

ON DIFFERENCES OF SEMI-CONTINUOUS FUNCTIONS

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ABSTRACT. Extrinsic and intrinsic characterizations are given for the class $DSC(K)$ of differences of semi-continuous functions on a Polish space K , and also decomposition characterizations of $DSC(K)$ and the class $PS(K)$ of pointwise stabilizing functions on K are obtained in terms of behavior restricted to ambiguous sets. The main, extrinsic characterization is given in terms of behavior restricted to some subsets of second category in any closed subset of K . The concept of a strong continuity point is introduced, using the transfinite oscillations $\text{osc}_\alpha f$ of a function f previously defined by the second named author. The main intrinsic characterization yields the following DSC analogue of Baire's characterization of first Baire class functions: a function belongs to $DSC(K)$ iff its restriction to any closed non-empty set L has a strong continuity point. The characterizations yield as a corollary that a locally uniformly converging series $\sum \varphi_j$ of DSC functions on K converges to a DSC function provided $\sum \text{osc}_\alpha \varphi_j$ converges locally uniformly for all countable ordinals α .

1. INTRODUCTION

The interest in functions of the first Baire class can be traced back to Baire's paper [Ba] in 1899. In the early twenties S. Mazurkiewicz and W. Sierpiński [Mazk], [Sie] were already studying the subclass of functions that are the difference of two semi-continuous functions. From a Banach space theory point of view, much of the interest originates from the ℓ^1 -theorem [R1]. Subsequently, the goal was to deduce properties of a given separable Banach space X from the topological class of the functions $f \in X^{**}$. The fundamental paper [HOR] follows this program. For example, it was proved in [HOR] that if K is a compact metric space and $F \in DSC(K) \sim \mathcal{C}(K)$, then whenever $(f_n) \subset \mathcal{C}(K)$ is a uniformly bounded sequence converging pointwise to F , we have that c_0 embeds into $[(f_n)]$, the closed linear span of (f_n) . These efforts culminate in the later result [R2], characterizing Banach spaces containing c_0 . For further structural results on various subspaces of first Baire class functions, see also [KL], [C], [CMR], [R3], [R4].

We concentrate here on the intrinsic nature of two subclasses of Baire-1 functions: $PS(K)$, the class of pointwise stabilizing functions, and $DSC(K)$, the class of differences of semi-continuous functions on a Polish space K . The definitions of these classes will be given in the next section. Note here that the assumption that K is a Polish space, is a special case of the well-behaved spaces of W. Kotze [K]. It would be interesting to see if the localization theorems presented here still hold in the more general framework of well-behaved spaces. We would like to recall here the Baire characterization theorem for a real-valued function f defined on K

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to be of the first Baire class in terms of its restriction to any closed subset F of K having a point of continuity relative to F . It is in this spirit that we characterize functions of the class $\text{PS}(K)$ in terms of their behavior when restricted to ambiguous sets, i.e., sets that are simultaneously F_σ and G_σ . More precisely, we prove in Theorem 3.1 that for a Baire-1 function F on a Polish space K to be pointwise stabilizing, it suffices that each $F|_{V^i}$ is continuous, where $(V^i)_{i=1}^\infty$ is a sequence of ambiguous sets such that $K = \bigcup_{i=1}^\infty V^i$. Note here that the only if part was already proved in [HOR]. As corollaries of this, we obtain that a Baire-1 function on a Polish space with a discrete range is pointwise stabilizing, and moreover such functions are uniformly dense in the first Baire class. However, there exists a Baire-1 function on $[0, 1]$ whose range is a convergent sequence, and which is not in $\text{PS}([0, 1])$. In section 4, we prove in Theorem 4.4 that for a bounded real-valued function f on a Polish space K to be a difference of semi-continuous function on K , it suffices that $f|_{A^j}$ is in $\text{DSC}(A^j)$ for all $j \geq 1$, where $(A^j)_{j=1}^\infty$ is a sequence of ambiguous sets such that $K = \bigcup_{j=1}^\infty A^j$. We then use this to give a characterization of DSC functions on a Polish space K in terms of their behavior when restricted to some subset of second category in closed subsets L of K . More precisely, we prove in Theorem 4.5 that f is in $\text{DSC}(K)$ if and only if for any closed subset L of K there exists a subset E of L of second category in L so that $f|_{\overline{E}}$ is in $\text{DSC}(E)$, where \overline{E} is the closure of E . An advantage of such a characterization is that if f is not in $\text{DSC}(\Delta)$ where Δ is the triadic Cantor set, then one can choose a closed perfect subset K of Δ so that: if $f(t) = \sum_{n=1}^\infty f_n(t)$ on K , with f_n continuous on K then $\sum_{n=1}^\infty |f_n(t)| < \infty$ only for t in a meager subset of K . These results deal with what we may call “localization”: the theorems say that a function $f : K \rightarrow \mathbb{R}$ is in a certain class, provided its restriction to certain smaller subsets of K are all in that class. They are also “extrinsic”; that is, one needs to know at some level, that the restrictions of the function in question are given as differences of semi-continuous functions.

The results in Section 5 yield intrinsic characterizations, based on the invariants introduced in [R2]. Note that for a given function f on a Polish space K , $x \in K$ is a point of continuity of f iff $\text{osc } f(x) = 0$, where $\text{osc } f$ is the oscillation of f . Letting $\text{osc}_\alpha f$ be the α^{th} transfinite oscillation introduced in [R2] (for α a countable ordinal), we define $x \in K$ to be a *strong continuity point of f* provided $\text{osc}_\alpha f(x) = 0$ for all α . Theorem 5.1 then yields the analogue for DSC functions of Baire’s famous characterization of Baire class one: *f is DSC iff $f|_L$ has a strong continuity point for every non-empty closed subset L of K* . The proof of 5.1 also yields an ordinal index for the “local DBSC complexity” of a DSC function f , as well as an effective intrinsic criterion for determining whether or not a given function is DSC. Theorem 5.2 yields an answer to the following problem: *if (f_n) is a sequence of DSC functions converging uniformly to f , what additional properties of the convergence will force f to also be DSC?* As noted above, *any* Baire -1 function can be so obtained, without such conditions. Theorem 5.2 yields the desired conclusion in terms of the existence of points of local rapidity of convergence of $(\text{osc}_\alpha f_n|_L)$ for arbitrary non-empty closed subsets L of K . This yields the corollary: *if $\varphi_1, \varphi_2, \dots$ are DSC functions with $\sum \varphi_j$ converging locally uniformly to f , then f is DSC provided $\sum \text{osc}_\alpha \varphi_j$ converges locally uniformly for all ordinals α* .

2. NOTATION

We use mostly the notation found in [HOR] or [R2], except that we do not restrict ourselves to bounded functions; we work only in the field of real scalars \mathbb{R} , although all our results easily extend to the complex scalars (where DSC functions are just those whose real and imaginary parts are DSC.) Let K be a Polish space. $\mathcal{C}(K)$ (resp. $C^b(K)$) denotes the class of continuous (resp. bounded continuous) scalar valued functions on K . $\mathcal{B}_1(K)$ denotes the class of scalar functions on K of the first Baire class and $\mathcal{B}_1^b(K)$ the class of bounded functions in $\mathcal{B}_1(K)$. That is,

$$\mathcal{B}_1(K) = \{f : K \rightarrow \mathbb{R} : \text{there exist } (f_n) \text{ in } \mathcal{C}(K) \text{ with } f_n \rightarrow f \text{ pointwise}\} .$$

It is easily seen that if $f \in \mathcal{B}_1^b(K)$, then there exists a uniformly bounded sequence (f_n) in $C^b(K)$ with $f_n \rightarrow f$ pointwise.

Recall that f is lower (resp. upper) semi-continuous if for any λ the set $\{x : f(x) > \lambda\}$ is open (resp. $\{x : f(x) \geq \lambda\}$ is closed). f is called semi-continuous if it is lower or upper semi-continuous.

This article deals with the following subclass of $\mathcal{B}_1(K)$:

$$\text{DSC}(K) = \{f : K \rightarrow \mathbb{R} : \text{there exists a sequence } (f_n) \subset \mathcal{C}(K) \text{ so that } f_n \rightarrow f \text{ pointwise and } \sum_{n=1}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty \text{ for } k \in K.\}$$

We let $\text{BDSC}(K)$ denote the bounded members of $\text{DSC}(K)$. Again, it is easily seen that such functions may be obtained as the pointwise limit of uniformly bounded sequences (f_n) in $C^b(K)$ satisfying $\sum |f_{n+1}(k) - f_n(k)| < \infty$ for all k .

By a classical theorem of Baire [H, p.274], $\text{DSC}(K)$ equals the set of all functions that are differences of semi-continuous functions on K .

Another subclass is the one of Baire-1 functions on K that are the difference of two bounded semi-continuous functions on K , and will be denoted by $\text{DBSC}(K)$. As in [HOR] we will adopt the following equivalent definition:

$$\text{DBSC}(K) := \{f : K \rightarrow \mathbb{R} : \text{there exist } (\phi_n) \text{ in } \mathcal{C}(K) \text{ with } f(k) = \sum_{n=1}^{\infty} \phi_n(k) \text{ and } \sum_{n=1}^{\infty} |\phi_n(k)| \leq C \text{ for all } k \in K \text{ and for some constant } C\}.$$

For $f \in \text{DBSC}(K)$ we then set

$$\|f\|_D = \inf \left\{ \sup_{t \in K} \sum_{n=0}^{\infty} |\phi_n(t)| : \phi_n \text{ is bounded continuous on } K \text{ for all } n \text{ and } f(t) = \sum_{n=0}^{\infty} \phi_n(t), t \in K \right\} .$$

As noted in [HOR], $\text{DBSC}(K)$ is then a Banach algebra under this norm.

Remarks.

1. It is proved in [R2] that for any DBSC function f on a Polish space, there exists a countable ordinal τ so that $f = u - v$, where

$$u = \frac{\| |f| + \text{osc}_\tau f \|_\infty - \text{osc}_\tau f + f}{2}, \quad \text{and} \quad v = \frac{\| |f| + \text{osc}_\tau f \|_\infty - \text{osc}_\tau f - f}{2},$$

u and v being non-negative lower semi-continuous functions; moreover, the D -norm of f is exactly given by $\|f\|_D = \|u + v\|_\infty$. the transfinite oscillation $\text{osc}_\alpha f$ of f is defined in Section 5 below.

2. The class $\text{DBSC}(K)$ is in general distinct from the class of Baire-1 functions. Mazurkiewicz [Mazk] gave a construction of a function that is in $\mathcal{B}_1(K) \sim \text{DBSC}(K)$ whenever K contains a homeomorphic copy of the countable compact ordinal $\omega^\omega + 1$. In particular, whenever K is uncountable, the inclusion of $\text{DBSC}(K)$ in $\mathcal{B}_1(K)$ is strict. Actually, this result follows via functional analytic reasoning. Indeed, fix K Polish and assume that K contains a copy of the (non-compact) ordinal ω^ω . Then the uniform and DBSC norms are *inequivalent* on $\text{DBSC}(K)$, hence there exists an f in the *uniform closure* of $\text{DBSC}(K)$, not belonging to $\text{DBSC}(K)$, (see [HOR], [R3]). Moreover the uniform closure of $\text{DBSC}(K)$, identified as $\mathcal{B}_{1/2}(K)$ in [HOR], is also then strictly contained in $\mathcal{B}_1^b(K)$, and also $\text{BDSC}(K)$ is then strictly larger than $\text{DBSC}(K)$. Moreover, if K is countable, $\text{BDSC}(K) = \mathcal{B}_1(K) = \ell^\infty(K)$ (see Corollary 3.3 below).

A subclass of $\text{DSC}(K)$ that we also treat here is the class of pointwise stabilizing functions on K , denoted $\text{PS}(K)$.

Definition. $\text{PS}(K)$ (resp. $\text{BPS}(K)$) is the class of all functions $f : K \rightarrow \mathbb{R}$ for which there exists a sequence (resp. uniformly bounded sequence) of functions $(f_n) \subset C(K)$ so that for any $k \in K$, there is an integer m with $f_n(k) = f(k)$ for all $n \geq m$.

Again, a simple truncation argument yields that $\text{BPS}(K)$ is precisely the class of bounded functions in $\text{PS}(K)$.

3. A CHARACTERIZATION OF $\text{PS}(K)$

Theorem 3.1. Suppose that $F : K \rightarrow \mathbb{R}$ is a given function on a Polish space K , and that $K = \bigcup_{i=1}^{\infty} W^i$ where each W^i is an ambiguous set. If $F|_{W^i}$ is continuous for all $i \in \mathbb{N}$, then F is pointwise stabilizing.

Proof. By considering $\widetilde{W}^j = W^j \sim \bigcup_{i=1}^{j-1} W^i$ and $\widetilde{W}^1 = W^1$ we can suppose without loss of generality that the W^i are disjoint. Let

$$W^j = \bigcup_{n=1}^{\infty} G_n^j,$$

where for all n , G_n^j is closed and $G_n^j \subset G_{n+1}^j$. Let $f_n : K \rightarrow \mathbb{R}$ be a continuous extension of $F|_{\bigcup_{j=1}^n G_n^j}$. These f_n 's exist by the Tietze extension theorem. Thus we have.

$$(1) \quad f_n|_{G_n^j} = F|_{G_n^j} \text{ for all } 1 \leq j \leq n.$$

Claim. $f_n \rightarrow F$ pointwise as $n \rightarrow \infty$, and the f_n 's stabilize.

Indeed, if $t \in K$ then $t \in W^j$ for some j . Fix such a j and let $n_0 \geq j$ large enough so that $t \in G_{n_0}^j$. Now for all $n \geq n_0$ we have:

$$f_n(t) = f_n|_{W^j}(t) = F|_{G_n^j}(t) = F(t)$$

since $t \in G_{n_0}^j$ and (1) holds for all $1 \leq j \leq n_0$. \square

We note that the result also holds if we assume that the W^j are just F_σ 's. Indeed, for each i , choose A_{ij} closed with $W^i = \bigcup_{j=1}^{\infty} A_{ij}$. But evidently $\bigcup_{i,j} A_{ij} = K$ and

$f|_{A_{ij}}$ is continuous for all i and j . Of course, the family $\{A_{ij} : i, j = 1, 2, \dots\}$ is countable and each A_{ij} is ambiguous, being closed: hence Theorem 3.1 applies. We also note that, since the ambiguous sets form an algebra of sets, the differences of closed sets are also ambiguous. We then obtain the following characterization of $\text{PS}(K)$ for Polish spaces K .

Theorem 3.2. *Let K be a Polish space and $F \in \mathcal{B}_1(K)$. Then the following are equivalent:*

1. $F \in \text{PS}(K)$.
2. *There exists a (disjoint) sequence $(V^i)_{i=1}^{\infty}$ of differences of closed subsets of K so that $\bigcup_{i=1}^{\infty} V^i = K$ and $F|_{V^i}$ is continuous for all i .*
3. *There exists a (disjoint) sequence $(V^i)_{i=1}^{\infty}$ of ambiguous subsets of K so that $\bigcup_{i=1}^{\infty} V^i = K$ and $F|_{V^i}$ is continuous for all i .*

Proof. To see that 1) \Rightarrow 2), recall that from Proposition 4.9 of [HOR], if $F \in \text{PS}(K)$ then there exist a sequence $(K_n)_{n=1}^{\infty}$ of closed sets of K so that $K_n \subset K_{n+1}$, $K = \bigcup_{n=1}^{\infty} K_n$ and $F|_{K_n}$ is continuous for all n . (The argument in [HOR] does not require that F be bounded.) It suffices then to set $V^i = K_i \sim K_{i-1}$.

2) \Rightarrow 3) follows immediately from the comments preceding the statement of Theorem 3.2. 3) \Rightarrow 1) follows immediately from the previous theorem. \square

Corollary 3.3. *Let K be a countable Polish space. Then every function on K belongs to $\text{PS}(K)$ and hence to $\text{DSC}(K)$.*

Proof. Let $F : K \rightarrow \mathbb{R}$ be given and let k_1, k_2, \dots be an enumeration of K . Of course $\{k_i\}$ is ambiguous and $F|_{\{k_i\}}$ is continuous for all i , so $F \in \text{PS}(K)$ by Theorem 3.2 and the obvious fact that $F \in \mathcal{B}_1(K)$ (e.g., $F = \sum_n F(k_n)\chi_{k_n}$ pointwise). \square

Corollary 3.4. *Let K be a Polish space and $F \in \mathcal{B}_1(K)$, so that $F(K)$ is a discrete set. Then $F \in \text{PS}(K)$.*

Proof. The hypothesis means that no point of $F(K)$ is a cluster point of $F(K)$. It follows easily that $F(K)$ is countable, say $F(K) = \{c_1, c_2, \dots\}$. Now a standard result [H] asserts that for $F \in \mathcal{B}_1(K)$, if $A \subset \mathbb{R}$ is open, then $F^{-1}(A)$ is an F_{σ} -set. For each $i \geq 1$, choose $\varepsilon_i > 0$ so that $(c_i - \varepsilon_i, c_i + \varepsilon) \cap F(K) = \{c_i\}$. Thus $A_i := F^{-1}\{c_i\} = F^{-1}(c_i - \varepsilon_i, c_i + \varepsilon_i)$ is an F_{σ} . Since the A_i 's are disjoint and partition K , it follows that the A_i 's are ambiguous; on the other hand $F|_{A_i}$ is trivially continuous for all i , hence $F \in \text{PS}(K)$ by Theorem 3.1. \square

Remark. In general, no weaker topological assumption on the range of F is possible, to ensure the validity of Corollary 3.4, when K is uncountable. Indeed, we may easily, for example, construct a function $f \in \text{DBSC}([0, 1]) \sim \text{PS}[0, 1]$ whose range is a convergent sequence along with its limit, as follows: let $\{d_i\}$ be a countable dense subset of $[0, 1]$ and let

$$f = \sum_{j=1}^{\infty} \frac{\chi_{\{d_j\}}}{2^j} .$$

Then f cannot be pointwise stabilizing, since there is no open non-empty subset \mathcal{U} of $[0, 1]$ so that $f|_{\mathcal{U}}$ is continuous (cf. Proposition 4.9 of [HOR]). Evidently $f \in \text{DBSC}([0, 1])$, since

$$\|f\|_{\text{DBSC}} \leq \sum_{j=1}^{\infty} \frac{\|\chi_{d_j}\|_{\text{DBSC}}}{2^j} \leq 2 .$$

Finally, $f([0, 1]) = \{0\} \cup \{2^{-j} := 1, 2, \dots\}$, a convergent sequence with its limit.

Following Hausdorff [H], let us say that $F \in \mathcal{B}_1(K)$ is a *step function* if $F(K)$ is discrete.

Corollary 3.5. *Let K be a Polish space and $F \in \mathcal{B}_1(K)$. Then F is a uniform limit of step functions. Hence $\text{DSC}(K)$ is uniformly dense in $\mathcal{B}_1(K)$.*

Proof. We first recall the classical

Fact 1. Given $G \subset F \subset K$, G a G_δ and F an F_σ , there exists an ambiguous set A with $G \subset A \subset F$ (cf. [Ku], Theorem 2, page 350).

Fact 2. $f : K \rightarrow \mathbb{R} \in \mathcal{B}_1(K) \Leftrightarrow f^{-1}(U)$ is an F_σ for all open $U \subset \mathbb{R}$ (see [H]).

Now fix $n \geq 2$ and for each $m \in \mathbb{Z}$, $1 \leq j \leq n$, choose A_j^m an ambiguous set with

$$(2) \quad G_j^m \stackrel{\text{def}}{=} f^{-1} \left(\left[\frac{m+j-1}{n}, \frac{m+j}{n} \right] \right) \subset A_j^m \subset f^{-1} \left(\frac{m+j-2}{n}, \frac{m+j+1}{n} \right) \stackrel{\text{def}}{=} F_j^m.$$

(This is possible by the Facts, since G_j^m is then a G_δ , F_j^m an F_δ .) Then evidently

$$(3) \quad \bigcup_{m=-\infty}^{\infty} \bigcup_{j=1}^n A_j^m = K.$$

Finally, by disjointifying the A_j^m 's, choose a sequence W_1, W_2, \dots of disjoint ambiguous sets with $K = \bigcup_{j=1}^{\infty} W_j$ so that for all i , there is an $m(i) \in \mathbb{Z}$ and $j(i)$, $1 \leq j(i) \leq n$ with

$$(4) \quad f(W_i) \subset \left(\frac{m(i) + j(i) - 2}{n}, \frac{m(i) + j(i) + 1}{n} \right).$$

Then let

$$(5) \quad f_n = \sum_{i=1}^{\infty} \left(\frac{m(i) + j(i)}{n} \right) \chi_{W_i}.$$

Now we have by Theorem 3.1 that $f_n \in \text{PS}(K)$; of course f_n is thus a step function. Now for any i and $x \in W_i$, $|f(x) - f_n(x)| \leq \frac{3}{n}$. But then $\|f - f_n\|_\infty \leq \frac{3}{n}$, whence $f_n \rightarrow f$ uniformly. \square

Remarks. 1. Corollaries 3.3 and 3.5 again reveal the immense difference between $\text{DBSC}(K)$ and $\text{BDSC}(K)$. Indeed, for general K , the uniform closure of $\text{DBSC}(K)$ equals $\mathcal{B}_{1/2}(K)$, a very thin subset of $\mathcal{B}_1^b(K)$ as long as K contains a homeomorphic copy of ω^ω . Of course Corollary 3.5 immediately yields for any Polish K , that the uniform closure of $\text{BDSC}(K)$ equals $\mathcal{B}_1^b(K)$. Moreover for K as above, $\text{DBSC}(K) \subsetneq \mathcal{B}_{1/2}(K)$, but if K is countable,

$$\text{BPS}(K) = \text{BDSC}(K) = \mathcal{B}_1^b(K) = \ell^\infty(K)$$

by Corollary 3.3.

2. Corollaries 3.3–3.5 are essentially given (with different reasoning) in [H] (see page 278).

4. EXTRINSIC CHARACTERIZATIONS OF $\text{DSC}(K)$

To motivate our theorems, we first establish the following:

Proposition 4.1. *Let K be a perfect Polish space. If U is a non-empty open subset of K and $\lambda \in \mathbb{R}$ then the set $\mathcal{B}_{U,\lambda} := \{f \in \mathcal{B}_1^b(K) : \text{there exist } (\phi_n) \subset C^b(K) \text{ with } \sum_{n=1}^{\infty} \phi_n = f \text{ and } \sum_{n=1}^{\infty} |\phi_n(t)| \leq \lambda \text{ for all } t \in U\}$ is of first category in $\mathcal{B}_1^b(K)$.*

Proof. Let $X = \{f \in \mathcal{B}_1^b(K) : f|_U \in \text{DBSC}(U)\}$. Then X is a linear subspace of $\mathcal{B}_1(K)$; we may introduce a norm on X by

$$\|x\| = \|x|_U\|_{\text{DBSC}(U)} + \|x|_{U^c}\|_{\infty}.$$

Now if $Y = \{f \in \text{DBSC}(K) : f \equiv 0 \text{ on } U^c\}$, then Y is a linear subspace of X , and the norms $\|\cdot\|$ and $\|\cdot\|_{\infty}$ are not equivalent on Y . Indeed, since U is open, U is again a perfect Polish space, and hence uncountable; it is trivial that Y is canonically isometric to $\text{DBSC}(U)$, so this follows from [HOR]. Hence, the norms $\|\cdot\|$ and $\|\cdot\|_{\infty}$ are not equivalent on X . Now evidently $\mathcal{B}_{U,\lambda} \subset X$, so the proposition follows from the easy:

Lemma 4.2. *If X is a linear subspace of a Banach space B so that $\|\cdot\|_B \leq \|\cdot\|_X$ where $\|\cdot\|_X$ is a norm on X which is not equivalent to the norm $\|\cdot\|_B$ of B , then X is of first category.*

Proof. Let $F_n := \overline{\{x \in X : \|x\|_X \leq n\}}$ (where the closure is taken in B). Then F_n has void interior, for if not, by the standard proof of the open mapping theorem, the norms are equivalent on X (and $X = B!$). So F_n is meager. Hence $\bigcup_{n=1}^{\infty} F_n$ is of first category. But $X \subset \bigcup_{n=1}^{\infty} F_n$, so X is of first category. \square

Corollary 4.3. *If K is a perfect Polish space, then $\text{BDSC}(K)$ is of first category in $\mathcal{B}_1^b(K)$.*

Proof. Let $f \in \text{BDSC}(K)$ and $(\phi_n)_{n=1}^{\infty} \subset C^b(K)$ with $f = \sum_{n=1}^{\infty} \phi_n$ and $\sum_{n=1}^{\infty} |\phi_n(t)| < \infty$ for all $t \in K$. Let $K_m := \{t \in K : \sum_{n=1}^{\infty} |\phi_n(t)| \leq m\}$. Then $K = \bigcup_{m=1}^{\infty} K_m$. But K is a Baire space, so there exist $m_0 \in \mathbb{N}$ and U a non-empty open subset of K with $U \subset K_{m_0}$. So $f \in \mathcal{B}_{U,m_0}$ and therefore, if $(U_n)_{n=1}^{\infty}$ is a basis of neighborhoods for K , then:

$$\text{BDSC}(K) \subset \bigcup_{n,m} \mathcal{B}_{U_n,m},$$

which proves that $\text{BDSC}(K)$ is of first category. \square

Remarks. 1. It also follows that if K is a perfect Polish space, then $\text{DSC}(K)$ is of first category in $\mathcal{B}_1(K)$, endowed with the topology of uniform convergence. ($\mathcal{B}_1(K)$ is a complete metric space in this topology, where e.g., we set $\rho(f,g) = \sup_{x \in K} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}$.) To see this, let U be a non-empty open subset of $\mathcal{B}_1(K)$ and let $D_U = \{f \in \mathcal{B}_1(K) : f|_U \in \text{DBSC}(U)\}$. Now since χ_U itself is in $\mathcal{B}_1^b(K)$, we have that $\mathcal{B}_1(K) = \mathcal{B}_1(U) \oplus \mathcal{B}_1(\sim U)$ via the obvious identifications. But then $D_U = \text{DBSC}(U) \oplus \mathcal{B}_1(\sim U)$, and so $\text{DBSC}(U)$ is first category in $\mathcal{B}_1^b(U)$ by Corollary 4.3. But $\mathcal{B}_1^b(U)$ is closed in $\mathcal{B}_1(U)$, so $\text{DBSC}(U)$ is first category in $\mathcal{B}_1(U)$, whence D_U is first category in $\mathcal{B}_1(K)$. But now the argument for 4.3 goes through (deleting the "b" superscripts) yielding that $\text{DSC}(K) \subset \bigcup_{n=1}^{\infty} D_{U_n}$, hence $\text{DSC}(K)$ is first category.

2. For any Polish space K , let $\mathcal{B}_{1/2}(K)$ denote the *uniform closure* of $\text{DBSC}(K)$ in $\mathcal{B}_1^b(K)$. An intrinsic equivalent definition may be found in [HOR], where it is shown that $\mathcal{B}_{1/2}(K) \neq \mathcal{B}_1^b(K)$ and $\mathcal{B}_{1/2}(K) \neq \text{DBSC}(K)$ if K contains a subset homeomorphic to $\omega^\omega + 1$ (see Proposition 5.3 of [HOR]). It then follows by the same argument as above that *if K is a perfect Polish space, then $\mathcal{B}_{1/2}(K) \cap \text{DBSC}(K)$ is of first category.* For the proof, simply replace “ $\mathcal{B}_1^b(K)$ ” by “ $\mathcal{B}_{1/2}(K)$ ” in Proposition 4.1 and its proof, and “ $\text{DBSC}(K)$ ” by “ $\text{DBSC}(K) \cap \mathcal{B}_{1/2}(K)$ ” in the proof of 4.3.

The proof of Corollary 4.3 actually shows that $\text{DSC}(K)$ is a subset of the class of functions $f : K \rightarrow \mathbb{R}$ so that, for any closed subset L of K , there exists a relatively open subset \mathcal{U} of L such that $f|_{\mathcal{U}}$ is in $\text{DBSC}(\mathcal{U})$ (since for any closed subset L of K we obviously have that $\text{DSC}(K)|_L \subset \text{DSC}(K)$). We will prove in our second intrinsic characterization theorem that this inclusion is in fact an equality. This theorem requires the following crucial decomposition result (our first intrinsic characterization).

Theorem 4.4. *Let K be a Polish space and $f : K \rightarrow \mathbb{R}$ be a given function. Suppose that $K = \bigcup_{j=1}^{\infty} A^j$, where A^j is ambiguous and $f|_{A^j} \in \text{DSC}(A^j)$ for all $j \geq 1$. Then $f \in \text{DSC}(K)$.*

Proof. Without loss of generality, we can suppose that the A^j 's are disjoint. By hypothesis,

$$(6) \quad f|_{A^j} = \lim_{n \rightarrow \infty} f_n^j \text{ with } \sum_{n=1}^{\infty} |f_{n+1}^j(t) - f_n^j(t)| < \infty \text{ for all } t \in A^j,$$

where f_n^j is continuous on A^j . Now each A^j is an F_σ , say $A^j = \bigcup_{n=1}^{\infty} A_n^j$ with the A_n^j 's closed and in addition, we can suppose that the A_n^j 's are increasing in n , i.e., $A_n^j \subset A_{n+1}^j \subset \dots$, while of course they are disjoint in j . The Tietze extension theorem then provides for each n a continuous function $f_n : K \rightarrow \mathbb{R}$ that extends $\sum_{j=1}^n f_n^j \chi_{A_n^j}$. In other words we have:

$$(7) \quad f_{n|A_n^j} = f_n^j \text{ for all } 1 \leq j \leq n.$$

Claim. $f_n \rightarrow f$ pointwise and $\sum_{n=1}^{\infty} |f_{n+1}(t) - f_n(t)| < \infty$ for all t .

Indeed, let $t \in K$; then t lies in some A^j . Pick $n_0 \geq j$ large enough so that $t \in A_{n_0}^j$. Now, if $n \geq n_0$ then $f_n(t) = f_n^j(t)$ by (7) since $n_0 \geq j$, and $f_n^j(t) \rightarrow f(t)$ as $n \rightarrow \infty$ by (6). Also for that same n_0 ,

$$\sum_{n \geq n_0} |f_{n+1}(t) - f_n(t)| = \sum_{n \geq n_0} |f_{n+1}^j(t) - f_n^j(t)| < \infty$$

by (6). \square

We are now ready for our second intrinsic characterization theorem.

Theorem 4.5. *Let $f : K \rightarrow \mathbb{R}$ be a given bounded function on some Polish space K . Then the following are equivalent:*

1. $f \in \text{DSC}(K)$.

2. For any closed non-empty subset L of K , there exists a closed relative neighborhood \mathcal{U} in L so that $f|_{\mathcal{U}}$ is in $\text{DSC}(\mathcal{U})$.
3. For any closed non-empty subset L of K , there exists a closed relative neighborhood \mathcal{U} in L so that $f|_{\mathcal{U}}$ is in $\text{DBSC}(\mathcal{U})$.
4. For any closed non-empty subset L of K , there exists a subset E of second category in L so that $f|_{\overline{E}}$ is in $\text{DSC}(E)$.

Recall that $\mathcal{U} \subset L$ is a relative neighborhood in L if \mathcal{U} has non-empty relative interior with respect to L .

Remark. Condition 4. means that for any closed subset L of K there exist a subset E of second category in L and a sequence of continuous functions (f_j) on \overline{E} , so that (f_j) converges to f pointwise on \overline{E} and $\sum_{j=1}^{\infty} |f_{j+1}(t) - f_j(t)| < \infty$ for all $t \in E$.

Proof. We will prove that: (1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (4) \Rightarrow (3).

(1) \Rightarrow (3): Let $(\phi_n)_{n=1}^{\infty} \subset \mathcal{C}(K)$ with $f = \sum_{n=1}^{\infty} \phi_n$ and $\sum_{n=1}^{\infty} |\phi_n(t)| < \infty$ for all $t \in K$. Let L be a closed subset of K . Then $L = \bigcup_{m=1}^{\infty} L_m$ where $L_m := \{t \in L : \sum_{n=1}^{\infty} |\phi_n(t)| \leq m\}$. Then L_m is closed for all m , and so there exists, by the Baire category theorem, an integer m_0 so that the interior of L_{m_0} is not empty. Take \mathcal{U} to be the closure of that interior. Then \mathcal{U} is a closed subset of L with a non-empty interior and $f|_{\mathcal{U}}$ is in $\text{DBSC}(\mathcal{U})$.

(3) \Rightarrow (2) is trivial.

To prove (2) \Rightarrow (1) we start by taking $L_0 = K$. By hypothesis there exists \mathcal{U}_0 a closed relative neighborhood in L_0 so that $f|_{\mathcal{U}_0}$ is in $\text{DSC}(\mathcal{U}_0)$. Let $v_0 := \overset{\circ}{\mathcal{U}}_0$ and set $L_1 := L_0 \sim V_0$. Then L_1 is closed and consequently there exists a closed relative neighborhood \mathcal{U}_1 so that $f|_{\mathcal{U}_1}$ is in $\text{DSC}(\mathcal{U}_1)$. Let $V_1 := \overset{\circ}{\mathcal{U}}_1$ and set $L_2 := L_1 \sim V_1$. Again L_2 is closed and so on. We thus construct a decreasing family of closed sets $(L_\alpha)_\alpha$ with $L_{\alpha+1} := L_\alpha \sim V_\alpha$; and at limit ordinals, $L_\alpha := \bigcap_{\beta < \alpha} L_\beta$. So, $V_\alpha = L_\alpha \sim L_{\alpha+1}$, and

$$(8) \quad f|_{V_\alpha} = f|_{L_\alpha \sim L_{\alpha+1}} \in \text{DSC}(V_\alpha) .$$

Since K is metrizable separable, $L_\eta = \emptyset$ for some $\eta < \omega_1$. Now

$$(9) \quad K = \bigcup_{\alpha < \eta} L_\alpha \sim L_{\alpha+1} .$$

To see this, take a $t \in K = L_0$. If t is not in L_1 , we are done; otherwise, let $\beta := \sup\{\alpha : t \in L_\alpha\}$. Then $t \in L_\beta \sim L_{\beta+1}$. Now, since η is countable, we can enumerate the ordinals that are less than η , as β_1, β_2, \dots . Thus, setting $M_n = L_{\beta_n} \sim L_{\beta_n+1}$ for all n , we have that

$$(10) \quad K = \bigcup_{n=1}^{\infty} M_n ,$$

and each set M_n is ambiguous, begin a difference of closed sets. Combining Theorem 4.4 with equations (8) and (10), we get that f must be in $\text{DSC}(K)$.

(1) \Rightarrow (4) is trivial.

(4) \Rightarrow (3): Let L be a closed set in K . By hypothesis, there exists a subset E of second category in L so that $f|_{\overline{E}}$ is in $\text{DSC}(E)$. Let then $(f_n) \subset \mathcal{C}(\overline{E})$ with

$f|_{\overline{E}}(t) = \lim_{n \rightarrow \infty} f_n(t)$ for all $t \in \overline{E}$, and $\sum_{n=1}^{\infty} |f_{n+1}(t) - f_n(t)| < \infty$ for all $t \in E$. Now let $F := \{t \in \overline{E} : \sum_{n=1}^{\infty} |f_{n+1}(t) - f_n(t)| < \infty\}$. Then obviously $E \subset F$. But $F = \bigcup_{m=1}^{\infty} F_m$ where $F_m := \{t \in \overline{E} : \sum_{n=1}^{\infty} |f_{n+1}(t) - f_n(t)| \leq m\}$. F_m is easily seen to be closed; and, since E is of second category in L , there exists an integer m_0 so that $F_{m_0} \neq \emptyset$. Setting then $\mathcal{U} = F_{m_0}$ ends the proof. \square

Remark. The following is another equivalent statement in Theorem 4.5:

4'. For any closed subset L of K there exists an F_σ subset E of L of second category in L so that $f|_E$ is in $\text{DSC}(E)$.

To see this simply let $E = \bigcup_{m=1}^{\infty} E_m$ where E_m are closed subsets of L . Since E is of second category in L , some E_m has a non-empty interior relative to L . But of course, if $f|_E$ is in $\text{DSC}(E)$, then $f|_{E_m} \in \text{DSC}(E_m)$.

5. INTRINSIC CHARACTERIZATIONS OF $\text{DSC}(K)$

We first recall the transfinite oscillations $\text{osc}_\alpha f$ of a given function f defined on a Polish space K , as introduced in [R2]. For any extended real valued function g on K and $x \in K$, $\overline{\lim}_{y \rightarrow x} g(y)$ denotes the ‘‘unrestricted’’ lim sup of g as y tends to x : $\overline{\lim}_{y \rightarrow x} g(y) = \inf_U \sup g(U)$, the inf over all open neighborhoods U of x . $\mathbf{U}g$ denotes the upper semi-continuous envelope of g : for $x \in K$, $\mathbf{U}g(x) = \overline{\lim}_{y \rightarrow x} g(y)$.

Definition. The α^{th} oscillation of f , $\text{osc}_\alpha(f)$, is defined by ordinal induction as follows: set $\text{osc}_0 f \equiv 0$. Suppose $\beta > 0$ is a given ordinal and $\text{osc}_\alpha f$ has been defined for all $\alpha < \beta$. If β is a successor, say $\beta = \alpha + 1$, we define

$$(11) \quad \widetilde{\text{osc}}_\beta f(x) = \overline{\lim}_{y \rightarrow x} (|f(y) - f(x)| + \text{osc}_\alpha f(y)) \quad \text{for all } x \in K .$$

If β is a limit ordinal, we set

$$(12) \quad \widetilde{\text{osc}}_\beta f = \sup_{\alpha < \beta} \text{osc}_\alpha f .$$

Finally, we set $\text{osc}_\beta f = \mathbf{U}\widetilde{\text{osc}}_\beta f$.

Evidently $\text{osc}_\alpha f$ is a $[0, \infty]$ -valued upper semi-continuous function for all α . A motivating result for the following: Theorem 3.5 of [R2] yields that f is locally in $\text{DBSC}(K)$ iff $\text{osc}_\alpha f$ is real-valued for all $\alpha < \omega_1$.

Classically, $\text{osc} f$ is defined by the equation

$$(13) \quad \text{osc} f(x) = \overline{\lim}_{y, z \rightarrow x} |f(y) - f(z)| = \overline{\lim}_{y \rightarrow x} f(y) - \underline{\lim}_{y \rightarrow x} f(y)$$

(where in the last identity, the meaningless term ‘‘ $\infty - \infty$ ’’ is replaced by ‘‘ ∞ ’’ if it occurs).

Now we easily have that $\text{osc}_1 f \leq \text{osc} f \leq 2 \text{osc}_1 f$; thus f is continuous at x iff $\text{osc}_1 f(x) = 0$.

Definition. $x \in K$ is called a strong continuity point of f if $\text{osc}_\alpha f(x) = 0$ for all α .

Our first intrinsic characterization theorem yields a DSC analogue of Baire’s theorem characterizing Baire-1 functions.

Theorem 5.1. *Let K be a Polish space and $f : K \rightarrow \mathbb{R}$ a given function. Then the following are equivalent:*

1. $f \in \text{DSC}(K)$.
2. For any closed non-empty subset L of K , $f|L$ has a strong continuity point.
3. For any closed non-empty subset L of K , the set of strong continuity points of $f|L$ contains a dense G_δ subset of L .
4. For any closed non-empty subset L of K , there exists an $x \in L$ so that $\sup_\alpha \text{osc}_\alpha f(x) < \infty$.

Proof. We show $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 4 \Rightarrow 1$. Of course $3 \Rightarrow 2 \Rightarrow 4$ are trivial.

$1 \Rightarrow 3$: Let $f = u - v$ with u and v upper semi-continuous. Now by Baire's famous theorem, if U, V denote the set of points of continuity of $u|L$ and $v|L$ respectively, U and V are both dense G_δ 's, hence $G \stackrel{\text{def}}{=} U \cap V$ is a dense G_δ subset of L . Now if $x \in G$, then $\text{osc } u|L(x) = \text{osc } v|L(x) = 0$. But for any $\alpha < \omega_1$, $\text{osc}_\alpha u|L = \text{osc } u|L$ and $\text{osc}_\alpha v|L = \text{osc } v|L$ since u, v are semi-continuous, by Proposition 3.4(d) of [R2]. Hence we have that

$$(14) \quad \begin{aligned} \text{osc}_\alpha f|L(x) &= \text{osc}_\alpha(u - v)|L(x) \leq \text{osc}_\alpha u|L(x) + \text{osc}_\alpha v|L(x) \\ &\quad \text{(by Proposition 3.4(b) of [R2])} \\ &= 0. \end{aligned}$$

Thus $\text{osc}_\alpha f|L(x) = 0$ for all α , so x is a strong point of continuity of $f|L$.

$4 \Rightarrow 1$: Let L be a closed non-empty subset of K . By Lemma 3.7 of [R2] there exists an $\eta < \omega_1$ so that $\text{osc}_\eta f|L = \text{osc}_\beta f|L$ for all $\beta > \eta$. Let $U = \{x \in L : \text{osc}_\eta f|L < \infty\}$. Then U is non-empty since we assume 4 holds. But U is a relatively open subset of L since $\text{osc}_\eta f|L$ is upper semi-continuous. But then by Proposition 3.4(c) of [R2], $(\text{osc}_\alpha f|L \pm f|L)$ are both upper semi-continuous, hence $f|L \in \text{DSC}(U)$. Thus condition 3 of Theorem 4.5 holds, whence $f \in \text{DSC}(K)$ by its conclusion. \square

Remark. The proof of this result yields an ‘‘effective’’ ordinal index for DSC functions on a Polish space K (as well as an ‘‘effective’’ intrinsic criterion for determining when a function is DSC), as follows. Following [R2], we define, for a general $f : K \rightarrow \mathbb{R}$, $i_D(f)$, the D -index of f , by $i_D(f) = \min\{\alpha < \omega_1 : \text{osc}_\alpha f = \text{osc}_{\alpha+1} f\}$. Now let $\eta_1 = \eta_1(f) = i_D(f)$ and $K_1(f) = \{x : \text{osc}_{\eta_1} f = \infty\}$. (Also set $K_0 = K$.) It follows from our argument for 5.1 (i.e., the cited results in [R2]) that $K \sim K_1 = \{x \in K : f \text{ is locally DBSC at } x\}$ (where f is locally DBSC at x if $\exists U$ a neighborhood of x with $f|U \in \text{DBSC}(U)$). Of course K_1 is closed (possibly empty). Now for each ordinal $\beta < \omega_1$ having defined K_β for all $\alpha < \beta$ set $K_\beta = \bigcap_{\alpha < \beta} K_\alpha$ if β is a limit ordinal. If not, let $\alpha + 1 = \beta$; if $K_\alpha = \emptyset$, set $K_\beta = \emptyset$. Otherwise, let $\eta_\beta = \eta_\beta(f) = i_D(f|K_\alpha)$ and let $K_{\alpha+1} = \{x \in K_\alpha : \text{osc}_{\eta_\beta} f|K_\alpha(x) = \infty\}$. Now since K is Polish, we may define $i_{\text{DSC}}(f)$ to be the least $\alpha < \omega_1$ so that $K_\alpha = K_{\alpha+1}$. It then follows that $f \in \text{DSC}(K)$ iff $K_\alpha = \emptyset$, where $\alpha = i_{\text{DSC}}(f)$. Indeed, when $K_\alpha = \emptyset$, $K = \bigcup_{\beta < \alpha} K_\beta \sim K_{\beta+1}$ and $f|(K_\beta \sim K_{\beta+1}) \in \text{DSC}(K_\beta \sim K_{\beta+1})$ for all β , whence $f \in \text{DSC}(K)$, by Theorem 4.4. In this case, the index $i_{\text{DSC}}(f)$ measures the ‘‘DBSC-complexity of the function f ’’. It seems very likely that if K is uncountable, then there exist BDSC functions on K with arbitrarily large countable DSC indexes.

Our last result yields an answer to the following problem: Suppose $f_n \rightarrow f$ uniformly on K , with f_n in $\text{DSC}(K)$ for all n . What additional properties of the convergence of the sequence (f_n) will force f to also belong to $\text{DSC}(K)$?

Theorem 5.2. *Let K be a Polish space, and let f, f_1, f_2, \dots be real-valued functions on K . Suppose for all $\alpha < \omega_1$ and non-empty closed subsets L of K , there exists a subsequence (f'_j) of (f_j) and a closed relative neighborhood U of L so that*

$$(i) \quad f'_j \rightarrow f \text{ uniformly on } U$$

and

$$(ii) \quad \sum_j \|\text{osc}_\alpha(f'_{j+1} - f'_j)|U)\|_\infty < \infty.$$

Then $f \in \text{DSC}(K)$.

We first require the following result from [R4], which for the sake of completeness we prove again here.

Lemma 5.3. *Let $\alpha < \omega_1$, K a Polish space, and $\varphi, \varphi_1, \varphi_2, \dots$ real-valued functions on K so that*

$$(i) \quad \sum_{j=1}^n \varphi_j \rightarrow \varphi \text{ uniformly on } K$$

and

$$(ii) \quad \sum \|\text{osc}_\alpha \varphi_j\|_\infty < \infty.$$

Then $\|\text{osc}_\alpha(\varphi - \sum_{j=1}^n \varphi_j)\|_\infty \rightarrow 0$ as $n \rightarrow \infty$.

Remark. It follows immediately that $\|\text{osc}_\alpha \varphi\|_\infty < \infty$. Indeed, for all n ,

$$\begin{aligned} \|\text{osc}_\alpha \varphi\|_\infty &\leq \left\| \text{osc}_\alpha \left(\varphi - \sum_{j=1}^n \varphi_j \right) \right\|_\infty + \left\| \text{osc}_\alpha \sum_{j=1}^n \varphi_j \right\|_\infty \\ &\leq \left\| \text{osc}_\alpha \left(\varphi - \sum_{j=1}^n \varphi_j \right) \right\|_\infty + \sum_{j=1}^{\infty} \|\text{osc}_\alpha \varphi_j\|_\infty. \end{aligned}$$

Hence in fact $\|\text{osc}_\alpha \varphi\|_\infty \leq \sum_{j=1}^{\infty} \|\text{osc}_\alpha \varphi_j\|_\infty < \infty$.

Proof of Lemma 5.3. Let $f_n = \sum_{j=1}^n \varphi_j$ for all n . Also let $f_0 = 0$. We prove by induction on $\gamma \leq \alpha$ that

$$(15) \quad \text{osc}_\gamma(\varphi - f_n)(x) \leq \sum_{j=n+1}^{\infty} (\text{osc}_\gamma \varphi_j)(x) \text{ for all } x \in K, \text{ all } n = 0, 1, 2, \dots$$

Of course (15) yields in particular that

$$\|\text{osc}_\gamma(\varphi - f_n)\|_\infty \leq \sum_{j=n+1}^{\infty} \|\text{osc}_\gamma \varphi_j\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty,$$

since $\text{osc}_\gamma g \leq \text{osc}_\alpha g$ for any function g .

Suppose then $0 \leq \gamma < \alpha$ and (15) has been proved for γ . Now fixing n and $\varepsilon > 0$, choose $q > n$ so that

$$(16) \quad \left| (\varphi - f_n)(y) - \sum_{j=n+1}^q \varphi_j(y) \right| \leq \varepsilon \text{ for all } y \in K$$

and

$$(17) \quad \sum_{j=q+1}^{\infty} \|\text{osc}_{\alpha} \varphi_j\|_{\infty} \leq \varepsilon .$$

Now, fixing $x \in K$, we have for any $y \in K$ that

$$\begin{aligned} & |(\varphi - f_n)(y) - (\varphi - f_n)(x)| + \text{osc}_{\gamma}(\varphi - f_n)(y) \\ & \leq \left| \sum_{j=n+1}^q \varphi_j(y) - \varphi_j(x) \right| + \sum_{j=n+1}^{\infty} \text{osc}_{\gamma} \varphi_j(y) + 2\varepsilon \\ & \quad \text{(by (16) and the induction hypothesis (15))} \\ & \leq \sum_{j=n+1}^q (|\varphi_j(y) - \varphi_j(x)| + \text{osc}_{\gamma} \varphi_j(y)) + 3\varepsilon \\ & \quad \text{by the triangle inequality and (17).} \end{aligned}$$

Thus by definition,

$$\begin{aligned} (18) \quad \widetilde{\text{osc}}_{\gamma+1}(\varphi - f_n)(x) & \leq \overline{\lim}_{y \rightarrow x} \sum_{j=n+1}^q (|\varphi_j(y) - \varphi_j(x)| + \text{osc}_{\gamma} \varphi_j(y)) \\ & \leq \sum_{j=n+1}^q \widetilde{\text{osc}}_{\gamma+1} \varphi_j(x) + 3\varepsilon \\ & \leq \sum_{j=n+1}^q \text{osc}_{\gamma+1} \varphi_j(x) + 3\varepsilon . \end{aligned}$$

Now since $\sum_{j=n+1}^q \text{osc}_{\gamma+1} \varphi_j$ is upper semi-continuous and x is an arbitrary point in K ,

$$(19) \quad \text{osc}_{\gamma+1}(\varphi - f_n) \leq \sum_{j=n+1}^{\infty} \text{osc}_{\gamma+1} \varphi_j + 3\varepsilon \text{ pointwise.}$$

Of course since $\varepsilon > 0$ is arbitrary, (15) is established for $\gamma + 1$.

Finally, suppose $\beta \leq \alpha$ is a limit ordinal and (15) is established for all $\gamma < \beta$. But then fixing $x \in K$, we have

$$\begin{aligned} (20) \quad \widetilde{\text{osc}}_{\beta}(\varphi - f_n)(x) & = \sup_{\gamma < \beta} \text{osc}_{\gamma}(\varphi - f_n)(x) \leq \sup_{\gamma < \beta} \sum_{j=n+1}^{\infty} \text{osc}_{\gamma} \varphi_j(x) \\ & \leq \sum_{j=n+1}^{\infty} \text{osc}_{\beta} \varphi_j(x) . \end{aligned}$$

Now again by taking the upper semi-continuous envelope, we obtain from (20) that (15) holds for $\gamma = \beta$, completing the proof of the lemma by transfinite induction. \square

Remark. Of course the lemma yields that if $X_{\alpha}(K) = X_{\alpha}$ is the class of bounded functions f on K with $\text{osc}_{\alpha} f$ bounded, then X_{α} is a Banach space under the norm

$\|f\|_{X_\alpha} = \max\{\|f\|_\infty, \|\text{osc}_\alpha f\|_\infty\}$. In fact the $X_\alpha(K)$'s are Banach algebras, with a rich structure connected with invariants for general Banach spaces; see [R4].

We are finally prepared for the

Proof of Theorem 5.2. Let L be a non-empty closed subset of K , and (as in the proof of Theorem 5.1), choose α a countable ordinal so that $\text{osc}_\alpha f|_L = \text{osc}_\beta f|_L$ for all $\beta > \alpha$. Now choose U and (f'_j) as in the hypotheses of Theorem 5.2. Then setting $\varphi_j = f'_{j+1} - f'_j$ for all $j > 1$, $\varphi_1 = f'_1$, we obtain by Lemma 5.3 that $\|\text{osc}_\alpha(f - f'_n)|U\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, whence (by the remark following 5.2), $\|\text{osc}_\alpha f|U\|_\infty < \infty$. It then follows that since $\text{osc}_\beta f|_L = \text{osc}_\alpha f|_L$ all $\beta > \alpha$, $\sup_\beta \|\text{osc}_\beta f|U\|_\infty = \|\text{osc}_\alpha f|U\|_\infty < \infty$, whence condition 4 of Theorem 5.1 holds, so $f \in \text{DSC}(K)$ by this result. \square

Theorem 5.2 yields various solutions to the problem mentioned before its statement. The following corollary is a ‘‘useable’’ such solution. Let us say that a series of $\mathbb{R} \cup \{\infty\}$ -valued functions (g_j) on a Polish space K converges *locally uniformly* if for all $x \in K$, there is a neighborhood U of X and an ν so that $g_j|U$ is real-valued for all $j \geq \nu$, and $\sum_{j \geq \nu} g_j$ converges uniformly on U .

Corollary 5.4. *Let K be a Polish space, (φ_j) a sequence in $\text{DSC}(K)$, and f a function on K so that*

(i) $\sum \varphi_j$ converges locally uniformly to f

and

(ii) $\sum \text{osc}_\alpha \varphi_j$ converges locally uniformly, for every $\alpha < \omega_1$.

Then $f \in \text{DSC}(K)$.

Proof. Fix $x \in K$, $\alpha < \omega_1$, and choose U a closed neighborhood of X and a ν so that $\sum_{j=\nu+1}^\infty \varphi_j$ and $\sum_{j=\nu+1}^\infty \text{osc}_\alpha \varphi_j$ converge uniformly on U . It follows that we may choose $\nu \leq n_1 < n_2 < \dots$ so that

$$(21) \quad \left\| \sum_{j=n_i+1}^{n_{i+1}} \text{osc}_\alpha \varphi_j|U \right\|_\infty < \frac{1}{2^i} \text{ for all } i = 1, 2, \dots .$$

Now set $g = f - \sum_{j=1}^{n_1} \varphi_j$ and $\psi_i = \sum_{j=n_i+1}^{n_{i+1}} \varphi_j$ for all i . Then of course $\sum \psi_i$ converges uniformly on U to g . But moreover, for all i ,

$$(22) \quad \text{osc}_\alpha \psi_i \leq \sum_{j=n_i+1}^{n_{i+1}} \text{osc}_\alpha \varphi_j \text{ pointwise.}$$

Hence by (21) and (23),

$$(23) \quad \sum_{i=1}^\infty \|\text{osc}_\alpha \psi_i|U\|_\infty < \infty .$$

It now follows by Theorem 5.2 that $g|U \in \text{DSC}(U)$. Indeed, for any non-empty closed subset L of U , any i , $\|\text{osc}_\alpha \psi_i|L \cap U\|_\infty \leq \|\text{osc}_\alpha \psi_i|U\|_\infty$, hence the hypotheses of Theorem 5.2 are fulfilled on U . Then since $\text{DSC}(U)$ is a linear space, also $f|U \in \text{DSC}(U)$. But this implies immediately that $f \in \text{DSC}(K)$, since f thus locally belongs to DSC . \square

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