide-(s) sequence into a β-wide-(s) sequence, where β depends only on λ. The proof also yields that T is non-strongly Tauberian iff given $\varepsilon > 0$, T maps some $(1+\varepsilon)$-wide-(s) sequence into a sequence of diameter less than ε (Proposition 22). Of course this yields immediately, via Proposition 20, that every super Tauberian operator is strongly Tauberian. We then use these discoveries about strongly Tauberian operators in an essential way, to localize Corollary 6. Thus we obtain in Corollary 25 that T is super Tauberian iff for all k every λ-wide-(s) sequence of length n has a subsequence of length k mapped to a β-wide-(s) sequence by T, where β depends only on λ, n only on λ and k.

We conclude with a rather delicate localization of the infinite perturbation result given in Corollary 17. This result, Proposition 26, apparently requires ultraproduct techniques for its proof. For standard facts about ultraproducts in Banach spaces, which we use without explicit reference, see [H].

§**2.**

We first summarize some of the basic concepts used here.

Definition 1. A semi-normalized sequence (b_j) in a Banach space is called

(i) a wide-(s) sequence if (b_j) is a basic sequence which dominates the summing basis; i.e., $\sum c_j$ converges whenever $\sum c_j b_j$ converges.

(ii) an (s)-sequence if (b_j) is weak-Cauchy and a wide-(s) sequence.

(iii) an ℓ^1-sequence if (b_j) is equivalent to the usual ℓ^1-basis.

(iv) non-trivial weak-Cauchy if (b_j) is weak-Cauchy but not weakly convergent.

As proved in Proposition 2.2 of [R3], every non-trivial weak-Cauchy sequence has an (s)-subsequence. It then follows immediately from the ℓ^1-theorem that

Proposition 1. *Every wide-(s) sequence has a subsequence which is either an (s)-sequence or an ℓ^1-sequence.*

(The ℓ^1-theorem refers to the author's result that every bounded non-trivial weak-Cauchy sequence has an ℓ^1-subsequence.)

Of course every sequence which is either an (s)-sequence, or an ℓ^1-sequence, is wide-(s). If we don't wish to distinguish between the two mutually exclusive possibilities of Definition 1 (ii), (iii), then wide-(s) sequences are more appropriate.

Throughout, let X, Y be Banach spaces.

Proposition 2.

(a) *A bounded subset W of X is non-relatively weakly compact iff W contains a wide-(s) sequence.*

(b) *$T \in \mathcal{L}(X, Y)$ is non-weakly compact iff there is a sequence (x_n) in X with (x_n) and (Tx_n) both wide-(s) sequences.*

Proof.

(a) First suppose W is non-relatively weakly compact. Choose (x_n) in W with no weakly convergent subsequence. If (x_n) has a weak-Cauchy subsequence (x'_n), then (x'_n) is of course non-trivial, hence (x'_n) has an (s)-sequence by Proposition 2.2 of [R3]. If (x_n) has no weak-Cauchy subsequence, (x_n) has an ℓ^1-subsequence by the ℓ^1-theorem.

Conversely, no wide-(s) sequence can have a weakly convergent subsequence, by Proposition 1, thus proving (a). Alternatively, rather than using Proposition 1, we may give the following elementary argument.

It is trivial that any subsequence of a wide-(s) sequence is also wide-(s).

Thus suppose to the contrary, that (x_n) is a wide-(s) sequence with (x_n) converging weakly to some x. We may assume without loss of generality that (x_n) is a basis for X, since x is in the closed linear span of the x_n's. *Define* $\underline{s}$, the summing functional in X^* by

$$\underline{s}\Big(\sum c_j x_j\Big) = \sum c_j \ .$$

Then since $\underline{s}(x_j) = 1$ for all j, $\underline{s}(x) = 1$, so $x \neq 0$. However no basic sequence can converge weakly to a non-zero element. Indeed, if (x_j^*) denotes the sequence of functionals biorthogonal to (x_j), then

$$x_i^*(x) = \lim_{j\to\infty} x_i^*(x_j) = 0 \ \text{ for all } i \ ; \ \text{since}$$

$$(*) \qquad x = \sum_{i=1}^{\infty} x_i^*(x) x_i \text{ because } (x_j) \text{ is a } \textit{basis} \text{ for } x \ , \ x = 0 \ .$$

Proof of (b). First suppose T is non-weakly compact. Then $W \overset{\text{df}}{=} T(Ba X)$ is non-relatively weakly compact. Choose then (x_j) in $Ba\,X$ with (Tx_j) a wide-(s) sequence. But then (x_j) cannot have a weakly convergent subsequence (x_j'), for else (Tx_j') would be wide-(s) and weakly convergent which is impossible. Hence (x_j) has a wide-(s) subsequence (x_j') by part (a); then also (Tx_j') is wide-(s).

Of course conversely if (x_j) and (Tx_j) are wide-(s) then (x_j) is a bounded sequence and (Tx_j) has no weakly convergent subsequence, so T is not weakly compact. $\square$

Corollary. *A Banach space is non-reflexive iff it contains a wide-(s) sequence.*

Remark. In Theorem 12 below, we prove a general result for selecting wide-(s) sequences, which immediately yields Proposition 2(a), and hence this corollary, without using the ℓ^1-Theorem.

We next review some permanence properties of wide-(s) sequences. First, a companion definition.

Definition 2. A seminormalized sequence (e_j) in a Banach space X is called

(i) a wide-(c) sequence if (e_j) is a basic sequence with bounded partial sums; i.e., $\sup_n \|\sum_{j=1}^n e_j\| < \infty$.

(ii) a (c)-sequence if (e_j) is a basic sequence with $(\sum_{j=1}^n e_j)_{n=1}^\infty$ a weak-Cauchy sequence.

(iii) the difference sequence of a sequence (b_j) if $e_j = b_j - b_{j-1}$ for all $j > 1$, $e_1 = b_1$.

Remark. In the definition of "wide-(s)", the term "(s)" stands for "summing." In the above, "(c)" stands for "convergent"; of course here, the series $\sum e_j$ is only weak Cauchy convergent; i.e., $\sum_j x^*(e_j)$ converges for all $x^* \in X^*$.

As noted in the introduction, wide-(s) and wide-(c) sequences were originally introduced by I. Singer [S1]. He used the terminology "(x_j) is of type P (resp. type P^*)" if (x_j) is a wide-(c) (resp. wide-(s)) basis for a Banach space X. The following result summarizes various permanence properties. (For the proof of Proposition 3(i)–(iii), see [S1] or Theorem 9.2 of [S2]; Proposition 3(iv) is given as Proposition 2.1 of [R3].)

Proposition 3. *Let (b_j) be a given sequence in X with difference sequence (e_j).*

(i) *(b_j) is wide-(s) iff (e_j) is wide-(c).*

(ii) *(b_j) is wide-(s) iff (b_j) is bounded, $(\|e_j\|)$ is bounded below, and (e_j) is a basic sequence.*

(iii) *Assuming (b_j) is a basic sequence with biorthogonal functionals (b_j^*) in $[b_j]^*$, then (b_j) is wide-(s) iff (b_j^*) is wide-(c) iff $(\sum_{j=1}^n b_j^*)_{n=1}^\infty$ is wide-(s).*

(iv) *(b_j) is (s) iff (e_j) is (c).*

(v) *If (b_j) is a convex block-basis of a wide-(s) sequence, then (b_j) is wide-(s).*

Remarks. 1. Proposition 3 (ii) yields that wide-(s) sequences are *characterized* as seminormalized basic sequences whose difference sequence is also a basic sequence.

2. The statements (ii)–(iv) of Proposition 3 may also be deduced from some arguments in [R3] (specifically, see the proofs of Propositions 2.1 and 2.4).

For the sake of completeness, we give the proof of Proposition 3(v). Suppose then (u_i)

is a wide-(s) sequence and there exist $0 \le n_1 < n_2 < \cdots$ and numbers $\lambda_1, \lambda_2, \ldots$ with

$$b_j = \sum_{i=n_j+1}^{n_{j+1}} \lambda_i u_i \ , \quad \lambda_i \ge 0 \ , \quad \sum_{i=n_j+1}^{n_{j+1}} \lambda_i = 1 \ \text{ for all } j \ .$$

Of course then (b_j) is a basic sequence, since it is a block basis of one. Since (u_i) is wide-(s), there is a number β so that

$$\Big| \sum_{i=1}^{k} c_i \Big| \le \beta \| \sum c_i u_i \| \text{ whenever } \sum c_i u_i \text{ converges.}$$

Now suppose scalars $c_1, c_2, \ldots$ given with only finitely many non-zero and let $k \ge 1$. But then

$$\begin{aligned} \Big| \sum_{j=1}^{k} c_j \Big| &= \Big| \sum_{j=1}^{k} c_j \sum_{i=n_j+1}^{n_{j+1}} \lambda_i \Big| \\ &= \Big| \sum_{j=1}^{k} \sum_{i=n_{j+1}}^{n_{j+1}} c_j \lambda_i \Big| \\ &\le \beta \Big\| \sum_j c_j \sum_{i=n_j+1}^{n_{j+1}} \lambda_i u_i \Big\| \\ &= \beta \Big\| \sum c_j b_j \Big\| \ . \end{aligned}$$ □

Of course Propositions 2 and 3 yield the following immediate consequence.

Corollary 4. *$T \in \mathcal{L}(X,Y)$ is non-weakly compact iff there is a sequence (e_j) in X so that (e_j) and (Te_j) are both wide-(c) sequences.*

For the next result, recall that $T \in \mathcal{L}(X,Y)$ is *Tauberian* if $T^{**}(X^{**} \sim X) \subset Y^{**} \sim Y$.

Theorem 5. *Let $T \in \mathcal{L}(X,Y)$. The following are equivalent.*

(i) *T is non-Tauberian.*
(ii) *There exists a wide-(s) sequence (x_j) in X with (Tx_j) norm-convergent.*
(iii) *There exists a wide-(c) sequence (e_j) in X with $\sum \|Te_j\| < \infty$.*

Corollary 6. *Again let $T \in \mathcal{L}(X, Y)$. The following are equivalent.*

(i) *T is Tauberian.*

(ii) *For every wide-(s) sequence (x_j) in X, there is a subsequence (x'_j) with (Tx'_j) wide-(s).*

(iii) *For every wide-(c) sequence (e_j), there exist $n_1 < n_2 < \cdots$ with $(\sum_{j=n_i+1}^{n_{i+1}} Te_j)_{i=1}^{\infty}$ a wide-(c) sequence.*

Proof of Theorem 5. Assume first T is non-Tauberian. it follows that there exists an $x^{**} \in X^{**} \sim X$ with $\|x^{**}\| = 1$ and $T^{**}x^{**} \overset{\mathrm{df}}{=} y$ in Y. We shall prove there exists a normalized wide-(s) sequence (x_j) in X with (Tx_j) converging in norm to y. It is convenient to isolate the following step of the proof.

Lemma 7. *Given $\varepsilon > 0$ and F a finite dimensional subspace of X^*, there exists an $x \in X$ with $\|x\| \le 1 + \varepsilon$, $\|Tx - y\| < \varepsilon$, and*

$$f(x) = x^{**}(f) \text{ for all } f \in F\ . \tag{1}$$

Proof. It is a standard result that given G a finite-dimensional subspace of X^*, there exists and x_G in X with $\|x_G\| < 1 + \varepsilon$ and

$$g(x) = x^{**}(g) \text{ for all } G\ . \tag{2}$$

But then letting $\mathcal{D}$ be the directed set of all finite-dimensional subspaces of X^* containing F, we obtain a net $(x_G)_{G \in \mathcal{D}}$ with $\lim_{G \in \mathcal{D}} x_G = x^{**}$ weak*, with x_G satisfying (1) for all such G. (Here, we regard $X \subset X^{**}$.) Hence $\lim_{G \in \mathcal{D}} Tx_G = y$ weakly. But then there exists x a convex combination of the x_G's with $\|Tx - y\| < \varepsilon$. But then x satisfies (1) since every x_G does. $\square$

Now continuing with the proof of Theorem 5, since $x^{**} \notin X$, $x^{**\perp} \overset{\mathrm{df}}{=} \{x^* \in X^* : x^{**}(x^*) = 0\}$ δ-norms X for some $\delta \ge 0$; i.e.,

$$\text{for all } x \in X\ ,\ \delta\|x\| \le \sup\{|z(x)| : z \in x^{**\perp}\ ,\ \|z\| \le 1\} \tag{3}$$

Next, choose $x^* \in X^*$ with $x^{**}(x^*) = 1$.

Now applying Lemma 8, we may inductively choose a sequence (x_n) in X and a sequence (F_n) of finite-dimensional subspaces of $x^{**\perp}$ satisfying the following properties for all n (where $F_0 = \{0\}$).

$$1 - \frac{1}{2^n} \leq \|x_n\| \leq 1 + \frac{1}{2^n} \tag{4}$$

$$\|Tx_n - y\| < \frac{1}{2^n} \tag{5}$$

$$F_n \ 2\delta\text{-norms}[x_1, \ldots, x_n] \tag{6}$$

$$F_{n-1} \perp x_n \tag{7}$$

$$x^*(x_n) = 1 \tag{8}$$

$$F_{n-1} \subset F_n \ . \tag{9}$$

Indeed, suppose $n \geq 1$ and F_{n-1} chosen. Applying Lemma 8 (and its obvious consequence that for an appropriate F, any x satisfying its conclusion must satisfy $\|x\| \geq 1 - \varepsilon$), we let $F = F_{n-1} + [x^*]$; then choose $x = x_n$ satisfying (1) with $\|x\| \geq 1 - \frac{1}{2^n}$. Since (1) holds, $x^{**}(f) = f(x) = 0$ for all $f \in F_{n-1}$; i.e., (7) holds.

Of course also (8) holds. Now since $x^{**\perp}$ δ-norms X, we may choose $F_n \supset F_{n-1}$ a finite dimensional subspace of $x^{**\perp}$ which 2δ-norms $[x_1, \ldots, x_n]$.

This completes the inductive construction. Now if we let $Z = \bigcup_{i=1}^\infty F_n$, then (6) yields that

$$Z \quad 2\delta\text{-norms}[x_j]_{j=1}^\infty \ . \tag{10}$$

Evidently (5) yields then $Tx_n \to y$. Now we claim

$$(x_j) \text{ has no weakly convergent subsequence.} \tag{11}$$

Indeed, if not, let (x_j') be a subsequence weakly convergent to x, say. Then by (8), $x^*(x) = 1$. But by (7), $f(x) = \lim_{n\to\infty} f(x_n) = 0$ for all $f \in Z$. Since Z 2δ-norms $[x_j]_{j=1}^\infty$, $x = 0$, a contradiction.

Since (11) is proved, (x_j) has a wide-(s) subsequence (x_j') by Proposition 2(a). Of course then $(x_j'/\|x_j'\|)$ is the desired normalized wide-(s) sequence whose image tends to y. Thus (i) $\Rightarrow$ (ii) is proved.

(ii) $\Rightarrow$ (iii). By passing to a subsequence, assume $\|Tx_j - Tx_{j-1}\| < \frac{1}{2^j}$. Then the difference sequence (e_j) of (x_j) satisfies (iii).

(iii) $\Rightarrow$ (ii). Let $b_n = \sum_{j=1}^n e_j$ for all n. Then $\lim_{j\to\infty} Tb_j = \sum_{j=1}^\infty Te_j$ in norm and (b_j) is wide-(s).

(ii) $\Rightarrow$ (i). Since (x_j) is wide-(s), it follows (e.g., by applying Proposition 2(a)), that there is a weak*-cluster point x^{**} of (x_j) in $X^{**} \sim X$. But then if $Tx_j \to y$ in X say, $T^{**}x^{**} = y$, so T is non-Tauberian. $\square$

Remarks. 1. Note the following immediate consequence of Theorem 5: T is Tauberian iff $T|Z$ is Tauberian for every separable $Z \subset X$ iff $T|Z$ is Tauberian for every $Z \subset X$ with a basis.

2. We prove a stronger quantitative version (called Theorem 5′) later on, in the last remark following the proof of Corollary 15.

3. As proved in [NR], if T is non-Tauberian, there exists Z a closed linear subspace of X with $T(Ba\,Z)$ non-closed. If we then choose $y \in \overline{T(Ba\,Z)} \sim T(Ba\,Z)$, we may choose (x_n) in $Ba(Z)$ with $Tx_n \to y$. It now follows that any weak*-cluster point G of (x_n) in Z^{**} does not belong to Z. Of course then (x_n) has no weakly convergent subsequence; it follows then by known results that (x_n) has either an (s)-subsequence or an ℓ^1-subsequence. We preferred here, however, to give a direct self-contained proof of the non-trivial implication in Theorem 5.

Proof of Corollary 6. (i) $\Rightarrow$ (ii). Let (x_j) be a wide-(s) sequence in X. Then it follows that (Tx_j) has no weakly convergent subsequence. Indeed, otherwise, there would exist (b_j) a convex block basis of (x_j) with (Tb_j) norm-convergent. But then (b_j) is wide-(s) by Proposition 3(v), hence T is non-Tauberian by Theorem 5, a contradiction. Thus by Proposition 2(a), there exists (x'_j) a subsequence of (x_j) (with (Tx_j) wide-(s); of course (x'_j) is still wide-(s), so this implication is proved.

(ii) $\Leftrightarrow$ (iii) is immediate from the permanence property Proposition 3(i) and the evident fact that if (b_j) is a given sequence and $0 \le n_0 < n_1 < \cdots$, then if (e_j) is the difference sequence of (b_j), $(\sum_{j=n_i+1}^{n_{i+1}} e_j)_{i=1}^\infty$ is the difference sequence of $(b_{n_i})_{i=1}^\infty$. Of course no wide-(s) sequence can be norm-convergent, so (ii) $\Rightarrow$ (i) follows immediately from Theorem 5. $\square$

Theorem 5 easily yields the following result:

Corollary 8. *Let $T \in \mathcal{L}(X,Y)$. The following are equivalent:*

(a) *T is Tauberian*

(b) $\ker(T+K)$ *is reflexive for all compact $K \in \mathcal{L}(X,Y)$.*

(c) $\ker(T+K)$ *is reflexive for all nuclear $K \in \mathcal{L}(X,Y)$.*

(The equivalence (a) $\Leftrightarrow$ (b) is due to González and Onieva [GO].) Indeed, (a) $\Rightarrow$ (b) is immediate since then $T+K$ is also Tauberian, and (b) $\Rightarrow$ (c) is trivial. Now suppose (c) holds yet (a) is false. Then by Theorem 5(ii), after passing to a subsequence, we can choose (x_j) a wide-(s) sequence in X with $\|Tx_{j+1} - Tx_j\| < \frac{1}{2^j}$ for all j. It then follows immediately that $T|[x_j]$ is nuclear. Indeed, letting (b_j) be the difference sequence for (x_j), then (b_j) is also a semi-normalized basis for $[x_j]$ and $\sum \|Tb_j\| < \infty$. But then there exists a nuclear operator $K : X \to Y$ with $K|[x_j] = T|[x_j]$. Hence $\ker(T-K) \supset [x_j]$ is non-reflexive, because (x_j) is a wide-(s)-sequence lying inside this kernel. $\square$

We next show that wide-(s) sequences immediately yield the triangular arrays in non-reflexive Banach spaces, discovered by R.C. James [J1], [J2].

Proposition 9. *Let (b_j) be a wide-(s) sequence in a Banach space X. There exist bounded sequences (f_j) and (g_j) in X^* satisfying*

$$\text{(i)} \qquad f_i(b_j) = 1 \textit{ for all } j \geq i$$
$$f_i(b_j) = 0 \textit{ for all } j < i$$

$$\text{(ii)} \qquad g_j(b_i) = 1 \textit{ for all } j \geq i$$
$$g_j(b_i) = 0 \textit{ for all } j < i \ .$$

Proof. We may assume without loss of generality, by the Hahn-Banach Theorem, that $X = [b_j]$. Let $\mathbf{s}$ be the summing functional for (b_j) and (b_j^*) be the sequence biorthogonal to (b_j). Then simply let $g_j = \sum_{i \leq j} b_i^*$ and $f_j = \mathbf{s} - \sum_{i<j} b_i^*$ for all j. Since (b_j) dominates the summing basis, (g_j) and hence (f_j) are uniformly bounded. $\square$

Remark. It follows immediately from the duality theory given in the Proposition, page 722 of [R3], that if $X = [b_j]$ as above, both sequences (f_j) and (g_j) are again basic; in fact both are wide-(s) sequences in X^*.

We now give some quantitative definitions, in order to obtain further permanence properties and obtain certain localizations of the preceding results.

(Recall: If (b_j) is a basic-sequence or a finite sequence, then (b_j) is called a λ-basic sequence if $\|\sum_{j=1}^k c_j b_j\| \leq \lambda \|\sum c_j b_j\|$ for all k and scalars $c_1, c_2, \ldots$ with $\sum c_j b_j$ convergent.)

Definition 3. A (finite or infinite) sequence (b_j) in Banach space is called λ-wide-(s) if

(a) (b_j) is a 2λ-basic sequence.
(b) $\|b_j\| \leq \lambda$ for all j.
(c) $|\sum_{j=k}^n c_j| \leq \lambda \|\sum_{j=1}^n c_j b_j\|$ for all $1 \leq k \leq n < \infty$ (resp. with n the length of (b_j) if finite), and scalars $c_1, c_2, \ldots, c_n$.

Definition 4. A (finite or infinite) sequence (e_j) in a Banach space is called λ-wide-(c) if

(a) (e_j) is a λ-basic sequence.
(b) $\|e_j\| \geq \frac{1}{\lambda}$ for all j.
(c) $\|\sum_{j=1}^k e_j\| \leq \lambda$ for all k.

Remarks. 1. Of course an infinite sequence (b_j) (resp. (e_j)) is wide-(s) (resp. wide-(c)) iff it is λ-wide-(s) (resp. λ-wide-(c)) for some $\lambda \geq 1$. Also, note that if (b_j) is λ-wide-(s), then trivially $\|b_j\| \geq \frac{1}{\lambda}$ and $\|b_j^*\| \leq 2\lambda$ for all j, where (b_j^*) is biorthogonal to (b_j) in $(b_j)^*$. Also immediately (e_j) λ-wide-(c) implies $\|e_j\| \leq 2\lambda$ for all j.

2. An easy refinement of the (easy) proof of Proposition 3(v) shows that any convex block-basis of a (finite or infinite) λ-wide-(s) sequence is again λ-wide-(s).

3. It is easily seen that if (b_i) is a wide-(s) sequence in c_0, then the basis-constant for (b_i) is at least 2. This is why we give the requirement that λ-wide-(s) sequences be 2λ-basic.

The following quantitative version of Proposition 3(i) follows immediately from the arguments in [R3].

Proposition 10. *Let (b_j) be a finite or infinite sequence with difference sequence (e_j). For all $\lambda \geq 1$, there exists a $\beta \geq 1$ so that*

(a) *If (b_j) is λ-wide-(s), then (e_j) is β-wide-(c).*

(b) *If (e_j) is λ-wide-(c), then (b_j) is β-wide-(s).*

The next result, immediately yields the quantitative triangular array result of R.C. James [J1], in virtue of Proposition 9.

Theorem 11. *Every non-reflexive Banach space contains for every $\varepsilon > 0$, a normalized $(1+\varepsilon)$-wide-(s) sequence.*

To deduce the quantitative result in [J1], suppose (b_i) is normalized and $(1+\varepsilon)$-wide-(s). Let $f_j = \sum_{i=j}^{\infty} b_i^* = \mathbf{s} - \sum_{i<j} b_i^*$ as in Proposition 9. Thus $\|f_j\| \leq 1+\varepsilon$ for all j. Now set $h_j = f_j/\|f_j\|$ for all j. Thus (b_j) and (h_j) are norm-one sequences satisfying

$$\begin{aligned} h_i(b_j) &\geq \frac{1}{1+\varepsilon} \quad \text{for all} \quad j \geq i \\ h_i(b_j) &= 0 \text{ for all } \ j < i \ . \end{aligned} \tag{12}$$

Remark. The work in [J2] (specifically Theorem 8) essentially yields Theorem 11. (I had overlooked the fundamental reference [J2] in the earlier version of this paper.) The treatment given here provides a general criterion for selecting $(1+\varepsilon)$-wide-(s) sequences out of a given sequence, which yields the apparently stronger result Theorem 11$'$ below.

Theorem 11 is proved by a refinement of the argument given for Proposition 2.2 in [R3]. We will in fact show the following result, which easily yields Theorem 11. (As usual, we regard $X \subset X^{**}$; for G in X^{**}, $\operatorname{dist}(G,X)$ denotes the distance of G to X; i.e., $\operatorname{dist}(G,X) = \inf\{\|G-x\| : x \in X\}$.

Theorem 12. *Let (x_j) be a bounded sequence in a Banach space X, having a weak*-cluster point G in $X^{**} \sim X$. Let $d = \operatorname{dist}(G,X)$, $\varepsilon > 0$, and $\lambda = \frac{1}{d} + \varepsilon$, $\beta = \frac{\|G\|+d}{d} + \varepsilon$. Then (x_j) has a subsequence (b_j) satisfying the following:*

1) *(b_j) is β-basic.*

2) *$|\sum_{j=k}^{n} c_j| \leq \lambda \|\sum_{j=1}^{n} c_j b_j\|$ for all $1 \leq k \leq n < \infty$ and scalars $c_1, \ldots, c_n$.*

Corollary. *If*

$$\gamma = \max\{\frac{1}{d}, \frac{\|G\|+d}{2d}, \sup_j \|b_j\|\}$$

then for any $\varepsilon > 0$*,* (x_j) *has a* $(\gamma+\varepsilon)$*-wide-*(s)*-subsequence.*

Of course Theorem 11 follows immediately from this result. Indeed, we may assume without loss of generality that X is separable non-reflexive.

Let $\varepsilon > 0$, and let $\delta > 0$ be decided. Choose (using Riesz's famous lemma) a G in X^{**} with $\|G\| = 1$ and $\operatorname{dist}(G, X) > 1-\delta$. Next choose (x_j) a normalized sequence in X having G as a ω^*-cluster point. Now if δ is such that $\frac{1}{1-\delta} < 1+\varepsilon$, then (x_j) has a $1+\varepsilon$-wide-(s) subsequence by the Corollary.

For the proof of Theorem 12 we first recall some standard ideas and results (cf. [R3]). Given $0 < \eta \le 1$ and Y a linear subspace of X^*, Y is said to η-norm X if

$$\eta\|x\| \le \sup_{y \in Ba\, Y} |y(x)| \quad \text{for all} \quad x \in X \ .$$

The next two lemmas summarize well known material.

Lemma 13. *Let* (x_j) *be a semi-normalized sequence in* X *and* Y *an* η*-norming subspace of* X^**. Assume that* $y(x_j) \to 0$ *as* $j \to \infty$ *for all* $y \in Y$*. Then given* $0 < \varepsilon < \eta$*,* (x_j) *has a* $\frac{1}{\eta-\varepsilon}$*-basic subsequence.*

Lemma 14. *Let* $G \in X^{**} \sim X$*,* $\delta = \operatorname{dist}(G/\|G\|, X)$*,* $G_\perp = \{x^* \in X^* : G(x^*) = 0\}$*. Then* $G_\perp$ $\frac{1+\delta}{\delta}$*-norms* X*.*

Remark. Lemmas 13 and 14 immediately imply the classical result of M.I. Kadec and A. Pełczyński [KP] that a semi-normalized sequence in a Banach space has a basic subsequence provided every weakly convergent subsequence converges weakly to zero. These lemmas are in fact a crystallization of the arguments in [KP].

Proof of Theorem 12. We have that $\operatorname{dist}(\frac{G}{\|G\|}, X) = \frac{d}{\|G\|}$. It follows immediately from Lemmas 13 and 14 that (x_j) has a β-basic subsequence, so let us assume then that (x_j) is already β-basic. Let $\eta > 0$ be decided later. We shall choose (b_j) a subsequence of (x_j)

and a sequence (f_j) in X^* satisfying the following conditions for all n:

$$\|f_n\| < \frac{1}{d} + \eta \tag{13}$$

$$f_i(b_j) = 0 \quad \text{for all} \quad 1 \le j < i \le n \tag{14}$$

$$G(f_i) = 1 \quad \text{for all} \quad 1 \le i \le n \tag{15}$$

$$|f_i(b_n) - 1| < \frac{\eta}{2^n} \quad \text{for all} \quad 1 \le i \le n\ . \tag{16}$$

First, by the Hahn-Banach theorem, choose $F \in X^{***}$ with

$$\|F\| = \frac{1}{d}\ , \quad F(G) = 1\ , \quad \text{and} \quad F(x) = 0 \quad \text{all} \quad x \in X\ . \tag{17}$$

Now choose f_1 in X^* satisfying (13) and (15) for $n = 1$. Suppose $n \ge 1$ and $f_1, \dots, f_n$, $b_1, \dots, b_{n-1}$ chosen satisfying (13)–(15); suppose also $b_{n-1} = x_k$ (if $n > 1$). Since (15) holds and G is a w^*-cluster point of the x_j's, we may choose an $\ell > k$ so that setting $b_n = x_\ell$, then (16) holds. Since the span of $b_1, \dots, b_n$ and G, $[b_1, \dots, b_n, G]$, is finite dimensional, by (17) we may choose f_{n+1} in X^* with $\|f_{n+1}\| < \frac{1}{d} + \eta$, so that f_{n+1} agrees with F on $[b_1, \dots, b_n, G]$; i.e., $G(f_{n+1}) = 1$ and $f_{n+1}(b_i) = 0$ all $1 \le i \le n$. This completes the inductive construction of (b_j) and (f_j).

Now let $1 \le k \le n$ and scalars $c_1, \dots, c_n$ be given with $\|\sum_{j=1}^n c_j b_j\| \le 1$.

$$\begin{aligned}
\Big|\sum_{j=k}^n c_j\Big| &= \Big|f_k\Big(\sum_{j=1}^n c_j b_j\Big) + \sum_{j=k}^n c_j(1 - f_k(b_j))\Big| \quad \text{by (14)} \\
&\le \frac{1}{d} + \eta + \sup_j |c_j|\eta \quad \text{by (13) and (16)} \\
&\le \frac{1}{d} + \eta + \sup_j \|b_j^*\|\eta \quad (\text{where } (b_j^*) \text{ is the sequence biorthogonal to } (b_j) \text{ in } [b_j]^*)\ .
\end{aligned}$$

Since (x_j) was assumed basic, so is (b_j), whence $\tau \overset{\text{df}}{=} \sup_j \|b_j^*\| < \infty$. Thus if η is such that $\eta(1 + \tau) \le \varepsilon$, the proof is finished. $\square$

The next result yields absolute constants in the selection of general basic sequences in Banach spaces.

Corollary 15. *Let (x_j) be a bounded non relatively compact sequence in the Banach space X. Given $\varepsilon > 0$, there exists an $x \in X$ and a subsequence (x'_j) of (x_j) so that the following holds, where $b_j = x'_j - x$ for all j.*

1) *If $\{x_1, x_2, \ldots\}$ is relatively weakly compact, then (b_j) is a $(1+\varepsilon)$-basic sequence.*
2) *If $\{x_1, x_2, \ldots\}$ is not relatively weakly compact, then (b_j) is a $(2+\varepsilon)$-basic sequence. Moreover there exists a block basis (u_j) of (b_j) and a $c > 0$ so that (cu_j) is a $(1+\varepsilon)$-wide-(s) sequence.*

Proof. Since (x_j) is non-relatively compact, we may by passing to a subsequence assume that for some $\delta > 0$, $\|x_i - x_j\| \geq \delta$ for all $i \neq j$.

In Case 1), choose $(\bar{x}_j)$ a subsequence of (x_j) and $x \in X$ with $\bar{x}_j \to x$ weakly. Then $(\bar{x}_j - x)_{j=1}^{\infty}$ satisfies the hypotheses of Lemma 13 for $\eta = 1$, where "Y" $= X^*$. The conclusion of 1) is now immediate from Lemma 13.

In Case 2), we may choose $G \in X^{**} \sim X$ with G a w^*-cluster point of (x_j). Now let $d = \operatorname{dist}(G, X^{**})$, $\eta > 0$ to be decided later, and choose $x \in X$ with

$$\|G - x\| < d + \eta \ . \tag{18}$$

Now setting $H = G - x$, we have by Riesz's famous argument that since

$$d \leq \|H\| \leq d + \eta \ , \tag{19}$$

$$\operatorname{dist}\left(\frac{H}{\|H\|}, X\right) \geq \frac{d}{d+\eta} \ . \tag{20}$$

Thus if $\frac{\eta}{d} < \varepsilon$, it follows from Lemmas 13, 14, and the fact that H is a weak*-cluster point of $(x_j - x)$, that (x_j) has a subsequence (x'_j) satisfying the first statement in (2). Finally, it follows by the techniques in [R1] that there exists a convex block basis (u_j) of (b_j) and a separable isometrically norming subspace Y of X^* so that

$$\|u_j\| < d + \eta \quad \text{for all} \quad j \tag{21}$$

and

$$y(u_j) \to H(y) \text{ as } j \to \infty \ , \quad \text{all } y \in Y \ . \tag{22}$$

Now (19), (21) and (22) yield that if F is any w^*-cluster point of (u_j) in X^{**}, then

$$\|F\| \leq d + \eta \quad \text{and} \quad \operatorname{dist}(F, X) \geq d \ . \tag{23}$$

Fixing such an F, let $c = \|F\|^{-1}$; then of course (cu_j) has $F/\|F\|$ as a w^*-cluster point. It now follows from the Corollary to Theorem 12 (still assuming $\eta < \varepsilon d$) that (cu_j) has a $(1+\varepsilon)$-wide-(s) subsequence. $\square$

Remarks. 1. We may now easily obtain the following quantitative refinement of the non-trivial part of Theorem 5.

Theorem 5′. *If $T \in \mathcal{L}(X,Y)$ is non-Tauberian, then for all $\varepsilon > 0$, there exists a $(1+\varepsilon)$-wide-(s) sequence (b_j) in X with (Tb_j) norm-convergent.*

Proof. We may deduce this by just quantifying our proof of Theorem 5 and Corollary 15. We prefer, however, to deduce the result directly from Theorem 5 and Corollary 15. Choose (x_j) wide-(s) with (Tx_j) norm-convergent, by Theorem 5. Now choose by Corollary 15, a convex block basis (u_j) of (x_j), an element $x \in X$, and a $c > 0$ so that $b_j \overset{\text{df}}{=} c(u_j - x)$ is $1+\varepsilon$-wide-(s). But of course since (Tx_j) is norm-convergent, so is (Tb_j). $\square$

2. We also obtain a quantitative refinement of Proposition 2(a), which moreover sharpens Theorem 11, namely

Theorem 11′. *If $T \in \mathcal{L}(X,Y)$ is non-weakly compact, then for all $\varepsilon > 0$, there exists a $(1+\varepsilon)$-wide-(s) sequence (b_n) in X with (Tb_n) wide-(s).*

Proof. By Proposition 2(a), first choose (x_j) in X with (x_j) and (Tx_j) wide-(s). By Corollary 15, $\varepsilon > 0$ given, there is a convex block basis (u_j) of (x_j), a u in X, and a $c > 0$, so that (b_j) is $(1+\varepsilon)$-wide-(s), where $b_j = c(u_j - u)$ for all j. But of course (Tu_j) is again wide-(s), so (Tb_j) cannot have a weakly convergent subsequence, hence finally we can choose a subsequence (b_j') of (b_j) with (Tb_j') also wide-(s). $\square$

3. It is easily seen that the $(1+\varepsilon)$-wide-(s) sequences in Theorem 5′ and Theorem 11′ may be chosen to be normalized. In fact, in most of the results formulated here, we can take our sequences normalized. Indeed, suppose (b_j) is an arbitrary semi-normalized

sequence in a Banach space. Then we may choose b'_j a subsequence and a positive number c so that $\|b'_j\|$ rapidly converges to c, say $\big|\,\|b'_j\| - c\big| < \frac{1}{2^j}$ for all j. Then $\eta > 0$ given, it follows from standard perturbation results that for some N, the sequence $(b'_j)_{j=N}^\infty$ is $1+\eta$-equivalent to $(cb'_j/\|b'_j\|)_{j=N}^\infty$. Now if (b_j) is λ-wide-(s), then of course $\frac{1}{\lambda} \le c \le \lambda$ and we obtain that $(b'_j/\|b'_j\|)_{j=N}^\infty$ is $(c \vee \frac{1}{c})(1+\eta)^2\lambda$-wide-($s$). So of course this normalized sequence is $\lambda^2(1+\eta)^2$-wide-(s). Thus if $\varepsilon > 0$ is given and $\lambda^2(1+\eta)^2 < 1+\varepsilon$, $\lambda > 1$, and (b_j) is λ-wide-(s) with (Tb_j) norm-convergent, then also (Tu_j) is norm-convergent, where $u_j = b'_{j+N}/\|b'_{j+N}\|$ all j, and (u_j) is $(1+\varepsilon)$-wide-(s).

4. For a uniformity estimate in the biorthogonal constant of uniformly separated bounded sequences, see [HKPTZ].

We next pass to the stability of wide-(s) sequences under perturbations. We first show that after passing to subsequences, triangular arrays and wide-(s) sequences are manifestations of the same phenomena.

Definition 5. Given $\lambda \ge 1$, a finite or infinite sequence (b_j) in a Banach space is called λ-triangular if there exists a sequence (f_j) in X^* so that

(i) $\quad f_i(b_j) = 1$ for all $j \ge i$

$\quad f_i(b_j) = 0$ for all $j < i$

and

(ii) $\quad \|f_j\|, \|b_j\| \le \lambda$ for all j .

Evidently Proposition 9 and Definition 3 yield immediately that *every* (finite or infinite) *λ-wide-(s) sequence (b_j) is λ-triangular*. Thus Theorem 11 yields immediately that every non-reflexive Banach space has, for every $\varepsilon > 0$, a normalized $(1+\varepsilon)$-triangular sequence. (This result is due to R.C. James; see Theorem 8 of [J2].) Now conversely, suppose (b_j) is an infinite λ-triangular sequence, and assume without loss of generality that $X = [b_j]$. Now let G be a w^*-cluster point of (b_j) in X^{**}. Then of course $\|G\| \le \lambda$, and $G(f_i) = 1$ for all i. Then letting F be a w^*-cluster point of (f_i) in X^{***}, it follows that $F(G) = 1$ and $F(x) = 0$ all $x \in X$. Hence

$$\mathrm{dist}(G, X) \ge \frac{1}{\lambda} \ . \tag{24}$$

The Corollary to Theorem 12 now immediately yields

Corollary 16. *If* (b_j) *is a* λ*-triangular sequence, then for every* $\varepsilon > 0$, (b_j) *has a* $(\frac{\lambda^2+1}{2} + \varepsilon)$*-wide-*$(s)$ *subsequence.*

A refinement of this reasoning now yields the following perturbation result.

Corollary 17. *Let* $\lambda \geq 1$ *be given. There exist* $\beta \geq 1$ *and* $\varepsilon > 0$ *so that if* (b_j) *and* (p_j) *are infinite sequences in a Banach space with* (b_j) λ*-wide-*(s) *and* $\|p_j\| \leq \varepsilon$ *for all* j*, then*

$$\text{the sequence } (b_j + p_j)_{j=1}^{\infty} \text{ has a } \beta\text{-wide-}(s) \text{ subsequence.} \tag{25}$$

Proof. Let $\varepsilon > 0$; we shall discover the appropriate bounds for ε and β in the course of the argument. Choose (f_j) in X^* satisfying (i), (ii) of Definition 5. Since (b_j) is λ-triangular, it follows as in the argument preceding Corollary 16 that if G is any w^*-cluster point of (b_j) in X^{**}, then

$$\|G\| \leq \lambda \quad \text{and} \quad \operatorname{dist}(G, Y) \geq \frac{1}{\lambda}\ , \tag{26}$$

where $Y = [b_j]$. But then (cf. Lemma 2.6 of [R2])

$$\operatorname{dist}(G, X) \geq \frac{1}{2\lambda}\ . \tag{27}$$

Of course if P is any w^*-cluster point of (p_j), then $\|P\| \leq \varepsilon$. Hence we obtain that if H is a weak*-cluster point of $(b_j + p_j)_{j=1}^{\infty}$, then

$$\|H\| \leq \lambda + \varepsilon \quad \text{and} \quad \operatorname{dist}(H, X) \geq \frac{1}{2\lambda} - \varepsilon\ . \tag{28}$$

Evidently we thus obtain from the Corollary to Theorem 12 that $(b_j + p_j)_{j=1}^{\infty}$ has a β-wide-(s) subsequence provided

$$\varepsilon < \frac{1}{2\lambda} \tag{29}$$

where

$$\beta = \varepsilon + \Big((\lambda^2 + \tfrac{1}{2}) \vee 2\lambda\Big)\Big/(1 - 2\varepsilon\lambda)\ . \tag{30}$$

Evidently if we just let $\varepsilon = \frac{1}{4\lambda}$, we obtain that $(b_j + p_j)$ has a $(2\lambda^2 + 5)$-wide-(s) subsequence. $\square$

Remarks.

1. If we assume the p_j's lie in the closed linear span of the b_j's, we obtain the better estimate

$$\varepsilon < \frac{1}{\lambda} \quad \text{and} \quad \beta = \frac{\lambda^2 + 1}{2(1 - \lambda\varepsilon)} + \varepsilon \ .$$

Indeed, this follows immediately from the Corollary to Theorem 12 and the fact that then in (28), we have the better estimate $\operatorname{dist}(H, X) \geq \frac{1}{\lambda} - \varepsilon$. (Thus if (b_j) is say $(1+\eta)$-wide-(s), η small, then for ε-sufficiently small, the perturbed sequence would have a $(1+2\eta)$-wide-(s) subsequence.

2. Of course the proof only requires that (b_j) is λ-triangular. Moreover the same qualitative conclusion holds if we just assume instead that (p_j) is uniformly bounded so that all weak*-cluster points have distance at most ε from X. That is, we obtain then the following generalization.

3. Most of our results here are of a "refined subsequence" nature. Nevertheless, wide-(s) sequences and triangular arrays may have large spans. For example, J.R. Holub [Ho] (cf. also [S2], pp.627–628) has obtained that $L^1([0, 1])$ has a wide-(s) basis; his argument yields also that $C(\Delta)$ has a wide-(s) basis, Δ the Cantor set, and hence $C(K)$ has such a basis, any uncountable compact metric space K.

This suggests the following problem.

Question 1. *Does every non-reflexive Banach space with a basis have a wide-(s) basis?*

Now triangular arrays may have large linear spans, even when the space has no basis. This suggests

Question 2. *Does every separable non-reflexive Banach space have a λ-triangular fundamental sequence for some $\lambda > 1$? For every $\lambda > 1$?*

(A sequence (b_j) in a Banach space B is called fundamental if $[b_j] = B$.)

Corollary 17′. *Let $\lambda \ge 1$, $\varepsilon < \frac{1}{2\lambda}$, and (b_j), (p_j) be sequences in X with (b_j) λ-triangular, and (p_j) bounded such that all weak*-cluster points of (p_j) in X^{**} have distance at most ε from X. Then assuming $\|p_j\| \le M$ all j, there exists a β depending only on λ, M and ε, so that $(b_j + p_j)$ has a β-wide-(s) subsequence.*

For the sake of definiteness, we note that the proof actually yields that given $\eta > 0$ (arbitrarily small), we may choose

$$\beta = \frac{(\lambda^2 + \lambda M + \frac{1}{2} - \lambda\varepsilon) \vee 2\lambda}{1 - 2\varepsilon\lambda} + \eta \ .$$

Of course an interesting special case occurs when (p_j) is weakly convergent or even a constant sequence; then we have (since $\varepsilon = 0$) that

$$\beta = \left(\left(\lambda^2 + \lambda M + \frac{1}{2}\right) \vee 2\lambda\right) + \eta \ ,$$

or in the case where the p_j's lie in $[b_j]$,

$$\beta = \left((\lambda + M) \vee \left(\frac{\lambda^2 + \lambda M + 1}{2}\right)\right) + \eta \ .$$

3. Combining the last observation in the preceding remark with Theorem 5′, we obtain that *if $T \in \mathcal{L}(X,Y)$ is non-Tauberian, then for all $\varepsilon > 0$ there exists (b_j) a $(2+\varepsilon)$-wide-(s) sequence in X with (Tb_j) norm-convergent and $\|Tb_j\| \le \varepsilon$ for all j.*

Indeed, simply choose (x_j) a $(1+\varepsilon)$-wide-(s) sequence in X with (Tx_j) norm-convergent, by Theorem 5′. Now given $\varepsilon > 0$, choose k with $\|Tx_n - Tx_k\| \le \frac{\varepsilon}{2}$ for all $n \ge k$. But then $(x_n - x_k)_{n=k+1}^{\infty}$ has a λ-wide-(s) subsequence, with

$$\lambda \le \left(\left(1 + \frac{\varepsilon}{2}\right)^2 + \frac{1}{2} + \varepsilon\right) \vee (2 + \varepsilon) = 2 + \varepsilon \text{ for } \varepsilon \text{ small enough.} \qquad \square$$

(We do not know if “2” can be replaced by “1” in the above assertion.)

We now localize some of our preceding results.

Proposition 18. *Given $\lambda \ge 1$, $\varepsilon > 0$, and k, there is an n so that every λ-triangular sequence of length at least n has a $(\frac{\lambda^2+1}{2} + \varepsilon)$-wide-$(s)$ subsequence of length k.*

Proof. We given an “old fashioned” compactness argument, just using Corollary 16. Were this false, we can find for every n a norm $\|\cdot\|_n$ on $\mathbb{R}^n$ so that if $(b_j)_{j=1}^n$ denotes the usual unit vectors, then setting $X_n = (\mathbb{R}^n, \|\cdot\|_n)$, and letting $f_j = \sum_{i=j}^n b_j^*$ for all n, we have that

$$\|b_j\|_n \le \lambda \ , \ \|f_j\|_n^* \le \lambda \text{ all } 1 \le j \le n \ , \tag{31}$$

and $(b_j)_{j=1}^n$ has no subsequence of length k which is β-wide-(s) in X_n, where $\beta = \frac{\lambda^2+1}{2} + \varepsilon$.

Now of course $(b_j^*)_{j=1}^n$, the functionals biorthogonal to (b_j), are bounded by 2λ. But then it follows that regarding $(b_j)_{j=1}^\infty$ instead as the usual unit basis of c_{00}, the set of all sequences which are ultimately zero, we may choose $k_1 < k_2 < \cdots$ so that for all $x \in c_{00}$,

$$\lim_{n\to\infty} \|x\|_{k_n} \stackrel{\text{df}}{=} \|x\| \ \text{ exists.} \tag{32}$$

Of course this limit will exist uniformly on $W_n \stackrel{\text{df}}{=} \{x \in \mathbb{R}^n : \|x\|_\infty \le 1\}$, for each n. Now let X be the completion of $(c_{00}, \|\cdot\|)$. But then it follows immediately that $(b_j)_{j=1}^\infty$ is λ-triangular in X, yet (b_j) has no $(\beta - \frac{\varepsilon}{2})$-wide-$(s)$ subsequence in X, of length k. But by Corollary 16, (b_j) *has* an infinite $(\beta - \frac{\varepsilon}{2})$-wide-$(s)$ subsequence. This contradiction completes the proof. $\square$

We next consider localized results which follow directly from our work above, known facts about ultraproducts, and certain results in [GM] to which we refer for all unexplained concepts. First, we briefly recall some ideas concerning ultraproducts. Let $\mathcal{U}$ be a non-trivial ultrafilter on N. $X_{\mathcal{U}}$, an ultraproduct of X, denotes the Banach space $\ell^\infty(X)/\mathcal{N}_{\mathcal{U}}(X)$: $\ell^\infty(X)$ denotes the Banach space of all bounded sequences in X, and $\mathcal{N}_{\mathcal{U}}(X)$ its subspace of sequences (x_j) with $\lim_{j\in\mathcal{U}} \|x_j\| = 0$. For any bounded sequence (x_j) in X, we denote its equivalence class in $X_{\mathcal{U}}$ by $[(x_j)]_{j=1}^\infty$. For such an object x, we have $\|x\| = \lim_{j\in\mathcal{U}} \|x_j\|$. Given $T \in \mathcal{L}(X,Y)$, the ultrapower $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ is defined by $T_{\mathcal{U}}(x) = [(Tx_j)]_{j=1}^\infty$ for all $x = [(x_j)]_{j=1}^\infty$ in $X_{\mathcal{U}}$. By the results in [GM], we may take as working definition: T is super weakly compact (resp. super Tauberian) provided $T_{\mathcal{U}}$ is weakly compact (resp. Tauberian). We note that (by results cited in [GM]), these definitions are independent of the chosen ultrafilter.

We first localize Proposition 2 and Corollary 4.

Proposition 19. *Let $T \in \mathcal{L}(X,Y)$. Then the following are equivalent.*

1.) *T is non-super weakly compact.*

2.) *There is a $\lambda \geq 1$ so that for all n, there exist $b_1,\dots,b_n$ in X with $(b_i)_{i=1}^n$ and $(Tb_i)_{i=1}^n$ λ-wide-(s).*

3.) *There is a $\lambda \geq 1$ so that for all n, there exist $e_1,\dots,e_n$ in X with $(e_i)_{i=1}^n$ and $(Te_i)_{i=1}^n$ λ-wide-(c).*

Proof. 2) $\Leftrightarrow$ 3) follows by Proposition 10. Now suppose first that condition 2) holds. Fix $\mathcal{U}$ a non-trivial ultrafilter on $\mathbb{N}$, and let $X_{\mathcal{U}}, Y_{\mathcal{U}}$ be the ultrapowers of X, Y respectively. For each n, let $b_1^n,\dots,b_n^n$ be chosen in X with $(b_i^n)_{i=1}^n$ and $(Tb_i^n)_{i=1}^n$ λ-wide-(s). For $n < i$, set $b_j^n = 0$, and now define (b_i) in $X_{\mathcal{U}}$ by $b_i = [(b_i^n)]_{n=1}^\infty$ for all i. Then it follows that (b_i) is λ-wide-(s) and $(T_{\mathcal{U}} b_i)$ is also λ-wide-(s). But then $T_{\mathcal{U}}$ is not weakly compact by Proposition 2, hence T is not super weakly compact. Conversely, if $T_{\mathcal{U}}$ is not weakly compact, then again applying Proposition 2, there exists a sequence (b_j) in $X_{\mathcal{U}}$ with (Tb_j) λ-wide-(s) in $Y_{\mathcal{U}}$. But now assuming (b_j) and (Tb_j) are both β-wide-(s) and letting $\lambda > \beta$, standard properties of ultraproducts allows us to deduce the existence of finite sequences satisfying 2) for all n. $\square$

Remark. It follows moreover from Theorem $11'$ and the remark following that in fact T *is non-super weakly compact iff there is a $\lambda \geq 1$ so that for all n and $\varepsilon > 0$ T maps a normalized $(1+\varepsilon)$-wide-(s) sequence of length n into a λ-wide-(s) sequence.*

We thus obtain immediately the following

Corollary. *The following assertions are equivalent, for all given Banach spaces X.*

1) *X is non-super reflexive.*

2) *There is a $\lambda \geq 1$ so that X contains λ-wide-(s) sequences of arbitrarily large length.*

3) *For all $\varepsilon > 0$, X contains normalized $(1+\varepsilon)$-wide-(s) sequences of arbitrarily large length.*

Proposition 20. *Let $T \in \mathcal{L}(X,Y)$. Then the following are equivalent.*

1.) *T is non-super-Tauberian.*

2.) *There is a* $\lambda \geq 1$, *so that for all* n, *all* $\varepsilon > 0$, *there exist* $b_1, \ldots, b_n$ *in* X *with* $(b_i)_{i=1}^n$ λ*-wide-*(s) *and* $\mathrm{diam}\{Tb_i : 1 \leq i \leq n\} < \varepsilon$.

3.) *For all* $\varepsilon > 0$ *and all* n *there exist* $b_1, \ldots, b_n$ *in* X *with* $(b_i)_{i=1}^n$ $(1+\varepsilon)$*-wide-*(s) *and* $\mathrm{diam}\{Tb_i : 1 \leq i \leq n\} < \varepsilon$.

4.) *There is a* $\lambda \geq 1$ *so that for all* n, *all* $\varepsilon > 0$, *there exist* $e_1, \ldots, e_n$ *in* X *with* $(e_i)_{i=1}^n$ λ*-wide-*(c) *and* $\sum_{i=1}^n \|Te_i\| < \varepsilon$.

Proof. Again 2) $\Leftrightarrow$ 4) follows by Proposition 10.

We now proceed with the same method as in the preceding case. Again, suppose condition 2) holds. Then for all n, we may choose $b_1^n, \ldots, b_n^n$ in X with $(b_i^n)_{i=1}^n$ λ-wide-(s) and $\mathrm{diam}\{Tb_i^n : 1 \leq i \leq n\} < \frac{1}{n}$. Again, letting $\mathcal{U}, X_{\mathcal{U}}, Y_{\mathcal{U}}$, and $T_{\mathcal{U}}$ be as in the preceding proof, and again defining $b_i = [(b_i^n)]_{n=1}^\infty$, we have that $(b_i)_{i=1}^\infty$ is a λ-wide-(s) sequence in $X_{\mathcal{U}}$. In this case, we have that $(T_{\mathcal{U}} b_i)$ is a constant sequence, hence $T_{\mathcal{U}}$ is non Tauberian by Theorem 5. (Thus T is non-super-Tauberian by the results in [GM].) Indeed, we have that for all j,

$$\|T_{\mathcal{U}} b_1 - T_{\mathcal{U}} b_j\| = \lim_{n \in \mathcal{U}} \|Tb_1^n - Tb_j^n\| = 0 \ . \tag{33}$$

Conversely, suppose T is non-super-Tauberian. Thus by [GM], $T_{\mathcal{U}}$ is non-Tauberian, so by Theorem 5$'$, for all $\varepsilon > 0$, there is a $1 + (\varepsilon/2)$-wide-(s) sequence (b_j) in $X_{\mathcal{U}}$ with $\mathrm{diam}\{Tb_i : 1 \leq i \leq n\} \leq \varepsilon/2$. Again it follows from standard properties of ultraproducts that 3) holds. Thus we have shown 2) $\Rightarrow$ 1) $\Rightarrow$ 3) and of course 3) $\Rightarrow$ 2) is trivial. $\square$

Remarks. 1. A proof of the above result may also be obtained by directly applying our Proposition 15 and Proposition 12 of [GM].

2. Our proof also yields (cf. the third remark following the proof of Corollary 16 above) that *if* T *is non-super-Tauberian, then for all* $\varepsilon > 0$ *and* n, *there exists* $(b_1, \ldots, b_n)$ *in* X *which is* $(2+\varepsilon)$*-wide-*(s) *and* $\|Tb_i\| < \varepsilon$ *for all* $1 \leq i \leq n$. Actually, Proposition 12 of [GM] and our Proposition 18 yield that “2” may be replaced by “1” in this assertion. We may deduce this from our work as follows: Choose any $(b_1^n, \ldots, b_n^n)$ in X with $(b_1^n, \ldots, b_n^n)$ 3-wide-(s) and $\|Tb_i^n\| < \frac{1}{n}$ all n, and let $b_j = [(b_j^n)]_{n=1}^\infty$ in $X_{\mathcal{U}}$ for all j ($\mathcal{U}$ a specified non-trivial ultrafilter on N). But then (b_j) is 3-wide-(s) in $X_{\mathcal{U}}$, yet $T_{\mathcal{U}} b_j = 0$ all j. But this

yields the result in [GM] that $\ker T_{\mathcal{U}}$ is non-reflexive. But then by Theorem 11, we have that given $\varepsilon > 0$, there exists (v_j) in $\ker T_{\mathcal{U}}$ with (v_j) $(1+\varepsilon)$-wide-(s). Of course now using standard ultraproduct techniques, we indeed obtain that *if T is non-super-Tauberian, then for all n and ε, there exists $(b_i, \dots, b_n)$ in X $(1+\varepsilon)$-wide-(s) and $\|Tb_i\| < \varepsilon$ for all i.* Since the ε here may be chosen arbitrarily, it also follows that in fact $(b_1, \dots, b_n)$ may be chosen normalized.

Proposition 20, 3) also yields the result of D.G. Tacon [T] that the super-Tauberian operators form an open set in $\mathcal{L}(X,Y)$ (with the same argument as the proof given in [GM]).

Corollary. *The non-super-Tauberian operators from X to Y are a closed subspace of $\mathcal{L}(X,Y)$.*

Proof. Assume $T_n \to T$, T_n non-super-Tauberian operators in $\mathcal{L}(X,Y)$ for all n. Now $\varepsilon > 0$ given, if T_n is fixed with $\|T_n - T\| < \varepsilon$, for all k, choose $b_1, \dots, b_k$ $(1+\varepsilon)$-wide-(s) with $\operatorname{diam}_{1 \le i \le k} T_n(b_i) < \varepsilon$. But then $\operatorname{diam}_{1 \le i \le k} T(b_i) < 2\varepsilon(1+\varepsilon) + \varepsilon$. Thus T is non-super-Tauberian by Proposition 20 part 3). $\square$

We now introduce a new class of operators, intermediate (as we shall see) between the classes of Tauberian and super Tauberian operators.

Definition. T in $\mathcal{L}(X,Y)$ is called *strongly Tauberian* provided its induced operator $\widetilde{T}$ from X^{**}/X to Y^{**}/Y is an (into) isomorphism.

(In this definition, letting $\pi : X^{**} \to X^{**}/X$ denote the quotient map, $\widetilde{T}$ is defined by $\widetilde{T}(\pi x^{**}) = \pi(T^{**}x^{**})$.) Evidently T is Tauberian precisely when its induced operator $\widetilde{T}$ is one–one. The stronger property given by Definition 6 immediately yields that the strongly Tauberian operators from an open semi-group; as we shall shortly see, this class is also closed under such natural operations as taking double adjoints. We now list several equivalences for these operators. (A word about notation: we regard $X \subset X^{**}$ and $X^{**} \subset X^{4*}$; for $Y \subset X$, $Y^\perp = \{x^* \in X^* : x^*(y) = 0 \text{ all } y \in Y\}$; thus e.g., $X^\perp$ is a subspace of X^{3*} so $X^{\perp\perp}$ is a subspace of X^{4*}. Recall that although $X^{\perp\perp}$ is canonically isometric to X^{**}, $X^{\perp\perp} \cap X^{**} = X$. Also, we denote T^{***} by T^{3*}, X^{***} by X^{3*} etc.)

Theorem 21. *Let $T \in \mathcal{L}(X,Y)$. Then the following are equivalent.*

1) *T is strongly Tauberian.*
2) *There is a $\delta > 0$ so that for all $x^{**} \in X^{**}$, $\operatorname{dist}(T^{**}x^{**}, Y) \geq \delta \operatorname{dist}(x^{**}, X)$.*
3) *$T^{3*}Y^{\perp} = X^{\perp}$*
4) *$T^{4*}|X^{*\perp}$ is an isomorphism.*
5) *T^{**} is strongly Tauberian.*
6) *For all $\lambda > 1$, there exists a $\beta > 1$ so that every (infinite) λ-wide-(s) sequence in X has a subsequence which is mapped into a β-wide-(s) sequence by T.*
7) *Same as 6), except replace "for all $\lambda > 1$") by "there exists $\lambda > 1$".*

Remark. It follows immediately from the definition that if $T, S \in \mathcal{L}(X,Y)$ are given with T strongly Tauberian and S weakly compact, then $T+S$ is strongly Tauberian, since $\widetilde{T+S} = \widetilde{T} + \tilde{S} = \widetilde{T}$. We obtain a simultaneous generalization of this observation, together with the openness of the set of strongly Tauberian operators in $\mathcal{L}(X,Y)$, as follows: *Given T strongly Tauberian, there exists $\varepsilon > 0$ so that if $S \in \mathcal{L}(X,Y)$ satisfies $\|\tilde{S}\| < \varepsilon$, then $S+T$ is strongly Tauberian.* Indeed, simply choose δ as in Theorem 21 part 2), then any $\varepsilon < \delta$ works.

Proof of Theorem 21. 1) $\Leftrightarrow$ 2) Immediate from Definition 6.

1) $\Leftrightarrow$ 3) $(X^{**}/X)^*$ is canonically identified with $X^{\perp}$; now we always have that $T^{3*}Y^{\perp} \subset X^{\perp}$ and moreover $(\widetilde{T})^*$ may be canonically identified with $T^{3*}|Y^{\perp}$. Now this equivalence follows from the observation that an operator is an isomorphism iff its adjoint is surjective.

For the next two equivalences, it is convenient to isolate out the following elementary fact.

Lemma. *Let Z, W, be Banach spaces, Z_i, W_i be closed linear subspaces of Z, W respectively, $1 \leq i \leq 2$, $S : Z \to W$ be a given bounded linear operator with $SZ_i \subset W_i$, $1 \leq i \leq 2$, and $Z = Z_1 \oplus Z_2$, $W = W_1 \oplus W_2$. Let $\tilde{S} : Z/Z_1 \to W/W_1$ be the induced operator defined by $\tilde{S}(\pi z) = \pi(Sz)$, π the appropriate quotient map. Then $\tilde{S}$ is an isomorphism iff $S|Z_2$ is an isomorphism.*

1) $\Leftrightarrow$ 4) We apply the Lemma to Z (resp. W) $= X^{4*}$ (resp. Y^{4*}), Z_1 (resp. W_1) $= X^{\perp\perp}$ (resp. $Y^{\perp\perp}$), $Z_2 = X^{*\perp}$ (resp. $Y^{*\perp}$), and $S = T^{4*}$. Now we have, by standard Banach space theory, that $X^{3*} = X^{\perp} \oplus X^*$ hence $X^{4*} = X^{\perp\perp} \oplus X^{*\perp}$. Thus the hypotheses of the Lemma are fulfilled. Of course for any operator U, U is an isomorphism iff U^{**} is an isomorphism. Now first assuming T is strongly Tauberian, we thus have that $(\widetilde{T})^{**}$ is an isomorphism. But $(X^{**}/X)^{**}$ may be canonically identified with $X^{4*}/X^{\perp\perp}$ and $(\widetilde{T})^{**}$ with $\tilde{S}$. Hence by the Lemma, $\tilde{S}|Z_2 = T^{4*}|X^{*\perp}$ is an isomorphism. But conversely, $T^{4*}|X^{*\perp}$ an isomorphism implies $(\widetilde{T})^{**}$ is an isomorphism, again by the Lemma, so T is strongly Tauberian.

4) $\Leftrightarrow$ 5) Now we apply our Lemma to the fact that $X^{4*} = (X^*)^{3*} = X^{*\perp} \oplus X^{**}$. Thus we obtain that $T^{4*}|X^{*\perp}$ is an isomorphism iff $\widetilde{T^{**}}$ is an isomorphism.

2) $\Rightarrow$ 6) Let T be strongly Tauberian, and choose $\delta > 0$ satisfying 2). Now let (x_j) be a λ-wide-(s) sequence in X, and let G be a w^*-cluster point of (x_j) in X^{**}. By an argument in the proofs of Corollaries 16, 17, we have that

$$\operatorname{dist}(G, X) \geq \frac{1}{2\lambda} \ . \tag{34}$$

Then $T^{**}G$ is a w^*-cluster point of (Tx_j), so by 2),

$$\operatorname{dist}(T^{**}G, Y) \geq \frac{\delta}{2\lambda} \ . \tag{35}$$

Hence by the Corollary to Theorem 12, (Tx_j) has a β-wide-(s) subsequence, where ($\varepsilon > 0$ given)

$$\beta = \left(\frac{2\lambda}{\delta} \vee \frac{\lambda^2 + \frac{\delta}{2}}{\delta} \vee \lambda \right) + \varepsilon \ . \tag{36}$$

6) $\Rightarrow$ 7) is trivial, so it remains to prove 7) $\Rightarrow$ 1). This is an immediate consequence of the following equivalence.

Proposition 22. *$T \in \mathcal{L}(X, Y)$ is non-strongly Tauberian iff for all $\varepsilon > 0$, there exists (x_j) a $(1+\varepsilon)$-wide-(s) sequence in X with* $\operatorname{diam}\{Tx_1, Tx_2, \ldots\} < \varepsilon$.

Proof of Proposition 22. It suffices to prove the direct implication. Indeed, if the second assertion of Proposition 22 holds, then (since trivially any $1+\eta$-wide-(s) sequence is $1+\varepsilon$-wide-(s) if $\eta<\varepsilon$), 6) of Theorem 21 fails, whence by 1) $\Rightarrow$ 6) of the latter, T is non-strongly Tauberian.

Now assume T is non-strongly Tauberian, and let $0<\eta<1$. Then we may choose G in X^{**} satisfying

$$\|G\|<1\ ,\ \ \mathrm{dist}(G,X)>1-\eta\ ,\quad \text{and}\quad \mathrm{dist}(T^{**}G,Y)<\eta\ . \tag{37}$$

Of course then we may also choose $y\in X$ with

$$\|T^{**}G-y\|<\eta\ . \tag{38}$$

Now standard techniques (cf. [R1]) yield

Lemma 23. *Let $L=\{x\in X:\|x\|<1$ and $\|Tx-y\|<\eta\}$. Then $G\in\tilde{L}$.*

(For $M\subset B$ a Banach space, $\widetilde{M}$ denotes the w^*-closure of M in B^{**}.)

Proof of Lemma 23. If not, by the Hahn-Banach separation theorem, choose $x^*\in X^*$ and $a<b\overset{\mathrm{df}}{=}G(x^*)$ so that

$$x^*(\ell)\le a\ \ \text{for all}\quad \ell\in L\ . \tag{39}$$

Now it follows that setting $W=\{x\in X:\|x\|<1$ and $x^*(x)>\frac{a+b}{2}\}$, then $G\in\widetilde{W}$ and of course $W\cap L=\emptyset$. Thus W is a convex set so that

$$T^{**}G\in\widetilde{TW}\ \ \text{and}\ \ \|Tw-y\|\ge\eta\ \ \text{all}\ \ w\in W\ . \tag{40}$$

Then again by the Hahn-Banach theorem, there exists a y^* in Y^* with $\|y^*\|=1$ and

$$y^*(Tw-y)\ge\eta\ \ \text{all}\ \ w\in W\ . \tag{41}$$

But since $T^{**}G-y$ is in the w^*-closure of $TW-y$, we obtain that

$$\|T^{**}G-y\|\ge\langle y^*,T^{**}G-y\rangle\ge\eta\ , \tag{42}$$

contradicting (38). (Here we have assumed $L \neq \emptyset$. However if $L = \emptyset$, instead let $W = \{x \in X : \|x\| < 1\}$; now (40) holds, and the rest of the argument following (40) again yields a contradiction to (38). $\square$

We may now complete the proof of Proposition 22 as follows. Let $\varepsilon > 0$, and let $\eta > 0$ be chosen, with $\frac{1}{1-\eta} < \varepsilon$. First, the proof of Theorem 12 yields that we may choose (b_j) a $\frac{1}{1-\eta}$-triangular sequence in L. (The proof of the existence of (b_j) in L, (f_j) in X^* satisfying (13)–(16) does not require the separability of X.) Here, we are just using that $G \in \tilde{L}$ and $\operatorname{dist}(G, X) > 1-\eta$. Then by Corollary (16), (b_j) has a $[\frac{(\frac{1}{1-\eta})^2+1}{2} + \eta]$-wide-$(s)$ subsequence (x_j). Of course we are now finished, as long as

$$\frac{(\frac{1}{1-\eta})^2 + 1}{2} + \eta < 1 + \varepsilon \quad \text{and} \quad \eta < \frac{\varepsilon}{2} \ .$$

Indeed, then since $x_j \in L$ for all j, $\|Tx_i - Tx_j\| < 2\eta < \varepsilon$ all i, j. $\square$

Remark. The proof of 2) $\Rightarrow$ 6) yields a non-linear estimate for the dependence of β on λ (36). However using Theorem 12 itself, we obtain that *if* δ *is as given in* 2) *of Theorem 21, then given* $\varepsilon > 0$, *every infinite* λ*-triangular sequence in* X *has a subsequence which is mapped into a* $(\frac{2\lambda}{\delta} + \varepsilon)$*-triangular sequence by* T which is also β-wide-(s), with β as in (36). Thus working with λ-triangular sequences, we recapture the best "δ" in 2), to within a factor of 2.

Of course Proposition 22 and Proposition 20, part 3) have the following immediate consequence.

Corollary 24. *Every super-Tauberian operator is strongly Tauberian.*

Remark. We thus have for a given $T \in \mathcal{L}(X, Y)$,

$$T \text{ super-Tauberian } \Rightarrow T \text{ strongly Tauberian } \Rightarrow T \text{ Tauberian.}$$

It is easily seen these implications are strict (in general). Indeed, there exist reflexive Banach spaces X which admit non-super Tauberian operators on them (e.g., $X = (\bigoplus_{n=1}^{\infty} \ell_n^1)_2$; of course any operator on a reflexive Banach space is strongly Tauberian. If

$X = (\bigoplus_{n=1}^{\infty} c_0)_2$, and $T((x_n)) = (\frac{1}{n}x_n)$ all $(x_n) \in X$, then T is Tauberian and any open neighborhood of T in $\mathcal{L}(X)$ contains a non-Tauberian operator, namely $Sx_n = \frac{1}{n}x_n$ for $n \le N$, $S(x_n) = 0$ for $n > N$, for suitable N. Hence T is non-strongly Tauberian since the strongly Tauberian operators are an open set in $\mathcal{L}(X)$, contained in the Tauberian ones. (I am indebted to A. Martínez for this conceptual proof that T is non-strongly Tauberian.) It is also known there exist Tauberian operators T with T^{**} non-Tauberian [AG].

Our next result, localizing Corollary 6, yields a "direct" characterization of super-Tauberian operators.

Corollary 25. *Let $T \in \mathcal{L}(X,Y)$. Then the following are equivalent.*

1) *T is super-Tauberian*
2) *For all $\lambda \ge 1$ there exists $\beta \ge 1$ so that for all positive integers k, there exists $n > k$ so that if $(x_1,\dots,x_n)$ is a λ-wide-(s) sequence in X; then $(Tx_{j_1},\dots,Tx_{j_k})$ is β-wide-(s) for some $j_1 < j_2 < \cdots j_k \le n$.*
3) *Same as 2), except replace "For all $\lambda \ge 1$" "there exists $\lambda \ge 1$".*

Remark. This corollary immediately yields "directly" the semi-group property: ST is super Tauberian if T, S are.

Now fix $\lambda \ge 1$, and let $\mathcal{U}$, $X_{\mathcal{U}}$, $Y_{\mathcal{U}}$, and $T_{\mathcal{U}}$ be as in the proof of Proposition 9. Since T is super-Tauberian, it follows easily that also $T_{\mathcal{U}}$ is super-Tauberian. Hence by Corollary 24, $T_{\mathcal{U}}$ is strongly Tauberian. Thus we may choose $\beta > 1$ so that

$$\text{Theorem 21 holds for "}X\text{"} = X_{\mathcal{U}},\ \text{"}T\text{"} = T_{\mathcal{U}}. \tag{43}$$

Now we claim that 2β works for the Corollary. If not, then there exists a positive k so that for all n, we may choose $(x_1^n,\dots,x_n^n)$ λ-wide-(s) in X, so that no subsequence of $(Tx_1^n,\dots,Tx_n^n)$ of length k is 2β-wide-(s). Now as usual, define (x_j) in $X_{\mathcal{U}}$ by $x_j = [(x_j^n)]_{n=1}^{\infty}$ for all j. Then it follows that (x_j) is λ-wide-(s) in $X_{\mathcal{U}}$. Hence by (43), there exists $j_1 < j_2 < \cdots$ with $(T_{\mathcal{U}}x_{j_i})$ a β-wide-(s) sequence. But then for n sufficiently large, $(Tx_{j_1}^n,\dots,Tx_{j_k}^n)$ is 2β-wide-(s), a contradiction. $\square$

We conclude with a perturbation result whose proof follows from ultraproduct considerations, which generalizes Proposition 18. This result is a localization of Corollary 17 (cf. the remark following its proof). We say that for sequences (x_j) and (y_j) in a Banach space, (y_j) is an ε-perturbation of (x_j) if $\|x_j - y_j\| \le \varepsilon$ all j.

Proposition 26. *Let $\lambda \ge 1$ be given. There exist $\beta \ge 1$ and $\varepsilon > 0$ so that for all k, there is an $n > k$ so that every ε-perturbation of a λ-triangular sequence of length n has a β-wide-(s) subsequence of length k.*

Proof. Let β and ε be chosen satisfying the conclusion of Corollary 17, and let $\beta' > \beta$. We shall show that β' and ε satisfy the conclusion of Proposition 26. Were this false, by simply taking the c_0-sum of some finite-dimensional Banach spaces, we may choose a Banach space X and for every n, sequences $(b_j^n)_{j=1}^n$ and $(p_j^n)_{j=1}^n$ in X with $(b_j^n)_{j=1}^n$ λ-triangular, $\|p_j^n\| \le \varepsilon$ all j, yet $(b_j^n + p_j^n)$ has no β'-wide-(s) subsequence of length k. But then letting $\mathcal{U}$ and $X_{\mathcal{U}}$ be as before, and defining $b_j = [(b_j^n)]_{n=1}^\infty$, $p_j = [(p_j^n)]_{n=1}^\infty$ for all j, then (b_j) is λ-triangular in $X_{\mathcal{U}}$ and $\|p_j\| \le \varepsilon$ for all j, hence $(b_j + p_j)$ has a β-wide-(s) subsequence in $X_{\mathcal{U}}$, by Corollary 17. But then for n sufficiently large, $(b_j^n + p_j^n)_{j=1}^n$ has a β'-wide-(s) subsequence of length k, contradicting our assumption. $\square$

References

[AG] T. Alvarez and M. González, *Some examples of tauberian operators*, Proc. Amer. Math. Soc. **111** (1991), 1023–1027.

[GM] M. González and A. Martínez-Abejón, *Supertauberian operators and perturbations*, Arch. Math. **64** (1995), 423–433.

[GO] M. González and V.M. Onieva, *Characterizations of Tauberian operators and other semigroups of operators*, Proc. Amer. Math. Soc. **108** (1990), 399–405.

[H] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. **313** (1980), 72–104.

[Ho] J.R. Holub, *Some problems concerning bases in Banach spaces*, Proc. Amer. Math. Soc. **23** (1969), 521–525.

[HKPTZ] C.W. Henson, N.J. Kalton, N.T. Peck, I. Terescak, and P. Zlatos, *Some Ramsey type theorems for normed and quasinormed spaces*, Studia Math. (to appear).

[J1] R.C. James, *Characterizations of reflexivity*, Studia Math. **23** (1964), 205–216.

[J2] R.C. James, *Weak compactness and reflexivity*, Israel J. Math. **2** (1964), 101–119.

[KP] M.I. Kadec and A. Pełczyński, *Basic sequences, biorthogonal systems and normings sets in Banach and Fréchet spaces*, Studia Math. **25** (1965), 297–323. (Russian)

[MM] D.P. Milman and V.D. Milman, *The geometry of nested families with empty intersection. The structure of the unit sphere of a non-reflexive space*, Matem. Sbornik **66** (1965), no. 108, 109–118 (Russian); Amer. Math. Soc. Translations **85** (1969), no. 2.

[NR] R. Neidinger and H. Rosenthal, *Norm-attainment of linear functionals on subspaces and characterizations of Tauberian operators*, Pac. J. Math. **118** (1985), 215–228.

[R1] H. Rosenthal, *Double dual types and the Maurey characterization of Banach spaces containing ℓ^1*, Longhorn Notes, The University of Texas at Austin Functional Analysis Seminar, 1983–84, pp. 1–37.

[R2] H. Rosenthal, *On non-norm-attaining functionals and the equivalence of the weak*-KMP with the RNP*, Longhorn Notes, The University of Texas at Austin Functional Analysis Seminar, 1985–86, pp. 1–12.

[R3] H.Rosenthal, *A characterization of Banach spaces containing* c_0, J. Amer. Math. Soc. **7** (1994), 707–748.

[S1] I. Singer, *Basic sequences and reflexivity of Banach spaces*, Studia Math. **21** (1962), 351–369.

[S2] I. Singer, *Bases in Banach Spaces I*, Springer-Verlag, New York, Heidelberg, Berlin, 1970.

[T] D.G. Tacon, *Generalized semi-Fredholm transformations*, J. Australian Math. Soc. Ser. A **37** (1983), 60–70.