ON AN INEQUALITY OF A. GROTHENDIECK CONCERNING OPERATORS ON L^1

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ABSTRACT. In 1955, A. Grothendieck proved a basic inequality which shows that any bounded linear operator between $L^1(\mu)$-spaces maps (Lebesgue-) dominated sequences to dominated sequences. An elementary proof of this inequality is obtained via a new decomposition principle for the lattice of measurable functions. An exposition is also given of the M. Lévy extension theorem for operators defined on subspaces of $L^1(\mu)$-spaces.

1. INTRODUCTION

Let μ, ν be measures on measurable spaces, and let $T : L^1(\mu) \to L^1(\nu$ be a bounded linear operator (here $L^1(\mu)$ denotes the real or complex Banach space of (equivalence classes of) μ-integrable functions). In [G], (see Corollaire, page 67) Grothendieck establishes the following fundamental inequality:

$$\left\{ \begin{array}{l} \text{Given } f_1, \ldots, f_n \text{ in } L^1(\mu), \text{ then} \\ \displaystyle\int \max_i |Tf_i| \, d\nu \le \|T\| \int \max_i |f_i| \, d\mu \ . \end{array} \right. \tag{1}$$

We first give some motivation for the inequality, then give a proof involving an apparently new principle concerning the lattice of measurable functions.

It follows easily from (1) that every such operator maps *dominated* (or *order bounded*) sequences into dominated sequences. In fact, it follows that

$$
\left\{
\begin{array}{l}
\text{\textit{if F is a family in $L^1(\mu)$ for which there exists a μ-integrable φ with}} \\
|f| \le \varphi \text{ a.e. \textit{for all f in F, then there exists a non-negative ν-integrable}} \\
\psi \text{ \textit{with} } \displaystyle\int \psi\, d\nu \le \|T\| \int \varphi\, d\mu \text{ \textit{so that} } |Tf| \le \psi \text{ a.e. \textit{for all f in F.}}
\end{array}
\right. \tag{2}
$$

This consequence of (1) (which is of course equivalent to (1)) is drawn explicitly by Grothendieck in [G] (see Proposition 10, page 66).

In the summer of 1979, during her research visit to the University of Texas at Austin, I suggested to Mireille Lévy that the inequality (1) might actually characterize those operators from a subspace of $L^1(\mu)$ to $L^1(\nu)$, which extend to an operator on all of $L^1(\mu)$. She indeed confirmed my conjecture [L]. Combining Lévy's result with (1) and a simple application of the closed graph theorem, we obtain the

Extension Theorem. *Let μ,ν be measures on measurable spaces, X a closed linear subspace of $L^1(\mu)$, and $T : X \to L^1(\nu)$ a bounded linear operator. Then the following assertions are equivalent:*

(a) *T maps dominated sequences to dominated sequences.*

(b) *There is a constant C so that*

$$
\left\{
\begin{array}{l}
\text{\textit{given n and $f_1,\dots,f_n$ in X, then}} \\
\displaystyle\int \max_i |Tf_i|\, d\nu \le C \int \max_i |f_i|\, d\mu\ .
\end{array}
\right. \tag{3}
$$

(c) *There is a bounded linear operator $\tilde{T} : L^1(\mu) \to L^1(\nu)$ with $\tilde{T}|X = T$.*

Moreover if α denotes the smallest C satisfying (3)*, then $\tilde{T}$ may be chosen with $\|\tilde{T}\| = \alpha$.*

A remarkable development of the setting for the Extension Theorem has recently been given in a series of papers by G. Pisier. In [P1], Pisier obtains an extension theorem for operators on H^1 to $L^1(\mu)$ which are also bounded from H^∞ to $L^\infty(\mu)$. In [P2, Theorem 3], he obtains the appropriate generalization of the Extension Theorem for operators from a subspace of $L^p(\mu)$ to $L^1(\nu)$, $1 \le p \le \infty$, and in fact in the more general setting of Banach lattices. Finally, in [P3, Theorem 3.5], Pisier obtains a non-commutative version of the Theorem. In Section 3, we give a proof of the Extension Theorem following the approach in [P1]. This also yields a rather quick alternate "functional-analytical" proof of (1). For a given subspace X of L^1, our exposition yields an explicit representation for elements of $X(L^\infty)$, the closure of $X \otimes L^\infty$ in $L^1(L^\infty)$ (see the Corollary towards the end of Section 3), which also suggests an open question regarding $X(L^\infty)$ (see the second Remark following the Corollary's statement).

We note one last motivating connection. Grothendieck's "L^1-inequality" (1) follows immediately from the classical Banach lattice result that every such operator T has an absolute value, or modulus, $|T|$, which is a linear operator from $L^1(\mu)$ to $L^1(\nu)$ with $\|(T)\| = \|T\|$ and

$$|Tf| \le |T|\,|f| \quad \text{for all } f \in L^1(\mu) \tag{4}$$

(cf. [S]). However the existence of $|T|$ may readily be deduced from (1), which thus certainly appears more basic and elementary.

2. A DECOMPOSITION PRINCIPLE FOR THE LATTICE OF MEASURABLE FUNCTIONS

We first formulate the principle for the case of real scalars.

Lemma 1. *Let $f_1, \dots, f_n$ be real valued measurable functions on a measurable space. There exist k (depending only on n) and non-negative measurable functions $h_1, \dots, h_k$ so that*

(i) $h_1 + \cdots + h_k = |f_1| \vee \cdots \vee |f_n|$;

(ii) *for all i, there exist $\varepsilon_{ij} \in \{0, 1, -1\}$ with $f_i = \sum_{j=1}^{k} \varepsilon_{ij} h_j$.*

Remark. We do not need the fact that the k in Lemma 1 depends only on n. Nevertheless, let $k(n)$ be the optimal choice for k. What is $k(n)$? The order of magnitude of $k(n)$? Shortly after circulating the original version of this paper, V. Mascioni completely solved this problem, proving that one may choose $k(n) = 2^n$, and this is best possible [M]. (Our proof below yields only that $k(n) \le e^{1/2}2^n n!$; also see the remark following Lemma 2.)

We first deduce the Grothendieck inequality for real scalars from Lemma 1. Given T and $f_1,\dots,f_n$ in $L^1(\mu)$, choose $h_1,\dots,h_k$ and the ε_{ij}'s as in the Lemma. Then for each i, we have

$$|Tf_i| = \Big|\sum \varepsilon_{ij} Th_j\Big| \le \sum_j |Th_j| \ . \tag{5}$$

Hence

$$\max_i |Tf_i| \le \sum_j |Th_j| \ . \tag{6}$$

Thus

$$\begin{aligned} \int \max_i |Tf_i|\, d\nu &= \int \sum_j |Th_j|\, d\nu \qquad \text{by (6)} \\ &= \sum_j \int |Th_j|\, d\nu \\ &\le \|T\| \sum_j \int h_j\, d\mu \qquad \text{since } h_j \ge 0 \text{ for all } j \\ &= \|T\| \int \sum_j h_j\, d\mu \\ &= \|T\| \int \max |f_i|\, d\mu \qquad \text{by (i) of the Lemma.} \end{aligned} \tag{7}$$

Proof of Lemma 1.

We prove the result by induction on n. Let $(\Omega, \mathcal{S})$ be the associated measurable space; i.e., $\mathcal{S}$ is a σ-algebra of subsets of Ω, and the f_i's are $\mathcal{S}$-measurable functions defined on Ω. For $n = 1$, let $h_1 = f_1^+$, $h_2 = f_1^-$ (where as usual, e.g., $f_1^+(\omega) = f_1(\omega)$ if $f_1(\omega) \geq 0$; $f_1^+(\omega) = 0$ otherwise). Of course then $|f_1| = h_1 + h_2$, $f_1 = h_1 - h_2$. Now let $n \geq 1$, and suppose the Lemma proved for n. Let $f_1, \dots, f_{n+1}$ be given measurable functions on Ω. Choose disjoint measurable sets $E_1, \dots, E_{n+1}$ so that $\Omega = \bigcup_{i=1}^{n+1} E_i$ and

$$|f_1|(\omega) \vee \cdots \vee |f_{n+1}|(\omega) = |f_i(\omega)| \text{ for all } \omega \in E_i, \text{ all } i. \tag{8}$$

Now fix i and apply the induction hypothesis to $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_{n+1}$ on E_i. We obtain $h_{i1}, \dots, h_{ik} \geq 0$ (k depends only on n) measurable functions so that

$$\sum_{j=1}^{k} h_{ij} = (|f_1| \vee \cdots \vee |f_{i-1}| \vee |f_{i+1}| \vee \cdots \vee |f_{n+1}|)\chi_{E_i} \overset{\text{df}}{=} \tau_i \tag{9}$$

and so that for each $j \neq i$, there are numbers $\varepsilon_{j\ell}^i$ in $\{0, 1, -1\}$ with

$$f_j \chi_{E_i} = \sum_{\ell=1}^{k} \varepsilon_{j\ell}^i h_{i\ell} \ . \tag{10}$$

Let $E_i^+ = \{\omega : f_i(\omega) \geq 0\}$. $E_i^- = \{\omega : f_i(\omega) < 0\}$. We now claim the following family of functions works, for our "h_i's" for $n + 1$:

$$\begin{cases} h_{i\ell}\chi_{E_i^+} \ , \ h_{i\ell}\chi_{E_i^-} \ , \\ (f_i - \tau_i)\chi_{E_i^+} \ , \ (-f_i - \tau_i)\chi_{E_i^-} \\ (1 \leq i \leq n+1 \ , \ 1 \leq \ell \leq k) \ . \end{cases} \tag{11}$$

Evidently if k' denotes the total number of functions listed in (11), then

$$k' = 2(n+1)(k+1) \ . \tag{12}$$

Now, all of these functions are non-negative (the last two types because $|f_i| \geq \tau_i$ on E_i, by (8)). To verify (i) of the Lemma, note that for each i,

$$|f_i|\chi_{E_i} = (f_i - \tau_i)\chi_{E_i^+} + \sum_{\ell=1}^{k} h_{i\ell}\chi_{E_i^+} + (-f_i - \tau_i)\chi_{E_i^-} + \sum_{\ell=1}^{k} h_{i\ell}\chi_{E_i^-} \ . \tag{13}$$

Thus, letting $h_1, \ldots, h_{k'}$ be the functions listed in (11), we have that

$$|f_1| \vee \cdots \vee |f_{n+1}| = \sum_{i=1}^{n+1} |f_i| \chi_{E_i} = \sum_{r=1}^{k'} h_r \ . \tag{14}$$

Finally, to verify (ii), fix j. Then

$$\begin{aligned} f_j \chi_{E_j} &= f_j \chi_{E_j^+} + f_j \chi_{E_j^-} \\ &= (f_j - \tau_j)\chi_{E_j^+} + \tau_j \chi_{E_j^+} - (-f_j - \tau_j)\chi_{E_j^-} - \tau_j \chi_{E_j^-} \\ &= (f_j - \tau_j)\chi_{E_j^+} + \sum_{\ell=1}^{k} h_{j\ell} \chi_{E_j^+} - (-f_j - \tau_j)\chi_{E_j^-} + \sum_{\ell=1}^{k} -h_{j\ell}\chi_{E_j^-} \ . \end{aligned} \tag{15}$$

Thus from (10) and (15), we obtain $\varepsilon_{jr} = 0, 1$, or -1 for all r so that

$$f_j = \sum_{i=1}^{n+1} f_j \chi_{E_i} = \sum_{r=1}^{k'} \varepsilon_{jr} h_r \ . \tag{16}$$

□

We next treat the case of complex scalars.

Lemma 2. *Let $f_1, \ldots, f_n$ be complex valued measurable functions on a measurable space. There exist k (depending only on n) and non-negative measurable functions $h_1, \ldots, h_k$ so that*

(i) $h_1 + \cdots + h_k = |f_1| \vee \cdots \vee |f_n|$.

(ii) *for all i, there exist measurable functions ε_{ij} so that $|\varepsilon_{ij}(\omega)| = 1$ or 0 for all j, with $f_i = \sum_{j=1}^{k} \varepsilon_{ij} h_j$*

Remark. Let $k_{\mathbb{C}}(n)$ denote the optimal choice for k. As in the real scalars case, we again ask what is the order of magnitude of $k_{\mathbb{C}}$? Our argument below yields that $k_{\mathbb{C}}(n) \le en!$. (V. Mascioni has also solved this problem, proving that $k_{\mathbb{C}}(n) = 2^n - 1$ [M].)

The deduction of the Grothendieck L^1-inequality involves the following

Corollary. *Let* $f_1,\dots,f_n$ *be as in Lemma 2, and let* $\varepsilon>0$*. There exist* $h_1,\dots,h_k$ *non-negative measurable functions satisfying (i) of Lemma 1 and*
(ii) for all i *there exist numbers* α_{ij} *with* $|\alpha_{ij}|=1$ *or* 0 *for all* j*, and*

$$\Big|f_i-\sum_{j=1}^{k}\alpha_{ij}h_j\Big|\le\varepsilon(|f_1|\vee\cdots\vee|f_n|)\ . \tag{17}$$

Comment. If the ε_{ij}'s in Lemma 2 can be chosen as simple functions (which is of course the case if the f_i's are simple), then the dependence of the α_{ij}'s on ε may be eliminated; i.e., we then have $f_i=\sum_j\alpha_{ij}h_j$ for all i. Note this is the case if the f_i's are all real-valued; thus Lemma 2 implies Lemma 1.

Proof of the Corollary using Lemma 2. Let the h_i's and ε_{ij}'s satisfy the conclusion of Lemma 2. We may choose disjoint measurable sets $F_1,\dots,F_r$ with $\Omega=\bigcup_{i=1}^r F_i$, so that for every ν, $1\le\nu\le r$, every i, $1\le i\le n$, and all j, $1\le j\le k$, there is a number ε_{ij}^ν, with $|\varepsilon_{ij}^\nu|=1$ or $\varepsilon_{ij}^\nu=0$, so that

$$|\varepsilon_{ij}(\omega)-\varepsilon_{ij}^\nu|\le\varepsilon\text{ for all }\omega\in F_\nu\ . \tag{18}$$

We now claim: *The family of functions*

$$h_i\chi_{F_\nu}\qquad 1\le i\le k\ ,\ 1\le\nu\le r\ ,$$

serves as our "h_ℓ's"; for each i*, the constant* ε_{ij}^ν *serves as our "$\alpha_{i\ell}$."* Indeed, we have that

$$\sum_{i,\nu}h_i\chi_{F_\nu}=|f_1|\vee\cdots\vee|f_n|\ . \tag{19}$$

Finally, fix i,ν. Then

$$\begin{aligned}\Big|f_i\chi_{F_\nu}-\sum\varepsilon_{ij}^\nu h_j\chi_{F_\nu}\Big|&=\Big|\sum_j(\varepsilon_{ij}-\varepsilon_{ij}^\nu)h_j\chi_{F_\nu}\Big|\quad\text{by Lemma 2(ii)}\\&\le\varepsilon\sum h_j\chi_{F_\nu}=\varepsilon(|f_1|\vee\cdots\vee|f_n|)\chi_{F_\nu}\ .\end{aligned}\tag{20}$$

Since the F_ν's are a partition of Ω, the result is proved. $\square$

Proof of Lemma 2. Again we proceed by induction. For any measurable complex valued function f, let

$$(\operatorname{sgn} f)(\omega) = \frac{f(\omega)}{|f(\omega)|} \text{ if } f(\omega) \neq 0 \ , \quad \operatorname{sgn} f(\omega) = 0 \text{ otherwise.}$$

Of course now the $n = 1$ case is "completely" trivial; simply let $h_1 = |f_1|$ and

$$\varepsilon_1(\omega) = \operatorname{sgn} f_1(\omega) \ .$$

Again, suppose Lemma 2 proved for n, and let $f_1, \ldots, f_{n+1}$ be given measurable functions. Choose the measurable partition $E_1, \ldots, E_{n+1}$ satisfying (8), and proceed exactly as in the case of Lemma 1. Thus, we obtain h_{ij}'s, $1 \le j \le k$ satisfying (9) (with τ_i as defined in (9)), so that for each $j \neq i$, there are measurable functions $\varepsilon^i_{j\ell}$ with $|\varepsilon^i_{j\ell}(\omega)| = 0$ or 1 for all ω, satisfying (10). Now we claim that the family of "h_i's" may be taken to be

$$h_{i\ell} \ , \quad (|f_i| - \tau_i)\chi_{E_i} \ , \quad 1 \le i \le n+1 \ , \ 1 \le \ell \le k \ . \tag{21}$$

Thus listing these as $h_1, \ldots, h_{k'}$, we have

$$k' = (k+1)(n+1) \ . \tag{22}$$

Lemma 2(i) now follows immediately, for

$$|f_i|\chi_{E_i} = (|f_i| - \tau_i)\chi_{E_i} + \sum_{\ell=1}^{k} h_{i\ell} \text{ for all } i \ . \tag{23}$$

It remains to verify (ii). Fix j. Then

$$(f_j - (\operatorname{sgn} f_j)\tau_j)\chi_{E_j} = (\operatorname{sgn} f_j)(|f_j| - \tau_j)\chi_{E_j} \ . \tag{24}$$

Thus

$$f_j\chi_{E_j} = (\operatorname{sgn} f_j)(|f_j| - \tau_j\chi_{E_j}) + \sum_{\ell=1}^{k}(\operatorname{sgn} f_j)h_{j\ell} \quad \text{by (9).} \tag{25}$$

Combining (10) and (25), we thus obtain our measurable functions $\varepsilon_{j1},\dots,\varepsilon_{jk'}$ valued in $\mathbb{T}\cup\{0\}$ with

$$\begin{aligned} f_j &= \sum_{i\neq j} f_j\chi_{E_i} + f_j\chi_{E_j} \\ &= (\operatorname{sgn} f_j)(|f_j| - \tau_j\chi_{E_j}) + \sum_{\ell=1}^{k}(\operatorname{sgn} f_j)h_{j\ell} + \sum_{i\neq j}\ \sum_{\ell=1}^{k}\varepsilon_{j\ell}^{i}h_{i\ell} \\ &= \sum_{\ell=1}^{k'}\varepsilon_{j\ell}h_\ell\ . \end{aligned} \tag{26}$$

□

We conclude Section 2 with a deduction of the complex Grothendieck L^1-inequality. Let then μ,ν be measures on measurable spaces, $T : L^1(\mu) \to L^1(\nu)$ be a bounded linear operator, and $f_1,\dots,f_n$ in $L^1(\mu)$ be given. Let $\varepsilon > 0$ be given, and choose $h_1,\dots,h_k$ and the complex numbers α_{ij} as in the conclusion of the Corollary to Lemma 2.

Now for each i, define p_i by

$$p_i = f_i - \sum_{j=1}^{k}\alpha_{ij}h_j\ . \tag{27}$$

Then we have that $f_i = \sum \alpha_{ij}h_j + p_i$ and moreover

$$|p_i| \le \varepsilon(|f_1| \vee \cdots \vee |f_n|) \quad \text{by (ii) of the Corollary.} \tag{28}$$

Thus

$$\begin{aligned} |Tf_i| &= \Big|\sum_j \alpha_{ij} Th_j + Tp_i\Big| \\ &\le \sum_j |Th_j| + |Tp_i| \;\text{ since } |\alpha_{ij}| \le 1 \text{ for all } j \\ &\le \sum_j |Th_j| + \sum_j |Tp_j| \ . \end{aligned} \tag{29}$$

Thus also

$$\max_i |Tf_i| \le \sum_j (|Th_j| + |Tp_j|) \ , \tag{30}$$

whence

$$\begin{aligned} \int \max_i |Tf_i|\, d\nu &\le \sum_j \int (|Th_j| + |Tp_j|)\, d\nu \\ &\le \|T\| \Big(\int \sum_j h_j\, d\mu + \int \sum |p_j|\, d\mu \Big) \\ &\le (1+n\varepsilon)\|T\| \int \max_i |f_i|\, d\mu \text{ by (i) of the Corollary and (28).} \end{aligned} \tag{31}$$

Since $\varepsilon > 0$ is arbitrary, the inequality (1) is proved. □

3. A proof of the Extension Theorem

As noted in the introduction, we follow the approach in [P1], thus obtaining an alternate proof of the Grothendieck L^1-inequality. (The approach, despite its brevity, seems considerably more sophisticated than the elementary proof given by our decomposition result, however.) Throughout, let μ, ν and T be as in the statement of the Extension Theorem. We shall also assume that ν is "nice enough" so that $(L^1(\nu))^* = L^\infty(\nu)$ (any $L^1(\nu)$ is isometric to $L^1(\nu')$ with ν' nice).

(a) ⇒ (b) For Y a subspace of $L^1(\mu)$ or $L^1(\nu)$, let Y_d denote the space of all dominated sequences (y_n) in Y, under the norm $\|(y_n)\|_d = \int \sup_n |y_n|\, d\mu$. We easily check that Y_d is a Banach space; evidently then T induces a linear operator S from Y_d to $(L^1(\nu))_d$, which has closed graph, since T itself is bounded. Thus S is bounded.

(c) ⇒ (a) follows immediately from Grothendieck's L^1-inequality (1). We give here an alternate proof of (1), using the set up in [P1]. We will freely use here some standard facts about $Y\widehat{\otimes}Z$, the projective tensor product of Banach spaces Y and Z. Let $L^1(\mu,Y)$ denote the space of Bochner-integrable Y-valued functions on Ω (where $(\Omega,\mathcal{S},\mu)$ is the measure space associated to μ). *Then* $L^1(\mu,Y)$ *is* (*canonically isometric to*) $L^1(\mu)\widehat{\otimes}Y$ (see Théorème 2, page 59 of [G]). It follows immediately that $T\otimes I$ yields a linear operator from $L^1(\mu)\otimes Y$ to $L^1(\nu)\otimes Y$ with $\|T\otimes I\| = \|T\|$. (Here, we assume "X" $= L^1(\mu)$; i.e., the hypotheses of (1).) We apply this fact to $Y = L^\infty(\nu)$. It follows that for any n, $f_1,\dots,f_n$ in $L^1(\mu)$, and $\varphi_1,\dots,\varphi_n$ in $L^\infty(\nu)$.

$$\Big\|\sum_{i=1}^n Tf_i\otimes\varphi_i\Big\| \le \|T\|\,\Big\|\sum_{i=1}^n f_i\otimes\varphi_i\Big\| \ . \tag{32}$$

Here, $g \overset{\text{df}}{=} \sum f_i\otimes\varphi_i$ denotes the element of $L^1(\mu,L^\infty(\nu))$ defined by $g(\omega) = \sum f_i(w)\varphi_i$, $\omega\in\Omega$; note that

$$\|g\| = \int \|g(\omega)\|\,d\mu(\omega) = \int \operatorname*{ess\,sup}_s \Big|\sum f_i(\omega)\varphi_i(s)\Big|\,d\mu(\omega) \ . \tag{33}$$

Now fixing $f_1,\dots,f_n$ and $\varphi_1,\dots,\varphi_n$ as above, we have

$$\begin{aligned}
&\Big|\int \sum (Tf_j)(s)\varphi_j(s)\,d\nu(s)\Big| \\
&\qquad \le \int \operatorname*{ess\,sup}_t \Big|\sum_{j=1}^n (Tf_j)(s)\varphi_j(t)\Big|\,d\nu(s) \\
&\qquad \le \|T\| \int \operatorname*{ess\,sup}_t \Big|\sum f_j(\omega)\varphi_j(t)\Big|\,d\mu(\omega) \qquad \text{by (32) and (33)} \\
&\qquad \le \|T\| \Big(\int \max_j |f_j(\omega)|\,d\mu(\omega)\Big) \Big\|\sum |\varphi_j|\,\Big\|_{L^\infty(\nu)} \ .
\end{aligned} \tag{34}$$

Now since ν is nice, a standard argument yields that we may choose $\varphi_1,\dots,\varphi_n$ in $L^\infty(\nu)$ with $\|\sum|\varphi_j|\,\|_{L^\infty(\nu)} = 1$ and

$$\int \max |Tf_j|(s)\,d\nu(s) = \int \sum (Tf_j)(s)\varphi_j(s)\,d\nu(s) \ . \tag{35}$$

Evidently (34) and (35) immediately yield Grothendieck's inequality (1).

It remains to prove (c) $\Rightarrow$ (d) and the "moreover" statement, i.e., M. Lévy's theorem. We closely follow the brief sketch given by Pisier in [P1], crystallizing some elements of the discussion. It is convenient to introduce one more condition in the Extension Theorem, which is explicitly used in [P1].

(d) *There is a constant C so that for any n, $f_1,\ldots,f_n$ in X, and simple $\varphi_1,\ldots,\varphi_n$ in $L^\infty(\nu)$,*

$$\Big|\sum_i \int (Tf_i)\varphi_i\,d\nu\Big| \le C\int \operatorname*{ess\,sup}_s \Big|\sum f_i(\omega)\varphi_i(s)\Big|\,d\mu(\omega)\ . \tag{36}$$

We first prove (d) $\Rightarrow$ (c). Consider the following general problem: Given Banach spaces Y, B, X a closed linear subspace of Y, $T : X \to B^*$ a bounded linear operator, and $C > 0$, when does there exist $\tilde T : Y \to B^*$ extending T, with $\|\tilde T\| \le C$? Is there a way of formulating this problem in terms of the Hahn-Banach Theorem? As e.g., developed in [G], $\mathcal{L}(Y,B^*)$ is indeed, naturally isometric to $(Y\hat\otimes B)^*$. The pairing is as follows: given $T : Y \to B^*$ a bounded linear operator and $\omega \overset{\text{df}}{=} \sum y_i\otimes b_i$ in $(Y\hat\otimes B)^*$ (with $\sum\|y_i\|\,\|b_i\| < \infty$), set

$$\langle T,\omega\rangle = \sum_i \langle Ty_i, b_i\rangle\ . \tag{37}$$

We then obtain the following result:

Lemma 3. *Given Y, B, X, and T as above, the following are equivalent:*

(i) *There is a linear operator $\tilde T : Y \to B^*$ extending T, with $\|\tilde T\| \le C$.*

(ii) *Let X_0, B_0 be dense linear subspaces of X and B respectively and regard $X_0\otimes B_0$ as a linear subspace of $Y\hat\otimes B$. Define F_T on $X_0\otimes B_0$ by $F_T(\omega) = \langle T,\omega\rangle$ for all ω in $X\otimes B$. Then*

$$\|F_T\| \le C\ . \tag{38}$$

To see this, note that (i) ⇒ (ii) is immediate. If (ii) holds, let $\tilde{F}_T$ be a Hahn-Banach extension of F_T to $Y\hat{\otimes}B^*$. Now simply let $\tilde{T}$ be the unique element of $\mathcal{L}(X, B^*)$ satisfying

$$\langle \tilde{T}, \omega\rangle = \tilde{F}_T(\omega) \text{ for some } \omega \in Y\hat{\otimes}B^* \ . \tag{39}$$

To obtain (d) ⇒ (c) of the Extension Theorem, let $X = X_0$, $Y = L^1(\mu)$, $B = L^\infty(\nu)$, and B_0 the subspace of B consisting of simple functions. Now condition (d) simply means that $\|F_T\| \le C$, where F_T is as in Lemma 3(ii). Thus by Lemma 3, we obtain a linear operator $\tilde{T} : L^1(\mu) \to L^1(\nu)^{**}$ extending T (where of course we regard $L^1(\nu) \subset L^1(\nu)^{**}$). The proof is completed by observing that there exists a norm-one linear projection P from $L^1(\nu)^{**}$ onto $L^1(\nu)$; then $P \circ \tilde{T}$ yields the desired operator extending T.

It remains to show that (b) ⇒ (d).

The argument for this implication involves a critical identification, due to M. Lévy [L], and appears to have been omitted from the sketch given in [P1].

Lemma 4. *Let B_0 denote the subspace of $L^\infty(\nu)$ consisting of simple functions, and let $g \in X \otimes B_0$. Then*

$$\|g\| = \min\left\{\int \max_j |f_j|\, d\mu \Big\| \sum_i |\varphi_i|\Big\|_\infty\right\} \tag{40}$$

the minimum taken over all n, $f_1, \dots, f_n$ in X, and $\varphi_1, \dots, \varphi_n$ in B_0 so that $g = \sum f_j \otimes \varphi_j$ (where $\|g\|$ is defined as in (33)).

Proof of Lemma 4. Suppose first $g = \sum f_j \otimes \varphi_j$ where $f_1, \dots, f_n$ are in $L^1(\mu)$, $\varphi_1, \dots, \varphi_n$ are in $L^\infty(\nu)$ (we do not need to assume here that the f_i's belong to X). We then have that for any ω and any s,

$$\Big|\sum_{j=1}^n f_j(\omega)\varphi_j(s)\Big| \le \max_j |f_j(\omega)| \sum_j |\varphi_j|(s) \ . \tag{41}$$

It follows immediately that

$$\|g\| \le \int \max |f_j(\omega)|\, d\mu(\omega) \Big\| \sum |\varphi_j| \Big\|_\infty \ . \tag{42}$$

Thus

$$\|g\| \le \inf \bigg\{ \int \max_j |f_j|\, d\mu \Big\| \sum |\varphi_j| \Big\|_\infty : g = \sum_{j=1}^n f_j \otimes \varphi_j \\ \text{with } f_j \in L^1(\mu) \text{ and } \varphi_j \in B_0 \text{ for all } j \bigg\} \ . \tag{43}$$

Now $g = \sum_{i=1}^\ell x_i \otimes \psi_i$ with the x_i's in X and the ψ_i's in B_0. We may then choose a ν-measurable partition $E_1, \ldots, E_m$ of S so that the ψ_i's are all $\mathcal{A}$-measurable, where $\mathcal{A}$ is the algebra generated by the disjoint sets $E_1, \ldots, E_n$. (Here, we assume $L^1(\nu) = L^1(S, \mathcal{E}, \nu)$.) It *then* follows that we may choose $z_1, \ldots, z_m$ in X with

$$g = \sum_{i=1}^m z_i \otimes \chi_{E_i} \ . \tag{44}$$

But then if $\omega \in \Omega$ and $s \in E_i$,

$$|g(\omega)(s)| = |z_i(\omega)| \ . \tag{45}$$

This shows

$$\|g(\omega)\|_{L^\infty(\mu)} = \max_i |z_i(\omega)| \ .$$

Hence

$$\begin{aligned} \|g\| &= \int \max_i |z_i|\, d\mu \\ &= \int \max_i |z_i|\, d\mu \Big\| \sum |\chi_{E_j}| \Big\|_\infty \ , \end{aligned} \tag{46}$$

proving (40). □

We finally show that (b) $\Rightarrow$ (d), thus completing the proof of the Extension Theorem. (The moreover assertion follows from the proof that (d) $\Rightarrow$ (c), for of course we show the same constant C in (b) works for (d).)

Let then $f_1, \dots, f_n$ be given in X, $\varphi_1, \dots, \varphi_n$ be simple elements of $L^\infty(\nu)$, and let C be as in (b).

By Lemma 4, we may choose $x_1, \dots, x_m$ in X and $\psi_1, \dots, \psi_m$ simple in $L^\infty(\nu)$ so that letting $g = \sum f_i \otimes \varphi_i$, then

$$g = \sum x_i \otimes \psi_i \tag{46)(i}$$

$$\|g\| = \int \max_i |x_i| \, d\mu \Big\| \sum |\psi_j| \Big\|_\infty . \tag{46)(ii}$$

Now

$$\begin{aligned} \Big| \sum_i \int (Tf_i)\varphi_i \, d\mu \Big| &= \Big| \sum_i \langle Tf_i, \varphi_i \rangle \Big| = \Big| \sum_i \langle Tx_i, \psi_i \rangle \Big| \text{ (by (46)(i))} \\ &\le \int \sum_i |Tx_i(s)\psi_i(s)| \, d\nu(s) \\ &\le \int \max_i |Tx_i|(s) \Big\| \sum_j |\psi_j| \Big\|_\infty d\nu(s) \\ &\le C \int \max |x_i(\omega)| \, d\mu(\omega) \Big\| \sum |\psi_j| \Big\|_\infty \text{ (by (b))} \\ &= C\|g\| \\ &= C \int \operatorname*{ess\,sup}_s \Big| \sum f_i(\omega)\varphi_i(s) \Big| \, d\mu(\omega) . \end{aligned} \tag{47}$$

This completes the proof of the Extension Theorem. □

The following representation result follows from the above proof of M. Lévy's theorem, and seems to be what's "really going on" (see also Lemma 1 of [L]).

Corollary. *Let X be a closed linear subspace of $L^1(\mu)$, and let $X(L^\infty(\nu))$ denote the closure of $X \otimes L^\infty(\nu)$ in $L^1(\mu, L^\infty(\nu))$. Then given $g \in X(L^\infty(\nu))$ and $\varepsilon > 0$, there exists a dominated sequence (x_j) in X and a sequence (φ_j) in $L^\infty(\nu)$ so that*

(i) *$\sum \varphi_j$ converges unconditionally in $L^\infty(\nu)$.*

(ii) $g = \sum x_j \otimes \varphi_j$.

(iii) $\int \sup_j |x_j(\omega)|\,d\mu(\omega) \big\| \sum_i |\varphi_i| \big\|_{L^\infty(\nu)} \le \|g\| + \varepsilon$.

Remarks. 1. If (x_j) in X is dominated and $\sum \varphi_j$ in $L^\infty(\nu)$ converges unconditionally, then $\sum x_j \otimes \varphi_j$ converges unconditionally in $L^1(\mu, L^\infty(\nu))$, to an element of $X(L^\infty(\nu))$. Indeed, for any choice of scalars (α_j) with $|\alpha_j| \le 1$ for all j and any $k \le \ell$, we have that

$$\Big\| \sum_{j=k}^{\ell} \alpha_j x_j \otimes \varphi_j \Big\| \le \int \max_j |x_j(\omega)|\,d\mu(\omega) \Big\| \sum_{j=k}^{\ell} |\varphi_j| \Big\|_\infty . \tag{48}$$

But $\sum \varphi_j$ converges unconditionally iff

$$\Big\| \sum_k^{\ell} |\varphi_j| \Big\|_\infty \to 0 \ \text{ as } \ k \to \infty \text{ with } \ell \ge k .$$

Hence $\sum \alpha_j x_j \otimes \varphi_j$ converges by (48).

2. Suppose (x_j) in X and (φ_j) in $L^\infty(\nu)$ satisfy

$$\int \sup_j |x_j|\,d\mu \Big\| \sum_i |\varphi_i| \Big\|_\infty \overset{\text{df}}{=} \tau < \infty .$$

(Equivalently, (x_j) is dominated and $\sum \varphi_j$ is weakly unconditionally summing in $L^\infty(\nu)$.) It then follows that for μ-almost all ω, $\sup_j |x_j(\omega)| < \infty$; for each such ω, we obtain that $\sum x_j(\omega)\varphi_j$ converges absolutely pointwise a.e. to an element of $L^\infty(\nu)$, and the function $g(\omega) \overset{\text{df}}{=} \sum x_j(\omega)\varphi_j$ belongs to $L^1(\mu, L^\infty(\nu))$ with $\|g\| \le \tau$. Does it then follow that g belongs to $X(L^\infty(\nu))$? This is indeed so provided X is isomorphic to a separable dual space, or more generally, a dual space with the Radon-Nikodym property.

Proof of the Corollary. Letting B_0 denote the space of the simple $L^\infty(\nu)$ functions as above, we have that $L^1(\mu) \otimes B_0$ is dense in $L^1(\mu) \hat{\otimes} L^\infty(\nu)$ since B_0 is dense in $L^\infty(\nu)$. Hence given $\varepsilon > 0$, we may choose a sequence (g_j) in $L^1(\mu) \otimes B_0$ with

$$\Big(\sum \|g_j\|^{1/2} \Big)^2 < \|g\| + \varepsilon \ \text{ and } \ g = \sum g_j . \tag{49}$$

Now by Lemma 4, for each i, we may choose finite sequences $(x_{ij})_{j=1}^{m_i}$ in X and $(\varphi_{ij})_{j=1}^{m_i}$ in B_0 with $g_i = \sum_j x_{ij} \otimes \varphi_{ij}$ and

$$\int \max_j |x_{ij}|\, d\mu(\omega) = \|g_i\|^{1/2} = \Big\| \sum_j |\varphi_{ij}| \Big\|_\infty . \tag{50}$$

Hence the series $\sum_i \sum_{j=1}^{m_i} x_{ij} \otimes \varphi_{ij}$ converges unconditionally to g. Now we have moreover that

$$\begin{aligned} \int \sup_i \max_j |x_{ij}|\, d\mu(\omega) &\le \sum_i \int \max_j |x_{ij}|\, d\mu(\omega) \\ &\le \sum |g_j|^{1/2} \text{ by (50).} \end{aligned} \tag{51}$$

Thus the sequence (x_{ij}) with $1 \le j \le m_i$, $i = 1, 2, \ldots$ is indeed dominated. Also $\sum_i \sum_{j=1}^{m_i} \varphi_{ij}$ converges unconditionally in $L^\infty(\nu)$ and

$$\Big\| \sum_i \sum_j |\varphi_{ij}| \Big\|_\infty \le \sum_i \Big\| \sum_j |\varphi_{ij}| \Big\|_\infty \le \sum |g_j|^{1/2} \text{ by (50).} \tag{52}$$

The Corollary now follows immediately from (49)–(52). ☐

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