ON CERTAIN EXTENSION PROPERTIES FOR THE SPACE OF COMPACT OPERATORS

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Abstract. Let Z be a fixed separable operator space, $X \subset Y$ general separable operator spaces, and $T : X \to Z$ a completely bounded map. Z is said to have the Complete Separable Extension Property (CSEP) if every such map admits a completely bounded extension to Y; the Mixed Separable Extension Property (MSEP) if every such T admits a bounded extension to Y. Finally, Z is said to have the Complete Separable Complementation Property (CSCP) if Z is locally reflexive and T admits a completely bounded extension to Y *provided* Y is locally reflexive and T is a complete surjective isomorphism. Let $\mathbf{K}$ denote the space of compact operators on separable Hilbert space and $\mathbf{K}_0$ the c_0 sum of $\mathcal{M}_n$'s (the space of "small compact operators"). It is proved that $\mathbf{K}$ has the CSCP, using the second author's previous result that $\mathbf{K}_0$ has this property. A new proof is given for the result (due to E. Kirchberg) that $\mathbf{K}_0$ (and hence $\mathbf{K}$) fails the CSEP. It remains an open question if $\mathbf{K}$ has the MSEP; it is proved this is equivalent to whether $\mathbf{K}_0$ has this property. A new Banach space concept, Extendable Local Reflexivity (ELR), is introduced to study this problem. Further complements and open problems are discussed.

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Introduction

The space $\mathbf{K}$ of compact operators on a separable infinite dimension Hilbert space H is often that thought of as the non-commutative analogue of c_0, the space of sequences vanishing at infinity. Indeed, if one regards $\mathbf{K}$ as matrices with respect to a fixed orthonormal basis of H, the *diagonal* matrices form a subalgebra isometric to c_0. In 1941, A. Sobcyk proved that c_0 has the Separable Extension Property (SEP) [S]: If $Z = c_0$, then given $X \subset Y$ separable Banach spaces and $T : X \to Z$ a bounded linear operator, there exists a bounded linear operator $\tilde{T} :\to Z$ extending T. In 1977, M. Zippin proved the (much deeper!) converse to this result [Z], *any infinite-dimensional separable Banach space Z with the* SEP *is isomorphic to c_0.* We continue here the study of operator space analogues of the SEP, initiated in [Ro2], with the goal in particular of specifying which of these analogues $\mathbf{K}$ satisfies. (For basic facts about operator spaces see [Pi3]; also see the Introduction to [Ro2] for a brief summary and orientation.)

Thus we consider a fixed operator space Z, and consider the following diagram:

$$\begin{tikzcd} Y \arrow[drr, dashed, "? \quad \tilde{T}"] & & \\ \cup & & \\ X \arrow[rr, "T"] & & Z \end{tikzcd}$$

Here, X and Y are (appropriately general) separable operator spaces and T is a completely bounded linear map.

Z is said to have the Complete Separable Extension Property (CSEP) if every such T admits a *completely bounded linear extension* T; the Mixed Separable Extension Property (MSEP) if T admits a *bounded linear extension* $\tilde{T}$, and the Complete Separable Complementation Property (CSCP) if T admits a bounded linear extension $\tilde{T}$ *provided* Z is separable locally reflexive, Y is also locally reflexive, *and* T is a complete surjective isomorphism. If $1 \leq \lambda$ is such that $\tilde{T}$ can be chosen with $\|\tilde{T}\|_{\mathrm{cb}} \leq \lambda \|T\|_{\mathrm{cb}}$ in the CSEP-case, we say Z has the λ-CSEP; if $\|\tilde{T}\| \leq \lambda \|T\|_{\mathrm{cb}}$ in the MSEP-case, we say Z has the λ-MSEP. It follows easily that if Z *has* the CSEP (resp. the MSEP), then X has the λ-CSEP (resp. the λ-MSEP) for some $\lambda \geq 1$.

Of course these properties are intimately connected with injectivity notions; thus Z is called (isomorphically) injective (resp. mixed injective) if this diagram admits a completely bounded solution (resp. bounded solution) T for arbitrary (not necessarily separable) operator spaces X and Y. As in the separable setting, if Z is injective (resp. mixed injective), there is a $\lambda \geq 1$ so that $\tilde{T}$ may always be chosen with $\|\tilde{T}\|_{\mathrm{cb}} \leq \lambda \|T\|_{\mathrm{cb}}$ (resp. $\|\tilde{T}\| \leq \lambda \|T\|_{\mathrm{cb}}$); if λ works, we say Z is λ-injective (resp. λ-mixed injective). We say Z is isometrically injective (resp. isometrically mixed injective) when $\lambda = 1$.

It is a fundamental theorem in operator space theory that $B(H)$ is isometrically injective for any Hilbert space H, where $B(H)$ denotes the space of bounded linear operators on H. It follows easily that if X is an operator space with $X \subset B(H)$ for some Hilbert space H, then X is isomorphically injective (resp. mixed injective) if and only if X is completely complemented (resp. complemented) in $B(H)$.

The separable extension properties we consider have their primary interest for $\lambda \geq 2$. Indeed, if Z is separable, then if $\lambda < 2$ and Z has the λ-CSEP, it is proved in [Ro2] that Z is λ-injective; we show analogously here that if Z has the λ-MSEP, Z is λ-mixed injective (and moreover Z is reflexive, whence by a result of G. Pisier, Z is actually Hilbertian (cf. [R])).

One of the main results of this work is that $\mathbf{K}$ has the CSCP. A result of E. Kirchberg yields that $\mathbf{K}$ *fails* the CSEP [Ki1]. We give a new proof and further complements in Section 4.

It is proved in [Ro2] that $\mathbf{K}_0$ has the CSCP, where $\mathbf{K}_0$ denotes the space of "small compact operators", namely the c_0-sum of $\mathcal{M}_n$'s, where $\mathcal{M}_n$ denotes the space of complex $n \times n$ matrices, identified with $B(\mathbb{C}^n)$ for all n, $\mathbb{C}^n$ being the standard n-dimensional complex Hilbert space. We obtain that $\mathbf{K}$ has the CSCP via the following route: in Section 1, we show that if $X \subset Y$ are given separable operator spaces, then *any complete isomorphism from X into $B(H)$ admits a complete isomorphic extension from Y into $B(H)$* (Theorem 1.1). It follows from this result that if $X \subset B(H)$ is fixed with X separable locally reflexive, then X *has the* CSCP *provided X is completely complemented in Y for any separable locally reflexive operator space Y when $X \subset Y \subset B(H)$* (see Corollary 1.8). Now it follows from the main result of section 2 (Theorem 2.1) that if $\mathbf{K} \subset Y \subset B(H)$ (where this is the *natural* embedding of $\mathbf{K}$ in $B(H)$) with Y separable, there is an absolute constant C and for all $\varepsilon > 0$, a projection P on $B(H)$ with $\|P\|_{\mathrm{cb}} < 1+\varepsilon$ with Y and $\mathbf{K}$ invariant under P so that $(I-P)Y \subset \mathbf{K}$ and $d_{\mathrm{cb}}(\mathbf{K}, \mathbf{K}_0) \leq C$. It then easily follows that $\mathbf{K}$ is completely complemented in Y provided Y is locally reflexive, from the fact that then $P\mathbf{K}$ has this property by the result in [Ro2]. We do not know if $\mathbf{K}$ has the MSEP. However Theorem 2.1 also yields that $\mathbf{K}$ has the MSEP if $\mathbf{K}_0$ has this property (Proposition 2.3).

We also obtain in Section 1 that if an operator space Z has the CSCP, it has the following stronger property: *there is a completely bounded operator $\tilde{T}$ completing the above diagram whenever Y is separable locally reflexive and X is locally complemented in Y* (Theorem 1.4). As shown in [Ro2], X "automatically" is locally complemented provided X is completely isomorphic to a nuclear C^*-algebra, or more generally, if X^{**} is isomorphically injective. (X is called locally complemented in Y provided there is a $C < \infty$ so that X is C-completely complemented in W for all $X \subset W \subset Y$ with W/X finite-dimensional). It was also previously proved in [Ro2] that $\mathbf{K}_0$ has this stronger property, and moreover one may drop the assumption that Y is locally reflexive.

The MSEP is studied in Section 3, where we introduce the following concept: Given operator spaces X and Y, X is called *completely semi-isomorphic to* Y if there is a completely bounded surjective map $T : X \to Y$ which is a Banach isomorphism; X is called completely semi-isometric to Y in case T can be chosen with $\|T\|_{\mathrm{cb}} = 1 = \|T^{-1}\|$. We then have the simple permanence property: mixed injectivity and the MSEP are both preserved under complete semi-isomorphisms (Proposition 3.9). The finite-dimensional isometrically mixed injectives are known up to Banach isometry; they are the ℓ^∞-direct sums of Cartan factors of type IV (see Theorem A, following Problem 3.2). This result suggests a possible classification of the isometrically injective finite-dimensional operator spaces; are all such completely semi-isometric to an ℓ^∞-direct sum of Cartan factors of types I–IV? (Problem 3.3). A remarkable factorization result of M. Junge's and the semi-isomorphism concept yield that the classification problem of the finite-dimensional mixed injectives is exactly analogous to the famous open commutative case; namely *if X is finite-dimensional and λ-mixed injective, then for all $\varepsilon > 0$, there is an n so that X is $(\lambda+\varepsilon)$-semi-isomorphic to some $(\lambda+\varepsilon)$-complemented subspace of $\mathcal{M}_n$* (Proposition 3.10).

To further penetrate the MSEP-problem for $\mathbf{K}$, we introduce a new pure Banach space concept in Section 3, that of Extendable Local Reflexivity (ELR). Several equivalences are given in Proposition 3.12; Theorem 3.13 yields the remarkable equivalence that a Banach space X is Extendably Locally Reflexive and has the bounded approximation property if and only if X^* has the bounded approximation property. The "if" part of this assertion is quite simple, and was discovered by the second author of this present paper in the fall of 1997. The remarkable "only if" part is due to W.B. Johnson and the first author of the present paper [JO]. Actually, ELR has a complete analogue, and the complete version of Theorem 3.13 also holds. The motivation for the introduction of this concept: *if X is a separable operator space so that X^{**} is isomorphically mixed injective and is (Banach)* ELR*, then X has the* MSEP (Theorem 3.14). In particular, $\mathbf{K}$ has the MSEP if $B(H)$ is ELR. The proof of this result involves a construction mixing Banach and operator space ideas, perhaps of interest in its own right (Lemma 3.16).

Section 4 establishes some necessary conditions for certain operator spaces to have the CSEP, yielding in particular a "qualitative" proof that $\mathbf{K}_0$ (and hence $\mathbf{K}$) fails the CSEP. It is proved for example that *if $Z_1, Z_2, \dots$ are finite-dimensional operator spaces, then if $(Z_1 \oplus Z_2 \oplus \cdots)_{c_0}$ has the* CSEP*, $\{Z_1, Z_2, \dots\}$ is of finite matrix type, and the Z_j's are all λ-injective for some λ* (Corollary 4.2). The converse to this result is established in Proposition 2.15 of [Ro2]. (See the beginning of Section 4 for the definition of *finite matrix type*.) It is further shown that if Z is a separable operator space so that $c_0(Z)$ has the CSEP, then Z is of finite matrix type (Corollary 4.4). We conjecture that if Z is separable with the CSEP, then Z itself is of finite matrix type.

Corollary 4.2 yields that $\mathbf{K}_0$ fails the CSEP, for $\{\mathcal{M}_1, \mathcal{M}_2, \dots\}$ is *not* of finite matrix type. We obtain a stronger quantitative result in Corollary 4.9: *For every n, there exists an operator space Y_n containing $\mathbf{K}_0$ so that $Y_n/\mathbf{K}_0$ is completely isometric to ℓ_n^∞, yet $\mathbf{K}_0$ is not λ-completely complemented in Y_n if $\lambda \le \sqrt{n}/2$.* It follows then from results in [Ro2] that Y_n, which is of course locally reflexive, is not λ-locally reflexive if $\lambda < (\sqrt{n}/2) - 3$. Putting these Y_n's together, we then obtain an operator space Y containing $\mathbf{K}_0$ so that $\mathbf{K}_0$ *is not completely complemented in Y, yet $Y/\mathbf{K}_0$ is completely isometric to c_0*. Thus Y cannot be locally reflexive since $\mathbf{K}_0$ has the CSCP (of course this also follows by its construction). It also then follows by results of E. Kirchberg ([Ki2]) that Y is not a nuclear operator space; however $\mathbf{K}_0$ and $Y/\mathbf{K}_0$ are obviously nuclear. We also show in Proposition 4.11 that any descending sequence of 1-exact finite dimensional Banach isometric spaces must be bounded below.

We finally show that $Z = \mathbf{K}_0$ fails to have a completely bounded solution $\tilde{T}$ to the above diagram if Y is general locally reflexive separable, X a general subspace. Actually, we obtain that there exists a subspace X of C_1 (the operator space of trace class operators) and a completely bounded linear map $T : X \to \mathbf{K}_0$ so that T has no completely bounded extension $\tilde{T}$ to C_1. (A remarkable result due to M. Junge yields that C_1 is 1-locally reflexive.) In fact, we establish in Proposition 4.14 that *if Z is separable and there is a completely bounded solution to the above diagram for arbitrary $X \subset C_1$, $Y = C_1$, then Z has the* CSEP.

Section 1
Extending complete isomorphisms into $B(H)$

The main result of this section is the following:

Theorem 1.1. *Let Y be a separable operator space, X a subspace of Y, and $T : X \to B(H)$ a complete isomorphic injection of X. There exists a complete isomorphic injection $\tilde{T} : Y \to B(H)$ extending T.*

Remarks. 1. We obtain

$$(*) \qquad \|\tilde{T}\|_{\rm cb} \le 3\|T\|_{\rm cb} \text{ and } \|\tilde{T}\|_{\rm cb}\|\tilde{T}^{-1}\|_{\rm cb} \le 12\|T\|_{\rm cb}\|T^{-1}\|_{\rm cb} + 6\ .$$

2. Our proof of the Theorem uses ideas from [LR]. In face, our argument may be refined to obtain the following stronger result, analogous to a result in [LR], showing that completely isomorphic separable sequences of $B(H)$ lie in the same position. *Let X, X' be separable operator subspaces of $B(H)$ and $T : X \to X'$ a complete surjective isomorphism. There exists a complete surjective isomorphism $\tilde{T} : B(H) \to B(H)$ extending T.*

We first give an operator-space version of a result of A. Pełczyński [Pe1], for which we use the following lemma (which is quite different than the argument in [Pe1]).

Lemma 1.2. *Let $X \subset Y$ and Z be Banach spaces and $T : Y \to Z$ a bounded linear operator so that $T|X$ is an (into) isomorphism. Let $\Pi : Y \to Y/X$ be the quotient map and define $\tilde{T} : Y \to Z \oplus Y/X$ by*

$$(1) \qquad \tilde{T}y = Ty \oplus \Pi y \quad \textit{for all } y \in Y\ .$$

Then $\tilde{T}$ is an into-isomorphism with $\tilde{T}|X = T$. In fact

$$(2) \qquad \|\tilde{T}\|\,\|\tilde{T}^{-1}\| \le 2\|T\|\,\|(T|X)^{-1}\| + 1$$

Remark. We put the ∞-norm on the direct sum $Z \oplus Y/X$. If $\tilde{X} = Tx$, $\tilde{Y} = Ty$, $(T|X)^{-1}$ refers to the inverse map from $\tilde{X}$ to X, T^{-1} the map corresponding inverse map from $\tilde{Y}$ to Y.

Proof. It is trivial that $\tilde{T}|X = Z$. (Of course we identity Z with $Z \oplus 0$.) We may assume without loss of generality that $\|T\| = 1$. Let $\delta = \|(T|X)^{-1}\|^{-1}$, and fix $y \in Y$ with $\|y\| = 1$. Set $\tau = \|\Pi y\|$; let $\varepsilon > 0$, and choose $x \in X$ with $\|x - y\| \le \tau + \varepsilon$. Then

$$\|x\| \ge 1 - \tau \ . \tag{3}$$

Hence

$$\|Tx\| \ge \delta(1-\tau) \ , \tag{4}$$

and so

$$\|\tilde{T}y\| \ge \delta(1-\tau) - (\tau + \varepsilon) \ . \tag{5}$$

Of course also

$$\|\tilde{T}y\| \ge \|\Pi y\| = \tau \ . \tag{6}$$

Hence

$$\begin{aligned} \|\tilde{T}y\| &\ge \max\{\delta - \varepsilon - (\delta+1)\tau, \tau\} \\ &\ge \frac{\delta - \varepsilon}{\delta + 2} \ \text{ for } \ 0 \le \tau \le 1 \ . \end{aligned}$$

Since $\varepsilon > 0$, we have proved that

$$\|\tilde{T}^{-1}\| \le 2\|(T|X)^{-1}\| + 1 \tag{7}$$

which establishes (2) and thus the Lemma. $\square$

The next result yields [Pe1, Proposition 1] when restricted to the Banach space category.

Proposition 1.3. *Let $X \subset Y$, $\tilde{X}$ be operator spaces and $T : X \to \tilde{X}$ a complete surjective isomorphism. There exists an operator space $\tilde{Y} \supset \tilde{X}$ and a complete surjective isomorphism $\tilde{T} : Y \to \tilde{Y}$ extending T, in fact satisfying*

$$\|\tilde{T}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}} \text{ and } \|\tilde{T}\|_{\mathrm{cb}}\|\tilde{T}^{-1}\|_{\mathrm{cb}} \le 2\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}} + 1 \ . \tag{8}$$

Remark. Proposition 1.3 (or rather its proof) is used in the proof of the main result of this section, Theorem 1.1. However if we use an alternate construction, due to G. Pisier (see 10b, p.137 of [Pi2]), we obtain considerably better quantitative information. We first formulate the result, then give the construction, leaving the details of proof to the interested reader.

Proposition. *Let $X \subset Y$, $\tilde{X}$ be operator spaces and $T : X \to \tilde{X}$ a completely bounded map. There exists an operator space $\tilde{Y} \supset \tilde{X}$ and a completely bounded map $\tilde{T} : Y \to \tilde{Y}$ extending T, satisfying the following:*

(i) $\|\tilde{T}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$.
(ii) *If T is an (into) isomorphism (resp. complete isomorphism) so is $\tilde{T}$, and $\|\tilde{T}^{-1}\| = \|T^{-1}\|$ (resp. $\|\tilde{T}^{-1}\|_{\mathrm{cb}} = \|T^{-1}\|_{\mathrm{cb}}$).*
(iii) *If T is surjective, so is $\tilde{T}$.*
(iv) *Y/X is completely isometric to $\tilde{Y}/\tilde{X}$.*

Corollary. *Let $X, Y, \tilde{X}$ be operator spaces with $X \subset Y$ and $\tilde{X}$ and X completely isomorphic; set $\beta = d_{\mathrm{cb}}(\tilde{X}, X)$. Then given $\varepsilon > 0$, there exists an operator space $\tilde{Y} \supset \tilde{X}$ and a complete bijection $T : Y \to \tilde{Y}$ with $TX = \tilde{X}$ and $\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}} < \beta + \varepsilon$.*

Pisier's construction (adapted to this setting) goes as follows: Let $Z = (Y \oplus \tilde{X})_1$ (endowed with its natural operator space structure, cf. [Pi3]). Assume first that $\|T\|_{\mathrm{cb}} = 1$. Let $\Gamma = \{x \oplus -Tx : x \in X\}$. Then Γ is a closed subspace of Z. Let $\tilde{Y} = Z/\Gamma$ and $\Pi : Z \to \tilde{Y}$ be the quotient map. Define $j : \tilde{X} \to \tilde{Y}$ and $\tilde{T} : Y \to \tilde{Y}$ by $j(\tilde{x}) = \Pi(0 \oplus \tilde{x})$ and $\tilde{T}(y) = \Pi(y \oplus 0)$ for all $\tilde{x} \in \tilde{X}$ and $y \in Y$.

One then has the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{T}} & \tilde{Y} \\ {\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle j} \\ X & \xrightarrow{T} & \tilde{X} \end{array}$$

where $i : X \to Y$ is the inclusion map. Then it follows that j is a *complete* (*into*) *isometry* and one now verifies all the details of the Proposition (identifying $\tilde{X}$ with $j(\tilde{x})$). In general, assuming $T \neq 0$, simply set $U = T/\|T\|_{\mathrm{cb}}$ and now carry out the construction for U instead, but then simply define $\tilde{T}$ by $\tilde{T}(y) = \|T\|_{\mathrm{cb}}\Pi(y \otimes 0)$ for all $y \in Y$. E.g., to verify the first part of (ii), let $\delta = 1/\|T^{-1}\|$ (resp. $1/\|T^{-1}\|_{\mathrm{cb}}$) (and assume still $\|T\|_{\mathrm{cb}} = 1$, so $\delta \leq 1$); if $y \in Y$,

$$\begin{aligned} \|\tilde{T}y\| &= \inf\{\|y - x\| + \|Tx\| : Y \in X\} \\ &\geq \inf\{\delta\|y - x\| + \delta\|x\| : x \in X\} \\ &\geq \delta\|y\| \ . \end{aligned}$$

Hence also $\|\tilde{T}\|^{-1} = \frac{1}{\delta} = \|T^{-1}\|$.

Proof of Proposition 1.3. We may assume $\tilde{X} \subset B(H)$ for a suitable Hilbert space H; also we may assume that $\|T\|_{\mathrm{cb}} = 1$. Using the isometric injectivity of $B(H)$, choose $T' : Y \to B(H)$ a linear map extending T with also $\|T'\|_{\mathrm{cb}} = 1$. Now we apply the result in Lemma 1.2 to $Z = B(H)$. Let $\Pi : Y \to Y/X$ be the quotient map and define $\tilde{T} : Y \to B(H) \oplus Y/X$ by

$$\tilde{T}y = T'y \oplus \Pi y \quad \text{for all} \quad y \in Y \ . \tag{9}$$

Now setting $\tilde{Y} = \tilde{T}(Y)$, it follows from Lemma 1.2 that $\tilde{Y}$ is a closed linear subspace of $B(H) \oplus Y/X$ and $\tilde{T}$ is an isomorphism from Y onto $\tilde{Y}$. We claim further that $\tilde{T}$ is indeed a complete isomorphism. Now it follows immediately from (9) and the complete contactivity of T' that $\|\tilde{T}\|_{\mathrm{cb}} = 1$. Let $\mathbf{X} = \mathbf{K} \otimes_{\mathrm{op}} X$, $\mathbf{Y} = \mathbf{K} \otimes_{\mathrm{op}} Y$, and $\mathbf{T} = I \otimes \tilde{T} : \mathbf{Y} \to \mathbf{K} \otimes_{\mathrm{op}} B(H) \overset{\mathrm{df}}{=} \mathbf{Z}$ where $I = \mathrm{Id}\,|\mathbf{K}$. It follows that $\mathbf{T}|\mathbf{X}$ is an isomorphism onto $\mathbf{K} \otimes_{\mathrm{op}} \tilde{X}$ and

$$\|(\mathbf{T}|\mathbf{X})^{-1}\| = \|T^{-1}\|_{\mathrm{cb}}, \|\mathbf{T}\| = \|\tilde{T}\|_{\mathrm{cb}} = 1 \ . \tag{10}$$

Thus we may apply Lemma 1.2. Let $\mathbf{\Pi} : \mathbf{Y} \to \mathbf{Y}/\mathbf{X}$ be the quotient map and define $\tilde{\mathbf{T}}$ as in (1). Thus Lemma 1.2 yields that $\tilde{\mathbf{T}}$ is an isomorphism onto $\mathbf{K} \otimes_{\mathrm{op}} \tilde{Y}$ and (2) yields

$$\|(\tilde{\mathbf{T}})^{-1}\| \leq 2\|(\tilde{\mathbf{T}}|\mathbf{X})^{-1}\| + 1 \ . \tag{11}$$

But also

$$\|(\tilde{\mathbf{T}})^{-1}\| = \|(\tilde{T})^{-1}\|_{\mathrm{cb}} \ . \tag{12}$$

(10)–(12) thus complete the proof. $\square$

The next result is a simple application of Proposition 1.3, following the concept introduced in [Ro2]: *A locally reflexive separable operator space Z is said to have the Complete Separable Complementation Property* (CSCP) *provided every complete isomorph of Z is completely complemented in every separable locally reflexive operator superspace.* Equivalently, given separable operator spaces $X \subset Y$ with Y locally reflexive and $T : X \to Z$ a complete surjective isomorphism, there exists a completely bounded $\tilde{T} : Y \to Z$ extending T. That is, we have the diagram

$$\begin{tikzcd} Y \arrow[drr, "\tilde{T}"] & & \\ \cup & & \\ X \arrow[rr, "T"] & & Z \end{tikzcd} \tag{13}$$
.

Our next result yields some equivalences for the CSCP We obtain a considerably stronger version of one of the equivalences at the end of this section (using Theorem 1.1).

Theorem 1.4. *Let X be a separable locally reflexive operator space. Then the following are equivalent:*

(a) *X has the CSCP.*
(b) *X is completely complemented in every separable locally reflexive superspace.*
(c) *X is locally complemented in every separable locally reflexive operator superspace, and whenever $\tilde{X}$ is a locally complemented subspace of a locally reflexive separable operator superspace $\tilde{Y}$ and $T : \tilde{X} \to X$ is a completely bounded map, there exists a completely bounded map $\tilde{T} : \tilde{Y} \to X$ extending T.*

Remarks.

1. If X and Y are operator spaces with $X \subset Y$, X is said to be *locally complemented* in Y if there is a C so that X is C-completely complemented in Z for all linear subspaces Z with $X \subset Z \subset Y$ and Z/X finite dimensional. (Then one says X is C-locally complemented in Y). A simple compactness argument yields that if X is $(C + \varepsilon)$-locally complemented in Y for all $\varepsilon > 0$, $X^{**} = X^{\perp\perp}$ is C-completely complemented in Y^{**} via a weak*-continuous projection. Conversely, it is proved in Sublemma 3.11 of [Ro2] that *if Y is locally reflexive and X^{**} is completely complemented in Y^{**}, X is locally complemented in Y.*

2. There are many situations in which the hypotheses for $\tilde{X}$ hold without $\tilde{X}$ necessarily being complemented. For example, if $\tilde{X}$ is completely isomorphic to a commutative C^*-algebra or more generally, a nuclear C^*-algebra, then if $\tilde{X} \subset \tilde{Y}$, with $\tilde{Y}$ locally reflexive, $\tilde{X}$ is "automatically" locally complemented in $\tilde{Y}$, but e.g., in the commutative case, there are examples where $\tilde{X}$ and $\tilde{Y}$ are actually commutative separable C^*-algebras with $\tilde{X}$ uncomplemented in $\tilde{Y}$. Perhaps the most general hypothesis on $\tilde{X}$ alone: *if $(\tilde{X})^{**}$ is an isomorphically injective operator space, then $\tilde{X}$ is locally complemented in any locally reflexive operator superspace* (cf. [Ro2]).

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (a): Let $\tilde{X}$ and $\tilde{Y}$ be operator spaces with $\tilde{X} \subset \tilde{Y}$, $\tilde{Y}$ locally reflexive separable, and $\tilde{X}$ completely isomorphic to X. Let $T : \tilde{X} \to X$ be a complete surjective isomorphism. By Proposition 1.3, there exists an operator space $Y \supset X$ and a complete surjective isomorphism $\tilde{T} : \tilde{Y} \to Y$ extending T. Then Y is also separable and locally reflexive, hence there exists a completely bounded projection $P : Y \to X$. Then $\tilde{P} \stackrel{\mathrm{df}}{=} (T|X)^{-1} P \tilde{T}$ is a completely bounded projection from $\tilde{Y}$ into $\tilde{X}$. Thus X has the CSCP.

(c) $\Rightarrow$ (a): The proof is the same. Indeed, letting $\tilde{X}$, $\tilde{Y}$ and $\tilde{T}$ be as above, Y is also separable and locally reflexive, hence since X is thus locally complemented in Y by assumption, X is completely complemented in Y, so $\tilde{X}$ is completely complemented in $\tilde{Y}$ as shown above.

(b) $\Rightarrow$ (c) is an immediate consequence of the following:

Lemma A. *Let Y and Z be locally reflexive operator spaces and X be a locally complemented subspace of Y. Let $T : X \to Z$ be a completely bounded linear operator. Then there exists a locally reflexive operator space $W \supset Z$ and a completely bounded linear operator $\tilde{T} : Y \to W$ with $\tilde{T}|X = T$.*

Remark. We obtain that in fact $\tilde{T}$ may be chosen with $\|\tilde{T}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$ and W is β-locally reflexive, where if X is $(C+\varepsilon)$-locally complemented in Y for all $\varepsilon > 0$, Y is λ-locally reflexive, and Z is γ locally reflexive, then

$$\beta = (\max\{\gamma, \lambda\})(C+1) .$$

We first deduce (b) $\Rightarrow$ (c) of Theorem 1.4 from the Lemma. Simply choose $W \supset X$ locally reflexive and $T' : \tilde{Y} \to W$ a completely bounded map extending T, by the Lemma, and then let $P : W \to X$ be a completely bounded surjective projection; $\tilde{T} = PT'$ is then the desired extension of T. $\square$

Comment. Of course the implication (b) $\Rightarrow$ (a) is superfluous in this entire argument, but its proof is considerably simpler than that of Lemma A.

To prove Lemma A, we require

Lemma B. *Let X be a locally complemented subspace of a locally reflexive operator space Y. Then Y/X is locally reflexive.*

Proof. Assume X is $(C+\varepsilon)$-locally complemented in Y, for all $\varepsilon > 0$. By the Remark following the statement of Theorem 1.4, choose a completely bounded projection $P : Y^{**} \to X^{**}$ with $\|P\|_{\mathrm{cb}} \le C$; let E denote the null space of P. It follows that if $\Pi : Y \to Y/X$ is the quotient map, then $\Pi^{**}|E \to (Y/X)^{**}$ is a complete surjective isomorphism with

$$\|(\Pi^{**}|E)^{-1}\|_{\mathrm{cb}} \le C + 1 .$$

Now let G be a finite-dimensional subspace of $(Y/X)^{**}$ and F be a finite dimensional subspace of $(Y/X)^* = X^\perp$; set $\tilde{G} = (\Pi^{**}|E)^{-1}G$. Now assuming that Y is λ-locally reflexive, given $\varepsilon > 0$, choose $\tilde{T} : \tilde{G} \to Y$ with

$$\|\tilde{T}\|_{\mathrm{cb}} < \lambda + \varepsilon \text{ and } \langle \tilde{T}\tilde{g}, f\rangle = \langle \tilde{g}, f\rangle \text{ for all } \tilde{g} \in \tilde{G},\ f \in F .$$

Finally, define $T : G \to Y/X$ by

$$T = \Pi\tilde{T}(\Pi^{**}|E)^{-1} .$$

Evidently then

$$\|T\|_{\mathrm{cb}} \le (\lambda + \varepsilon)(C+1) .$$

Finally, suppose $g \in G$ and $f \in F$. Then

$$\begin{aligned}
\langle \Pi\tilde{T}(\Pi^{**}|E)^{-1}g, f\rangle &= \langle \tilde{T}(\Pi^{**}|E)^{-1}g, f\rangle \quad (\text{since } f \in X^\perp) \\
&= \langle (\Pi^{**}|E)^{-1}g, f\rangle \\
&= \langle \Pi^{**}(\Pi^{**}|E)^{-1}g, f\rangle \quad (\text{again since } f \in X^\perp) \\
&= \langle g, f\rangle
\end{aligned}$$

$\square$

Remarks. 1. Of course we obtain that Y/X is $\lambda(C+1)$-locally reflexive. Actually, if we assume instead that X^{**} is C-completely cocomplemented in Y^{**}, we have that Y/X is (λC)-locally reflexive. In particular, *if Y is 1-locally reflexive and X^{**} is completely contractively cocomplemented in Y^{**}, Y/X is 1-locally reflexive.*

2. After the "final" draft of our paper was finished, we learned that S-H. Kye and Z-J. Ruan had already obtained a variant of Lemma B in [KR] (see Proposition 5.4 there), as well as an interesting

converse. The work in [KR] contains moreover some remarkable characterizations of λ-injectivity for dual operator spaces.

Proof of Lemma A. We use the construction of G. Pisier mentioned in the Remarks following the statement of Proposition 1.3. Let X, Y, Z and T be as in Proposition 1.3; we may assume that $\|T\|_{\mathrm{cb}} = 1$. Assume then that C, λ and γ are as in the Remark following the statement of Lemma A. Let $E = (Y \oplus Z)_\infty$ and $\Gamma = \{x \oplus -Tx : x \in X\} \subset E$; let $W = E/\Gamma$, $\Pi : E \to W$ the quotient map, and define $j : Z \to W$ and $\tilde{T} : Y \to W$ by $j(z) = \pi(0 \oplus z)$ and $\tilde{T}(y) = \pi(y \oplus 0)$ for all $z \in Z$ and $y \in Y$. Then j is a complete (into) isometry and $jT = \tilde{T}i$, where $i : X \to Y$ is the inclusion map. Thus $\tilde{T}$ is the desired extension with $\|\tilde{T}\|_{\mathrm{cb}} = 1$ also.

Now we have that E is $\max\{\lambda, \gamma\}$ locally reflexive. We claim that Γ *is locally complemented in* E. To see this, let P be the natural projection of E onto Y with nullspace Z and let Λ be a linear subspace of E containing Γ with Γ of finite-codimension in Λ. Now $P\Gamma = X$ and so X is of finite-codimension in $P\Lambda$. Thus given $\varepsilon > 0$, there is a surjective linear projection $Q : P\Lambda \to X$ with $\|Q\|_{\mathrm{cb}} < C + \varepsilon$. Now defining $U : X \to \Gamma$ by $U(x) = x \oplus Tx$ for all $x \in X$, also $\|U\|_{\mathrm{cb}} = 1$ and of course $UX = \Gamma$; we claim that $R \overset{\mathrm{df}}{=} UQP|\Lambda$ is the desired projection onto Γ. Clearly $RE \subset \Gamma$. But let $x \oplus -Tx$ be a a typical element of Γ. Then

$$(UQP)(x \oplus -Tx) = UQx = Ux = x \oplus -Tx \ .$$

We have thus proved that also Γ is $(C+1)$-locally complemented. Thus W is indeed locally reflexive by Lemma B. $\square$

We now proceed with the proof of Theorem 1.1. For the remainder of this section, we assume H is separable; we identify H with ℓ^2 and $B(H)$ with $\mathcal{M}_\infty$, the set of all infinite matrices yielding bounded linear operators on H. $\mathcal{N} \subset \mathcal{M}_\infty$ will be called a *special copy* of $B(H)$ provided there exist infinite pairwise-disjoint subsets $M_1, M_2, \ldots$ of N so that letting $M = \bigcup_{j=1}^\infty M_j$, and letting $m_1^j, m_2^j, \ldots$ be an increasing enumeration of M_j for each j, then $A = (a_{ij})$ in $\mathcal{M}_\infty$ belongs to $\mathcal{N}$ provided

(i) $a_{ij} = 0$ if i or $j \notin M$
(ii) for all $i, j \in M$, if $i \in M_r$ and $j \in M_s$, there exist numbers b_{rs} so that $a_{ij} = b_{rs}$ if $i = m_k^r$ and $j = m_k^s$ for some k, $a_{ij} = 0$ if $i = m_k^r$ and $j = m_\ell^s$ with $k \neq \ell$.

Evidently $\mathcal{N}$ is then a WOT-closed $*$-subalgebra of $B(H)$, $*$-isomorphic to $B(H)$. (We could also define $\mathcal{N}$ "intrinsically" as follows: let $e_1, e_2, \ldots$ be the usual basis for ℓ^2; let $H_j = [e_i]_{i \in M_j}$. For each i, j let E_{ij} be the partial isometry in $B(H)$ with initial domain H_j and final domain H_i, so that $E_{ij}(e_{m_\ell^j}) = e_{m_\ell^i}$, $\ell = 1, 2, \ldots$. Then $\mathcal{N}$ is the WOT closed linear span of the E_{ij}'s.)

Lemma 1.5. *Let Z be a separable closed subspace of $B(H)$. There exists a special copy $\mathcal{N}$ of $B(H)$ so that $Z \oplus \mathcal{N}$ is a complete direct decomposition.*

Remark. In fact we show that letting Q be the projection from $Z \oplus \mathcal{N}$ onto $\mathcal{N}$ with kernel Z, then

$$\|Q\|_{\mathrm{cb}} \le 2 \ . \tag{14}$$

Let us first give the

Proof of Theorem 1.1. Let X, Y and T be as in the statement of 1.1. Since $B(H)$ is 1-injective, we may choose $T' : Y \to B(H)$ a completely bounded linear map extending T with $\|T'\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$. Now set $Z = \overline{T'Y}$. Choose $\mathcal{N}$ a special copy of $B(H)$ satisfying (14) (by Lemma 1.5). Since $\mathcal{N}$ is completely isometric to $B(H)$ and Y/X is separable, we may choose $V : Y/X \to \mathcal{N}$ a complete (into) isometry. Let $\Pi : Y \to Y/X$ denote the quotient map and define $\tilde{T} : Y \to B(H)$ by

$$\tilde{T}y = T'y + V\Pi y \quad \text{for all} \quad y \in Y \ . \tag{15}$$

Now letting $W = Y/X$, we have that the map $U : Z \oplus W \to Z + V(W)$ is a complete isomorphism, where

$$U(z \oplus w) = z + V(w) \quad \text{for all} \quad z \in Z \quad \text{and} \quad w \in W \ .$$

Indeed, letting I denote the identity on $\mathbf{K}$ and P the projection from $Z \oplus W$ onto W with kernel Z and $R = Id - P$, then $U = VP + R$, so

$$\|I \otimes U\| \le \|I \otimes VP\| + \|I \otimes R\| \le 2 \ . \tag{16}$$

Also if $\overline{Q}$ is the projection from $Z + V(W)$ onto $V(W)$ with kernel Z, and $\tau \in \mathbf{K} \otimes V(W)$, then

$$\begin{aligned} \|I \otimes U^{-1}(\tau)\| &= \|I \otimes U^{-1}\overline{Q}\tau + I \otimes U^{-1}(I - \overline{Q})\tau\| \\ &= \max\left\{\|I \otimes U^{-1}\overline{Q}(\tau)\|, \|I \otimes U^{-1}(I - \overline{Q})\tau\|\right\} \\ &\le 3\|\tau\| \quad \text{by (14).} \end{aligned} \tag{17}$$

Thus we have by (16) and (17)

$$\|U\|_{\mathrm{cb}}\|U^{-1}\|_{\mathrm{cb}} \le 6 \ . \tag{18}$$

Now if we instead define $\tilde{\tilde{T}} : Y \to Z \oplus Y/Z$ by $\tilde{\tilde{T}}(y) = T'y \oplus \Pi y$, then by the proof of Proposition 1.3, $\tilde{\tilde{T}}$ is a complete into isomorphism, hence also $\tilde{T} = U\tilde{\tilde{T}}$ is a complete into isomorphism, and we have by (8) that $(*)$ holds. $\square$

We now proceed with the proof of Lemma 1.5. For $M \subset \mathbb{N}$, let $H_M = [e_i : i \in M]$ and let $B_M = PB(H)P$ where P is the orthogonal projection on H_M.

Lemma 1.6. *Let G be a separable subspace of $\mathbf{K}^\perp$. There exists an infinite M with $G \perp B_M$.*

Proof. Say that L, M are almost disjoint if $(L \sim M) \cup (M \sim L)$ is finite. Now if L, M are almost disjoint subsets of $\mathbb{N}$ and $\mu \in \mathbf{K}^\perp$, then

$$\|\mu\| \ge \|\mu|B_L\| + \|\mu|B_M\| \ . \tag{19}$$

Indeed, we may choose $L' \subset L$, $M' \subset M$ with L', M' disjoint and $L \sim L'$, $M \sim M'$ finite. Since $\mu \in \mathbf{K}^\perp$, we have e.g., $\mu|B_{L\sim L'} = 0$, so $\mu|H_L = \mu|B_{L'}$, $\mu|B_M = \mu|B_{M'}$, whence

$$\|\mu|B_L\| + \|\mu|B_M\| = \|\mu|B_{L'}\| + \|\mu|B_{M'}\| = \|\mu|B_{L'\cup M'}\| \le \|\mu\|. \tag{20}$$

It follows immediately that if $M_1, \ldots, M_n$ are pairwise almost disjoint subsets of $\mathbb{N}$, then

$$\|\mu\| \ge \sum_{i=1}^{n} \|\mu(B_{n_i})\| \ . \tag{21}$$

Now by an ancient classical fact, there exists an uncountable family $(M_\alpha)_{\alpha\in\Gamma}$ of almost disjoint infinite subsets of $\mathbb{N}$. (21) yields that for $\mu \in \mathbf{K}^\perp$,

$$\mu|B_{M_\alpha} = 0 \quad \text{for all but countably many } \alpha\text{'s.} \tag{22}$$

Since G is separable, it now follows that for some α, $M|B_{M_\alpha} = 0$ for all $M \in G$, yielding 1.6. $\square$

We need one more ingredient to obtain Lemma 1.5. Let $\mathbf{Ca}$ denote the Calkin algebra $B(H)/\mathbf{K}$, and now let $\Pi : B(H) \to \mathbf{Ca}$ denote the quotient map.

Lemma 1.7. *Let $\mathcal{N}$ be a special copy of $B(H)$. Then $\Pi|\mathcal{N}$ is an (into) $*$-isomorphism. In particular, $\Pi|\mathcal{N}$ is a complete isometry.*

Proof. Since Π is a $*$-isomorphism, we need only observe that $\mathcal{N}$ contains no non-zero compact operators, which is trivial since E_{ij} is a non-finite rank partial isometry for all i, j as in the alternate definition of special $\mathcal{N}$. $\square$

We are finally prepared for the

Proof of Lemma 1.5. Since Z is separable, so also is $\mathbf{K} \otimes \overline{\Pi Z}$; thus we may choose G a separable subspace of $\mathbf{K}^\perp$ so that

$$\mathbf{K} \otimes G \text{ isometrically norms } \mathbf{K} \otimes \overline{\Pi Z} \text{ via the canonical pairing.} \tag{23}$$

Now by Lemma 1.6, choose M an infinite subset of $\mathbb{N}$ so that

$$G \perp B_M \ . \tag{24}$$

Finally, let $M_1, M_2, \ldots$ be infinite disjoint sets with $M = \bigcup M_j$, and let $\mathcal{N}$ be the special copy of $B(H)$ corresponding to the M_j's. Of course then $\mathcal{N} \subset B_M$, and so by (24),

$$g(x) = g(\Pi x) = 0 \ \text{ for all } g \in G \text{ and } x \in \mathcal{N} \ . \tag{25}$$

Now let τ be an element of $\mathbf{K} \otimes (Z \oplus \mathcal{N})$, say

$$\tau = \sum_{i=1}^{k} L_i \otimes (z_i \oplus x_i) \ \text{ where } \ L_i \in \mathbf{K}\ ,\ z_i \in Z \text{ and } x_i \in \mathcal{N} \text{ for all } i\ .$$

It follows from Lemma 1.7 that then

$$\|\sum L_i \otimes x_i\| = \|\sum L_i \otimes \Pi x_i\| \ . \tag{26}$$

Now by (23), given $\varepsilon > 0$, we may choose $S_1, \ldots, S_\ell$ in $\mathbf{K}$ and $g_1, \ldots, g_\ell$ in G so that $\|\sum S_j \otimes g_j\| = 1$ and

$$\begin{aligned} \|\sum L_i \otimes \Pi(z_i)\| &\le (1+\varepsilon)|\langle \sum S_j \otimes g_j, \sum L_i \otimes \Pi z_i\rangle| \\ &= (1+\varepsilon)|\langle \sum S_j \otimes g_j, \sum_i L_i \otimes \Pi(z_i \oplus x_i)\rangle| \text{ (by (25)} \\ &\le (1+\varepsilon)\|\sum L_i \otimes \Pi(z_i \oplus x_i)\| \end{aligned} \tag{26}$$

(where $\langle \sum S_j \otimes g_j, \sum L_i \otimes T_i\rangle = \sum_{i,j} g_i(T_i) S_j \otimes L_i$).
Thus

$$\begin{aligned} \|\sum L_i \otimes x_i\| &= \|\sum L_i \otimes \Pi x_i\| \text{ (by Lemma 1.7)} \\ &\le (2+\varepsilon)\|\sum L_i \otimes \Pi(z_i \oplus x_i)\| \text{ by (26)} \\ &\le (2+\varepsilon)\|\sum L_i \otimes (z_i \oplus x_i)\| \ . \end{aligned} \tag{27}$$

Since $\varepsilon > 0$ is arbitrary, we have indeed proved that if $Q(z+x) = x$ for all $z \in Z$ and $x \in \mathcal{N}$, then $\|Q\|_{\text{cb}} \le 2$. $\square$

We conclude this section with a considerable strengthening of Corollary 1.4.

Corollary 1.8. *Let X be a locally reflexive separable operator space so that X^{**} is completely isomorphic to $B(H)$. Then X has the* CSCP *provided X is completely complemented in every separable locally reflexive Y with*

$$X \subset Y \subset X^{**} \ .$$

Proof. Let $Z \subset W$ be operator spaces with W separable locally reflexive and Z completely isomorphic to X. We must show that Z is completely complemented in W. Let $T : Z \to X$ be a complete surjective isomorphism. By Theorem 1.1, since X^{**} is completely isomorphic to $B(H)$, we may choose Y, $X \subset Y \subset X^{**}$ and $\tilde{T} : W \to Y$ a complete surjective isomorphism extending T. Then Y is locally reflexive, hence there is a completely bounded projection P from Y onto X. Then $Q = T^{-1} P \tilde{T}$ is a completely bounded projection from W onto Z. □

Section 2
An operator space construction on certain subspaces of $\mathcal{M}_\infty$

Definition. Let $W \subset \mathbb{N} \times \mathbb{N}$. M_W denotes all A in $\mathcal{M}_\infty$ with $a_{ij} = 0$ if $(i,j) \notin W$. $\mathbf{K}_W$ denotes $M_W \cap \mathbf{K}$. We *define* an operation on all $\mathbb{N} \times \mathbb{N}$ matrices, denoted P_W, as follows: for any A, $P_W A = B$ where $b_{ij} = a_{ij}$ if $(i,j) \in W$, $b_{ij} = 0$ otherwise. In case $P_W \mathcal{M}_\infty \subset \mathcal{M}_\infty$, it follows immediately that $P_W | \mathcal{M}_\infty$ is bounded; then we set

$$\|P_W\| = \|P_W | \mathcal{M}_\infty\| \ \text{ and } \ \|P_W\|_{\mathrm{cb}} = \|P_W | \mathcal{M}_\infty\|_{\mathrm{cb}}$$

(which a-priori might be infinite).

We may now formulate the main result of this section, which involves the construction of a certain W for which P_W is completely bounded.

Theorem 2.1. *Let Y be a separable closed subspace of $\mathcal{M}_\infty$ with $\mathbf{K} \subset Y$, and let $\varepsilon > 0$. There exists an absolute constant C, a subset W of $\mathbb{N} \times \mathbb{N}$, and a subspace $\tilde{Y}$ of $B(H)$ satisfying the following:*

- (i) $\|P_W\|_{\mathrm{cb}} \le 2$.
- (ii) $d_{\mathrm{cb}}(\mathbf{K}_W, \mathbf{K}_0) \le C$.
- (iii) *There is a complete surjective isomorphism $T : Y \to \tilde{Y}$ with*
  - (a) $\|T\|_{\mathrm{cb}} \|T^{-1}\|_{\mathrm{cb}} \le 1 + \varepsilon$
  - (b) $\|Ty - y\| \le \varepsilon \|y\|$ *for all $y \in Y$*
  - (c) $T|K = I|K$.
- (iv) $\tilde{Y}$ *is invariant under P_W.*
- (v) $P_{\sim W} \tilde{Y} \subset \mathbf{K}$.

Before proving this result, we give two applications.

Theorem 2.2. $\mathbf{K}$ *has the CSCP.*

Proof. By Corollary 1.8, it suffices to prove that if Y is locally reflexive separable and

$$\mathbf{K} \subset Y \subset \mathcal{M}_\infty \ , \tag{28}$$

then $\mathbf{K}$ is completely complemented in Y. Now let $\varepsilon > 0$, and choose W and $\tilde{Y}$ as in Theorem 2.1; also let T satisfy (iii) of Theorem 2.1. Then by (iii)(c) and (iv),

$$\mathbf{K} \subset \tilde{Y} \ \text{ and } \ \mathbf{K}_W \subset \tilde{Y} \ . \tag{29}$$

By (ii) and (iii), $\mathbf{K}_W$ is completely isomorphic to K_0 and $\tilde{Y}$ is separable locally reflexive. Hence by the results in [R], there is a completely bounded projection Q from $\tilde{Y}$ onto $\mathbf{K}_W$. Then using (29) and (iv), (v) of Theorem 2.1, $\tilde{P}$ is a completely bounded projection from $\tilde{Y}$ onto $\mathbf{K}$, where

$$\tilde{P} = (QP_W + P_{\sim W}) | \tilde{Y} \ . \tag{30}$$

Finally, $P \overset{\mathrm{df}}{=} T^{-1} \tilde{P} T$ is a completely bounded projection from Y onto $\mathbf{K}$, completing the proof. □

For our second application, we briefly introduce the concept to be developed in the next section.

Definition. An operator space Z has the *Mixed Separable Extension Property* (MSEP) if for all separable operator spaces Y, subspaces X, and completely bounded maps $T : X \to Z$, there exists a bounded linear map $\tilde{T} : Y \to Z$ extending T.

As we show in Section 3, a separable Z has the MSEP iff Z is complemented in Y for every separable operator space Y with $Z \subset Y \subset B(H)$ (for $Z \hookrightarrow B(H)$ a fixed complete embedding). As noted in the introduction, we do not know if $\mathbf{K}$ has this property. The next result reduces this problem to $\mathbf{K}_0$.

Proposition 2.3. $\mathbf{K}$ *has the* MSEP *if and only if* $\mathbf{K}_0$ *does.*

Proof. If $\mathbf{K}$ has the MSEP so does $\mathbf{K}_0$, because it is completely complemented in $\mathbf{K}$. For the non-trivial implication, suppose $\mathbf{K}_0$ has the MSEP and let Y be a separable operator space satisfying (28). Again, for fixed $\varepsilon > 0$, choose $W, \tilde{Y}$, and T as in Theorem 2.1; then choose Q a bounded linear projection from $\tilde{Y}$ onto $\mathbf{K}_W$. Again, letting $\tilde{P}$ and P be as in the proof of 2.2, it now follows that P is a bounded linear projection from Y onto $\mathbf{K}$. $\square$

We now proceed with the proof of Theorem 2.1. We first isolate part of the proof in the following result:

Lemma 2.4. *Let* $W \subset \mathbb{N} \times \mathbb{N}$ *be described as follows: there exists* (m_j) *in* $\mathbb{N} \cup \{0\}$ *that* $1 = m_0 < m_1 < m_2 < \cdots$, $m_{j+1} - m_j \to \infty$, *so that for all* $(i, j) \in \mathbb{N} \times \mathbb{N}$, $(i, j) \in W$ *iff the following are all satisfied for some* $k = 1, 2, \ldots$;

(a) $m_{k-1} < i \le m_k$ *and* $j \le m_{k+1}$;
(b) $m_{k-1} < j \le m_k$ *and* $i \le m_{k+1}$;
(c) $i = 1$ *and* $j \le m_1$ *or* $j = 1$ *and* $i \le m_1$.

Then $\|P_W\|_{\mathrm{cb}} \le 2$ *and* $d_{\mathrm{cb}}(\mathbf{K}_W, \mathbf{K}_0) \le C$ *for some absolute constant* C.

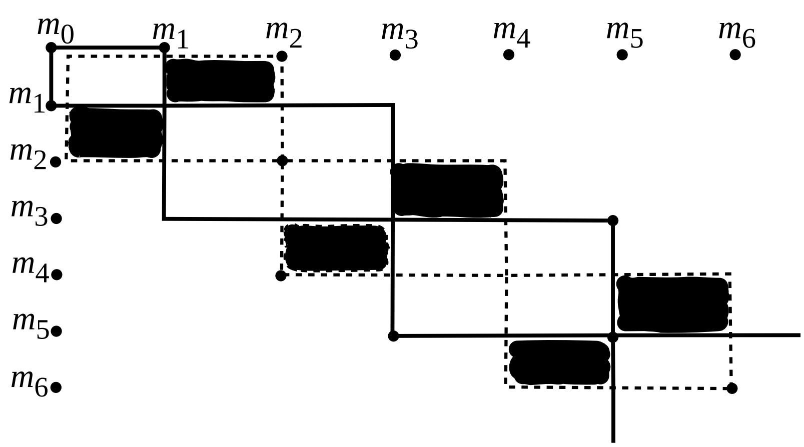
In the following we use interval notation to denote intervals in $\mathbb{N} \cup \{0\}$.

Proof. Let $A_j = (m_{2j-1}, m_{2j+1}] \times (m_{2j-1}, m_{2j+1}]$ for $j \ge 1$,
$A_0 = [m_0, m_1] \times [m_0, m_1]$.
Let $B_j = (m_{2j}, m_{2(j+1)}] \times (m_{2j}, m_{2(j+1)}]$ for $j \ge 0$.
Let $A = \bigcup_{j=0}^{\infty} A_j, \quad B = \bigcup_{j=0}^{\infty} B_j$.

We claim that

$$W = A \cup B\ . \tag{31}$$

The following diagram intuitively illustrates why this is so: the heavy lines denote the A_j's, the dotted lines denote the B_j's. $\bigcup_{j=0}^{2} B_j \sim A$ is shaded in the diagram; the regularity of $\bigcup_{j=0}^{\infty} B_j \sim A$ used in showing that P_W is completely bounded.



First, if (i,j) satisfy (c) of 2.4, then $(i,j)\in A_0$. Now suppose $(i,j)\in W$, and $i>1$, $j>1$. Now if $i\le m_1$ or $j\le m_1$, then $(i,j)\in(m_0,m_2)\times(m_0,m_2]\subset B$ by (a) and (b) of 2.4.

Suppose then $i>m_1$ and $j>m_1$. But then if $i\le m_2$ or $j\le m_2$, $(i,j)\in(m_1,m_3]\times[m_1,m_3]\subset A$. Continuing by induction, we obtain that

$$W\subset A\cup B\ . \tag{32}$$

Next suppose $(i,j)\in[m_0,m_j]\times[m_0,m_1]$. By (c), we may assume $i>1$ and $j>1$. But then i and j satisfy both (a) and (b) for $k=1$, so $(i,j)\in W$. Now suppose $(i,j)\in(m_0,m_2]\times[m_0,j_2]\sim[m_0,m_1]\times[m_0,m_1]$. Then if $1<i\le m_1$, $m_1<j\le m_2$, whence (a) holds for $k=1$ and also (b) holds vacuously for $k=1$, while (a) holds vacuously for $k=2$ and (b) holds for $k=2$.

If $m_1<i\le m_2$, we get that $1<j\le m_1$, so by symmetry again (a) and (b) both hold for $k=1$ and $k=2$. Thus $(m_0,m_2]\times(m_0,m_2]\subset W$. Carrying this one more step for the pattern, now suppose

$$(i,j)\in(m_1,m_3]\times(m_1,m_3]\sim(m_0,m_2]\times(m_0,m_2]\ .$$

Thus if $m_1<i\le m_2$, $m_2<j\le m_3$, whence (a) holds for $k=2$, vacuously for $k=3$, and (b) holds for $k=2$ and $k=3$.

If $m_2<i\le m_3$, then $m_1<j\le m_2$, so again (a) and (b) hold for $k=2$ and $k=3$ by symmetry.

Thus by induction, we obtain that

$$(m_{2j},m_{2j+2}]\times(m_{2j},m_{2j+2}]\ \text{ and }\ (m_{2j+1},m_{2j+3}]\times(m_{2j+1},m_{2j+3}]\subset W$$

for all j, whence

$$A\cup B\subset W\ . \tag{33}$$

Of course (31) is now established via (32) and (33).

Now it is evident that $\|P_E\|_{\mathrm{cb}}=1$ for $E=A,B$, and $A\cap B$. This gives the "easy" estimate

$$\|P_W\|_{\mathrm{cb}}\le 3\ , \tag{34}$$

since $P_W=P_A+P_B-P_{A\cap B}$. We are indebted to T. Schlumprecht for the following better estimate:

$$\|P_{B\sim A}\|_{\mathrm{cb}}=1\ . \tag{35}$$

To see this, fix $j\ge 0$. Then

$$\begin{aligned}(m_{2j},m_{2j+2}]&\times(m_{2j},m_{2j+2}]\sim A\\ &=(m_{2j},m_{2j+1}]\times(m_{2j+1},m_{2j+2}]\cup(m_{2j+1},m_{2j+2}]\times(m_{2j},m_{2j+1}]\ .\end{aligned} \tag{36}$$

Resorting to a simple picture, we thus have that he matrices in $M_{B_j\sim A}$ have the form

$$T=\begin{bmatrix}0 & C\\ D & 0\end{bmatrix}$$

whence $\|T\|=\max\{\|C\|,\|D\|\}$ and so $\|P_{B_j\sim A}\|_{\mathrm{cb}}=1$. Since B is the union of the disjoint blocks $B_1,B_2,\dots$, (35) follows.

It remains to prove the final assertion of 2.4. In the following, the letter "c" denotes absolute constants, which may vary from line to line.

First, via the Pełczyński decomposition method, we obtain the following

Fact. Let (n_j) be a sequence of positive integers with $\sup_j n_j = \infty$. Then

$$d_{\mathrm{cb}}\left(\Big(\bigoplus_{j=1}^{\infty} M_{n_j}\Big)_{c_0}, \mathbf{K}_0\right) \le c\ .$$

(In fact, here one may take $c = 2$.) It then follows immediately that

$$d_{\mathrm{cb}}(\mathbf{K}_A, \mathbf{K}_0) \le c \tag{37}$$

(for c in the Fact). Next, we define C, D by

$$C = \bigcup_{j=0}^{\infty} (m_{2j}, m_{2j+1}] \times (m_{2j+1}, m_{2j+2}] \tag{38i}$$

$$D = \bigcup_{j=0}^{\infty} (m_{2j+1}, m_{2j+2}] \times (m_{2j}, m_{2j+1}]\ . \tag{38ii}$$

Again by the Fact, we obtain

$$d_{\mathrm{cb}}(\mathbf{K}_E, \mathbf{K}_0) \le c \quad \text{for} \quad E = C \text{ or } D\ . \tag{39}$$

Finally, we have

$$K_W = K_A \oplus K_C \oplus K_D\ . \tag{40}$$

Indeed, $\|P_E\|_{\mathrm{cb}} = 1$ for $E = A$, C or D, and $W = A \cup C \cup D$ by (31) and (38). We then have that

$$d_{\mathrm{cb}}(\mathbf{K}_W, (\mathbf{K}_A \oplus \mathbf{K}_C \oplus \mathbf{K}_D)_\infty) \le 3 \tag{41}$$

and by the Fact

$$d_{\mathrm{cb}}((\mathbf{K}_A \oplus \mathbf{K}_C \oplus \mathbf{K}_D)_\infty, \mathbf{K}_0) \le c \tag{42}$$

completing the proof. $\square$

Remark. The following intriguing problem arises: *characterize the sets $W \subset \mathbb{N} \times \mathbb{N}$ so that P_W is bounded*. A related problem: *if P_W is bounded, is it completely bounded?*

Proof of Theorem 2.1.
Let $\Pi : Y \to Y/\mathbf{K}$ be the quotient map; without loss of generality, $Y/\mathbf{K}$ is infinite-dimensional. Choose $y_1, y_2, \dots$ in Y so that (y_j) is bounded and (Πy_j) is a bounded biorthogonal system spanning $\Pi(Y)$. Thus, we may choose $M < \infty$ and (y_j^*) in $Y^\perp$ so that for all j and k,

$$y_j^*(y_k) = \delta_{jk} \quad , \quad \|y_j^*\| \le M \text{ and } \|y_j\| \le M\ . \tag{43}$$

Let $\varepsilon > 0$ be given and set $m_0 = 0$. We shall construct a sequence (m_j) in $\mathbb{N}$ and certain sequences $(y_i^{(j)})$ in Y.

Step 1. Choose $m_1 \in \mathbb{N}$ so that

$$\|P_{\{1\}\times(m_1,\infty)} y_1\| + \|P_{(m_1,\infty)\times\{1\}} y_1\| < \frac{\varepsilon}{2}\ . \tag{44}$$

Now define $y_1^{(1)} = y_1$ and $y_j^{(1)} = P_{(m_1,\infty)\times(m_1,\infty)} y_j$ for all $j > 1$.

Step 2. Choose $m_2 > m_1$ so that

$$\|P_{[1,m_1]\times(m_2,\infty)} y_i^{(1)}\| + \|P_{(m_2,\infty)\times[1,m_1]} y_i^{(1)}\| < \frac{\varepsilon}{2^2} \tag{45}$$

for $1 \le i \le 2$.

Now set $y_i^{(1)} = y_i^{(2)}$ for $i \le 2$,

$$y_i^{(2)} = P_{(m_2,\infty)\times(m_2,\infty)} y_i^{(1)}) \ \text{ for } i > 2 \ .$$

Step j. Suppose $j > 2$ and $m_1 < \cdots < m_{j-1}$, $(y_i^{(s)})_{i=1}^\infty$ have been chosen, for all $1 \le s \le j-1$. Choose $m_j > m_{j-1}$ so that

$$\|P_{[1,m_{j-1}]\times(m_j,\infty)} y_i^{(j-1)}\| + \|P_{(m_j,\infty)\times[1,m_{j-1}]} y_i^{j-1}\| < \frac{\varepsilon}{2} \ , \tag{46}$$

for all i, $1 \le i \le j$.

Now set $y_i^{(j)} = y_i^{(j-1)}$ for $1 \le i \le j$, then set

$$y_i^{(j)} = P_{(m_j,\infty)\times(m_j,\infty)} y_i^{(j-1)} \ \text{ for all } i > j \ . \tag{47}$$

This completes the inductive construction. Then we have that for all $j \ge 1$,

$$y_i^{(j)} = P_{(m_j,\infty)\times(m_j,\infty)} y_i \ \text{ for all } \ i > j \ . \tag{48}$$

Hence

$$y_j^{(j)} = P_{(m_{j-1},\infty)\times(m_{j-1},\infty)} y_j \ \text{ for all } \ j > 1 \text{ and } y_1^{(1)} = y_1 \ . \tag{49}$$

Thus it follows that $y_j - y_j^{(j)}$ is a finite rank operator for all j, whence

$$\Pi y_j^{(j)} = \Pi y_j \ \text{ and } \ \|y_j^{(j)}\| \le M \text{ for all } j \ . \tag{50}$$

Now let W be defined as in Lemma 2.4. That is, instead defining $C_0, C_1, C_2, \dots$

$$C_k = (m_{k-1}, m_k] \times (m_{k+1}, \infty) \cup (m_{k+1}, \infty) \times (m_{k-1}, m_k] \ \text{ for } k \ge 1 \ , \tag{51}$$

$$C_0 = \{1\} \times (m_1, \infty) \cup (m_1, \infty) \times \{1\} \ , \tag{52}$$

then

$$W = \sim C \ \text{ where } \ C = \bigcup_{j=0}^\infty C_j \ . \tag{53}$$

Now fix $1 \le i$. Then by (44)–(46), (49) and (51)–(52),

$$\|P_{C_{j-1}} y_i^{(i)}\| < \frac{\varepsilon}{2^j} \ \text{ if } \ j \ge i \ . \tag{54}$$

But by (49),

$$P_{C_{j-1}} y_i^i = 0 \ \text{ if } \ i > j \ . \tag{55}$$

We thus obtain for all i, from (54) and (55), that

$$\|P_C y_i^{(i)}\| = \Big\| \sum_{j=i}^{\infty} P_{C_{j-1}} y_i^{(i)} \Big\| \le \sum_{j=i}^{\infty} \|P_{C_{j-1}} y_i^i\| < \frac{\varepsilon}{2^{i-1}} \ . \tag{56}$$

We next define the operator T specified in the statement of 2.1. First, let Y_0 denote the *linear span* of $\mathbf{K}$ and the $y_i^{(i)}$'s. Note that the $y_i^{(i)}$'s are linearly independent over $\mathbf{K}$. We first define T on Y_0. For $S \in \mathbf{K}$ and scalars $c_1, \dots, c_n$, set

$$y = \sum_{j=1}^{n} c_j y_j^{(j)} \tag{57}$$

and define

$$T(S+y) = S + P_W y \ . \tag{58}$$

Now if we assume that

$$\|S+y\| = 1 \ , \tag{59}$$

then

$$|c_j| = |y_j^*(S+y)| \le M \quad \text{for all} \quad j \ . \tag{60}$$

But then

$$\|T(S+y) - (S+y)\| = \Big\| \sum_{j=1}^{n} c_j P_C y_j^{(j)} \Big\| = 2M\varepsilon \text{ by (56) and (59).} \tag{61}$$

Now it follows immediately that T extends to a bounded linear operator (also denoted T) from Y into $B(H)$, satisfying

$$\|Ty - y\| \le 2M\varepsilon \quad \text{for all} \quad y \in Y \ . \tag{62}$$

Now if we assume (as we may) that $2M\varepsilon < 1$, then setting $\tilde{T} = TY$, $\tilde{Y}$ is a closed linear subspace of $B(H)$ and T maps Y one-to-one onto $\tilde{Y}$. Moreover, since TY_0 is invariant under P_W, so is $\tilde{Y}$. We now have that (i), (ii), (iv) of Theorem 2.1 hold (by Proposition 2.2), and furthermore (b) and (c) of (iii) hold. It remains to verify (iii)(a) and (v) of 2.1. Now (v) is easy, for suppose $z \in Y_0$, $z = S + y$, $S \in K$, y as in (57). Then $P_{\sim W} TZ = P_{\sim W} S + P_{\sim W} y$, but $P_{\sim W} y$ is actually an absolutely converging series of finite rank operators by (56). Thus $P_{\sim W} TZ \in K$, proving 2.1(v).

Finally, for each j, define a rank-one operator $F_j : Y \to B(H)$ by

$$F_j(y) = y_j^*(y) P_C y_j^{(j)} \ . \tag{63}$$

Then it follows from (56) and (43) that

$$\|F_j\|_{\mathrm{cb}} < \frac{M\varepsilon}{2^{j-1}} \quad \text{for all} \quad j \ . \tag{64}$$

Then setting $Q = \sum F_j$, Q is of course also completely bounded, and

$$\|Q\|_{\mathrm{cb}} < 2M\varepsilon \ . \tag{65}$$

Now an inspection of the definition of T on Y_0 yields that

$$Tz = z - Qz \quad \text{for all} \quad z \in Y \ . \tag{66}$$

But then we obtain

$$\|T\|_{\mathrm{cb}} \|T^{-1}\|_{\mathrm{cb}} < \frac{2M\varepsilon}{1 - 2M\varepsilon} \tag{67}$$

which of course (qualitatively) yields 2.1(iii)(a). $\square$

Remark. Let us say that $T \in M_\infty$ is a *generalized block diagonal* (gbd) if there exists a W of the form given in Proposition 2.2 so that $T = P_W T$. The following is a byproduct of our proof of Theorem 2.1: *Every operator in M_∞ is (for every $\varepsilon > 0$) a perturbation of a gbd operator by a compact operator of norm less than ε.*

Section 3

The λ-Mixed Separable Extension Property and Extendably Locally Reflexive Banach spaces

We first give the quantitative version of the property introduced in the preceding section.

Definition. *Let $\lambda \geq 1$. An operator space Z has the λ-Mixed Separable Extension Property* (λ-MSEP) *if for all separable operator spaces Y, subspaces X, and completely bounded maps $T : X \to Z$, there exists a bounded linear map $\tilde{T} : Y \to Z$ extending T with $\|\tilde{T}\| \leq \lambda \|T\|_{\text{cb}}$.*

Next, we give a simple result summarizing various permanence properties of the MSEP for separable operator spaces Z.

Proposition 3.1. *Let Z be a separable operator space and assume $Z \subset B(H)$ for some H. Then the following are equivalent.*

(a) *Z has the* MSEP.
(b) *Z is complemented in Y for all separable spaces Y with $Z \subset Y \subset B(H)$.*
(c) *Z has the λ-*MSEP *for some $\lambda \geq 1$.*

Moreover, fixing $\lambda \geq 1$, then the following are equivalent.

(a′) *Z has the λ-*MSEP.
(b′) *Z is λ-complemented in Y for all Y as in (b).*
(c′) *Z is λ-complemented in every separable operator superspace.*

Proof. (a) $\Rightarrow$ (b) is essentially trivial, for let $T : Z \to Z$ be the identity map; a bounded linear extension $\tilde{T} : Y \to Z$ is a bounded projection onto Z.

(b) $\Rightarrow$ (c) and (b′) $\Rightarrow$ (a′). We first observe that there is a λ' so that (b′) holds. If not, choose for every n, Y_n a separable superspace of Z contained in $B(H)$ so that Z is not n-complemented in Y_n. Then letting $Y = [Y_j : j = 1, 2, \ldots]$, Y is a separable superspace of Z contained in $B(H)$, and Z is uncomplemented in Y, contradicting (b). Now assuming (b′), it suffices to show that (a′) holds.

Suppose then X, Y are separable operator spaces with $X \subset Y$ and $T : X \to Z$ is completely bounded. By the isometric operator injectivity of $B(H)$, we may choose $S : Y \to B(H)$ with $\|S\|_{\text{cb}} = \|T\|_{\text{cb}}$ and $S|X = T$. Then letting $\tilde{Y} = [S(Y), Z]$, $\tilde{Y}$ is a separable superspace of Z and hence there is a projection $P : \tilde{Y} \to Z$ onto Z with $\|P\| \leq \lambda$. Then $\tilde{T} \overset{\text{df}}{=} PS$ is the desired extension of T with $\|\tilde{T}\| \leq \lambda \|T\|_{\text{cb}}$. This completes the proof, in view of the triviality of the implications (a′) $\Rightarrow$ (c′) $\Rightarrow$ (b′). $\square$

Although we are mainly interested in the separable case, we next note that the equivalence (a) $\Rightarrow$ (c) of Proposition 2.1 holds in general.

Proposition 3.2. *Let Z be an operator space with the* MSEP. *Then Z has the λ-*MSEP *for some $\lambda \geq 1$.*

Proof. If not, we may choose for every n, operator spaces X_n and Y_n with $X_n \subset Y_n$ and $T_n : X_n \to Z$ with $\|T_n\|_{\text{cb}} = \frac{1}{n^2}$ so that

$$\|\tilde{T}_n\| > n \text{ for any } \tilde{T}_n : Y_n \to Z \text{ with } \tilde{T}_n | X_n = T_n \ . \tag{68}$$

Let now $Y = (Y_1 \oplus Y_2 \oplus \cdots)c_0$ and $X = (X_1 \oplus X_2 \oplus \cdots)c_0$ endowed with the standard operator space structure. Of course Y is separable. Define $T : X \to Z$ by

$$T(x_j) = \sum T_j x_j \ . \tag{69}$$

T is well defined, since Z is a Banach space, and if $(x_j) \in X$, then $\sum \|T_j x_j\| \leq \sum \frac{1}{n^2} \|x\|$.

Now given m, $K_1, \dots, K_m$ in $\mathbf{K}$, and $z', \dots, z^m$ in X, we have that

$$\begin{aligned} \Big\| \sum K_i \otimes Tz^i \Big\| &\le \sum_j \Big\| \sum_i K_i \otimes Tz^i_j \Big\| \\ &\le \sum_j \|T_j\|_{\mathrm{cb}} \Big\| \sum_i K_i \otimes Tz^i_j \Big\| \\ &\le \sum \frac{1}{n^2} \max_j \Big\| \sum_i K_i \otimes Tz^i_j \Big\| \\ &= \sum \frac{1}{n^2} \Big\| \sum K_i z^i \Big\| \ . \end{aligned} \tag{70}$$

Hence T is completely bounded, but there is no bounded linear extension $\tilde{T} : Y \to Z$ since for such a presumed extension, $\tilde{T}/Y_n$ extends T_n, whence $\|\tilde{T}|Y_n\| > n$. □

Of course the MSEP is related to a more restrictive injectivity property.

Definition. An operator space Z is *mixed injective* if for all operator spaces Y, subspaces X, and completely bounded maps $T : X \to Z$, there is a bounded linear map $\tilde{T} : Y \to Z$ extending T. If, for $\lambda \ge 1$, $\tilde{T}$ can always be chosen with $\|\tilde{T}\| \le \lambda \|T\|_{\mathrm{cb}}$, we say Z is *λ-mixed injective*. Finally, if Z is 1-mixed injective, we say that Z is *isometrically mixed injective*.

We then have the following result, whose simple proof (via the isometric operator injectivity of $B(H)$) is left to the reader.

Proposition. *Let Z be an operator space with $Z \subset B(H)$ for some H. Then Z is mixed injective iff Z is complemented in $B(H)$. Hence Z is λ-mixed injective for some $\lambda \ge 1$. Moreover if $\lambda \ge 1$, then Z is λ-mixed injective if Z is λ-complemented in $B(H)$.*

As pointed out in the Introduction, we not not know if $\mathbf{K}_0$ has the MSEP. The next result shows this problem is equivalent to the question of whether $\mathbf{K}_0$ is complemented in Y for all separable Y with $\mathbf{K}_0 \subset Y \subset \mathbf{K}_0^{**}$, in virtue of the fact that $\mathbf{K}_0^{**}$ is an (isometrically)-injective operator space.

Proposition 3.3. *Let Z be a separable operator space so that Z^{**} is mixed injective. Then Z has the* MSEP *iff*

$$(*) \qquad Z \text{ is complemented in } W \text{ for all separable spaces } W \text{ with } Z \subset W \subset Z^{**}$$

Proof. One implication is trivial. For the slightly less trivial assertion, let $X \subset Y$ be separable operator spaces and $T : X \to Z$ a completely bounded map. Choose $\tilde{T} : Y \to Z^{**}$ a bounded linear extension of χT, where $\chi : Z \to Z^{**}$ is the canonical injection. Let $Y = [\chi Z, \tilde{T}(Y)]$ and let $P : Y \to Z$ be a surjective bounded linear projection (where Z is of course identified with χZ). Then $P\tilde{T}$ is the desired operator extending T. □

Remark. Of course Proposition 3.2 "reduces" the problem of the MSEP for $\mathbf{K}_0$, to a pure Banach space question: See [JO] for a study of the family of separable Banach spaces Z satisfying $(*)$, particularly in the case where $Z = (E_n)_{c_0}$, $E_1, E_2, \dots$ finite-dimensional.

The next perhaps surprising result shows that the MSEP and mixed injectivity are equivalent for operator spaces complemented in their double duals.

Proposition 3.4. *Let X be an operator space which is β-complemented in X^{**} and suppose X has the λ-MSEP. Then X is $\lambda\beta$-mixed injective.*

Corollary. *Every reflexive operator space with the* MSEP *is mixed injective.*

Proof of Proposition 3.4. Let Y be an operator super space of X. By Proposition 3.1, it suffices to prove that X is $\lambda\beta$-complemented in Y.

First, fix F a finite-dimensional subspace of X. We shall prove:

$$\text{there exists a linear operator } T_F = T,\ T : Y \to X^{**}, \quad \text{with } \|T\| \leq \lambda \text{ and } T|F = I|F\ . \tag{71}$$

Let $\mathcal{G}$ be the family of finite-dimensional subspaces of Y containing F, directed by inclusion. For each $G \in \mathcal{G}$, since X has the λ-MSEP, choose $T_G : G \to X$ a linear operator with $\|T_G\| \leq \lambda$ and $T_G|F = I|F$. Then define $\tilde{T}_G : Y \to X$ by $\tilde{T}_G(y) = 0$ if $y \notin G$, $\tilde{T}_G(y) = T_G(y)$ if $y \in G$. Well, $\tilde{T}_G$ is neither continuous nor linear. However the weak*-compactness of the λ-ball of X^{**} in its weak*-topology allows us by the Tychonoff theorem to select a subnet $(\tilde{T}_{G_\beta})_{\beta\in\mathcal{D}}$ of the net $(\tilde{T}_G)_{G\in\mathcal{G}}$ so that

$$\lim_{\beta\in\mathcal{D}} \tilde{T}_{\alpha_\beta}(y) \stackrel{\mathrm{df}}{=} T(y) \tag{72}$$

exists weak* in X^{**} for all $y \in Ba(Y)$. Since we do have that $\tilde{T}_G(\lambda y) = \lambda\tilde{T}_G(y)$ for all $y \in Y$, we obtain that the limit in (72) exists weak* for all y in Y, and in fact we discover that T as defined by (72) is indeed a linear operator with $\|T\| \leq \lambda$. Finally, if $f \in F$, then $\tilde{T}_\alpha(f) = f$ for all f, whence also $Tf = f$. Thus (71) is proved.

Finally, let $\mathcal{F}$ be the family of finite-dimensional subspaces of X directed by inclusion. For each $F \in \mathcal{F}$, choose T_F satisfying (71). Again exploiting the weak*-compactness of the λ-ball of X^{**}, we find a subnet $(T_{F_\beta})_{\beta\in\mathcal{D}}$ of the net $(T_F)_{F\in\mathcal{F}}$ so that

$$\lim_{\beta\in\mathcal{D}} T_{F_\beta}(y) \stackrel{\mathrm{df}}{=} S(y) \tag{73}$$

exists weak* in X^{**} for all $y \in Ba(y)$. Now it follows that $S : Y \to X^{**}$ is a linear operator with $\|S\| \leq \lambda$. But if $x \in X$, then "eventually", $x \in F_\beta$ for $\beta \in \mathcal{D}$, whence $T_{F_\beta}(x) = x$, so also $S(x) = x$. Finally, letting $Q : X^{**} \to X$ be a surjective projection with $\|Q\| \leq \beta$, we obtain that $P = QS$ is the desired projection from Y onto X of norm at most $\beta\lambda$. $\square$

Remarks. 1. Of course the proof shows that if X is complemented in X^{**}, then X is mixed injective if X has the formally weaker property that for some $\lambda \geq 1$ and for all finite-dimensional operator spaces $F \subset G$ and linear maps $T : F \to X$, there is a linear extension $\tilde{T} : G \to X$ with $\|\tilde{T}\| \leq \lambda\|T\|_{\mathrm{cb}}$.

2. The same compactness argument also yields that if X is an operator space with X completely complemented in X^{**}, then if X has the CSEP, X is injective. (This strengthens Proposition 2.10 of [Ro2].) Indeed, as noted in [Ro2], it follows that X has the λ-CSEP for some $\lambda \geq 1$. But then just replacing "bounded" by "completely bounded" in the above proof, one obtains that if X is β-completely complemented in X^{**}, then X is $\beta\lambda$-completely complemented in Y.

We next note that certain results in [R] carry over almost word for word to the mixed category.

Proposition 3.5. *Let X be a non-reflexive operator space. If X is mixed injective, X has a subspace Banach-isomorphic to ℓ^∞. If X is separable with the* MSEP*, X has a subspace Banach-isomorphic to c_0.*

Proof. If X satisfies the hypothesis in the second statement, X is isomorphic (in fact completely isomorphic) to a complemented subspace of some C^*-algebra. The second assertion now follows from results of H. Pfister [Pf] and A. Pełczyński [Pe2]. If X satisfies the first hypothesis, X is completely isomorphic to a complemented subspace of some von-Neumann algebra. The first assertion now follows from these results and the result of [Ro1] that any non-weakly compact operator from ℓ^∞ into some Banach space fixes a copy of ℓ^∞. The argument itself is word for word as the proof of Proposition 2.8 of [Ro2], deleting the word "completely" in all its occurrences. $\square$

Finally, we note the analogue of Proposition 2.22 of [Ro2].

Proposition 3.6. *Let X be a separable operator space with the* λ-MSEP. *If $\lambda < 2$, then X is reflexive (and hence is λ-mixed injective by Proposition 3.3).*

Proof. The argument is essentially the same as that for Proposition 2.22 of [Ro2]. We give this argument however, for the sake of completeness. Suppose to the contrary that X is not reflexive. Then X contains a subspace isomorphic to c_0 by Proposition 3.5. Now let $\varepsilon > 0$, to be decided later, and choose (using the "folklore" result, proved in Proposition 2.22 of [Ro2]) a subspace Z of X which is Banach $(1+\varepsilon)$-isomorphic to c_0 and $(1+\varepsilon)$-complemented in X. Now let Y be a separable subspace of Z^{**} with $Z \subset Y$ and let $i : Z \to X$ be the identity injection, and also let $P : X \to Z$ be a surjective projection with $\|P\| < 1+\varepsilon$. Since X has the λ-MSEP, letting Y have its natural operator space structure, we find a bounded linear extension $\tilde{\imath} : Y \to X$ with $\|\tilde{\imath}\| \le \lambda$. But then letting $Q = P\tilde{\imath}$, Q is a projection from Y onto Z and

$$\|Q\| < (1+\varepsilon)\lambda \ . \tag{74}$$

Since Z is $(1+\varepsilon)$-isomorphic to c_0, it now follows that if $\tilde{Y}$ is separable with $c_0 \subset \tilde{Y} \subset \ell^\infty$, then

$$c_0 \text{ is } (1+\varepsilon)^2\lambda\text{-complemented in } \tilde{Y} \ . \tag{75}$$

This implies c_0 itself has the $(1+\varepsilon)^2\lambda$ SEP, whence by a result of Sobczyk [S], $(1+\varepsilon)^2\lambda \ge 2$. Of course this is a contradiction for ε small enough. $\square$

We now give some examples of operator spaces with the MSEP. Evidently any complemented subspace of an operator space with the MSEP also has the MSEP. The next result is thus an immediate consequence of a result in [Ro2].

Proposition 3.7. *Let X be a λ-complemented subspace of $c_0(R \oplus C)$. Then X has the* 2λ-MSEP.

Proof. It is proved in [Ro2] that $c_0(R\oplus C)$ has the 2-CSEP, hence trivially the 2-MSEP. $\square$

Of course $c_0(R\oplus C)$ is Banach isomorphic to $c_0(\ell^2)$, and the infinite-dimensional complemented subspaces of $c_0(\ell^2)$ have been isomorphically classified in [BCLT]; there are exactly six of them.

Problem 3.1. *Let X be an infinite-dimensional separable operator space with the* MSEP. *Is X Banach isomorphic to one of the spaces*

$$c_0, \quad (\ell^2_n)_{c_0}, \quad c_0(\ell^2), \quad \ell^2, \quad c_0 \oplus \ell^2 \ , \quad \textit{or} \ \ (\ell^2_n)_{c_0} \oplus \ell^2 \ ? \tag{76}$$

Of course if $\mathbf{K}$ (or equivalently $\mathbf{K}_0$) *has* the MSEP, the answer is negative, and the list must be much bigger than this. It is worth pointing out, however, that work of G. Pisier yields immediately that *every reflexive mixed injective operator space is Hilbertian*, i.e., Banach isomorphic to a Hilbert space (cf. [R]). Thus the list in (76) *is* complete in the separable reflexive case; ℓ^2 is the only example!

We next give some examples of 1-mixed injective operator spaces. Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ and $n, m \in \mathbb{N}^*$. Recall that $\mathcal{M}_\infty$ denotes $B(\ell^2)$ regarded as matrices operating on the natural basis.

Now the following are all 1-mixed injective.

I. $\mathcal{M}_{n,m}$
II. $\mathcal{S}_n$, the $n \times n$ symmetric matrices
III. $\mathcal{AS}_n$, the anti-symmetric $n \times n$ matrices.

(If A^t denotes the transpose of A, then $A \in \mathcal{S}_n$ iff $A = A^t$; $A \in \mathcal{AS}_n$ iff $A = -A^t$.)

Neither $\mathcal{S}_\infty$ nor $\mathcal{AS}_\infty$ are injective, while of course $\mathcal{M}_{n,m}$ is 1-injective for all n and m. However *another* family of 1-mixed injectives occurs; the *spin factors*.

Definition. A closed subspace X of $B(H)$ is called a *spin factor* if

(a) *X is self-adjoint*
(b) $\dim X > 1$
(c) the square of every element of X is a scalar.

It is known that spin-factors are 1-mixed injective [ES] and Hilbertian. Moreover, in the separable case, X is a spin-factor iff there exists a sequence $S_1, S_2, \ldots$ of anti-commuting self-adjoint unitaries with $X = [S_n]$. Here, (S_n) is either finite of length at least 2, or infinite. X, as an operator space, is determined up to complete isometry by its dimension (i.e., the length of this sequence (S_n)). For $n \in \mathbb{N}^* \sim \{1\}$, let $\mathcal{S}p(n)$ denote a spin factor of dimension n for $n < \infty$ (resp. separable infinite-dimensional if $n = \infty$).

Standard constructions yield that for all n, $\mathcal{S}p(n)$ is 1-completely isometric to a (necessarily contractively complemented) subspace of $\mathcal{M}_{2^{n/2}}$ if n is even, $\mathcal{M}_{2^{[n/2]}} \oplus \mathcal{M}_{2^{[n/2]}}$ if n is odd.

However the following result yields that $\mathcal{S}p(\infty)$ is not completely isomorphic to a subspace of K.

Proposition 3.8. *Let X be an operator space so that $X \otimes_{\mathrm{op}} X$ is completely isomorphic to a subspace of X. If $\mathcal{S}p(\infty)$ completely embeds in X, then ℓ^1 Banach embeds in X, hence X^* is non-separable.*

Proof. By a result of U. Haagerup [H] (see also [Pa]), if (S_i) is an infinite spin system in $B(H)$, $(S_i \otimes S_i)$ is Banach-equivalent to the usual ℓ^1-basis. Now if Y is a subspace of X which is completely isomorphic to $\mathcal{S}p(\infty)$, $Y \otimes_{\mathrm{op}} Y$ is completely isomorphic to $\mathcal{S}p(\infty) \otimes_{\mathrm{op}} \mathcal{S}p(\infty)$, hence ℓ^1 embeds in $X \otimes_{\mathrm{op}} X$ by Paulsen's results. $\square$

We next give a remarkable, simple permanence property of mixed injectivity and the Mixed Separable Extension Property. We first need the following (apparently new) concept.

Definition. Given operator spaces X and Y, X is *completely semi-isomorphic* to Y if there exists a completely bounded map $T : X \to Y$ which is a Banach isomorphism, i.e., (since X, Y are assumed complete), so that T is 1–1 and onto. We call such a map T a *complete semi-isomorphism* from X onto Y. In case $\|T\|_{\mathrm{cb}} = 1 = \|T^{-1}\|$, T is called a *complete semi-isometry* and X is said to be *completely semi-isometric* to Y in case there exists a complete semi-isometry mapping X onto Y. In case $\|T\|_{\mathrm{cb}} \|T^{-1}\| \leq \lambda$, we say X is λ-completely semi-isomorphic to Y. Finally, we set $d_s(X, Y) = \inf\{\lambda : X \text{ is completely semi-isomorphic to } Y\}$.

This relation is trivially reflexive and is also (quantitatively) transitive: if X is λ-completely semi-isomorphic to Y and Y is β-completely semi-isomorphic to Z, then X is $\lambda\beta$-completely semi-isomorphic to Z. The relation is of course not symmetric in general; e.g., $R \cap C$ is completely semi-isomorphic to R but R is not completely semi-isomorphic to $R \cap C$. It can be shown that the relation does not yield a partial order on operator spaces modulo complete isomorphism. In fact, there exist non-completely isomorphic operator spaces X and Y so that each is completely semi-isometric to the other. However if X and Y are each completely semi-isomorphic to the other, then X and Y *are* completely isomorphic if one of them, say X, is homogeneous, i.e., if every bounded operator on X is completely bounded. Indeed, suppose X is λ-homogeneous (i.e., $\|U\|_{\mathrm{cb}} \leq \lambda \|U\|$ for all $U \in \mathcal{L}(X)$) and suppose $T : X \to Y$ and $S : Y \to X$ are surjective complete semi-isomorphisms. But then $T^{-1} S^{-1}$ is completely bounded, hence so is $T^{-1} = T^{-1} S^{-1} S$, and $\|T^{-1}\|_{\mathrm{cb}} \leq \lambda \|T^{-1}\| \, \|S^{-1}\| \, \|S\|_{\mathrm{cb}}$. We thus obtain that $d_{\mathrm{cb}}(X, Y) \leq \lambda d_s(X, Y) d_s(Y, X)$.

We now give the permanence property mentioned above: mixed injectivity and the MSEP are both preserved under complete semi-isomorphisms; i.e., if X is completely semi-isomorphic to Y and Y has one of these properties, so does X.

Proposition 3.9. *Let $\lambda, \beta \geq 1$ and let Z and $\tilde{Z}$ be operator spaces with $\tilde{Z}$ β-completely semi-isomorphic to Z. Then if Z is λ-mixed injective (resp. has the λ-*MSEP*), $\tilde{Z}$ is $\beta\lambda$-mixed injective (resp. has the $\lambda\beta$-*MSEP*).*

Proof. Choose $S : \tilde{Z} \to Z$ a surjective complete semi-isomorphism with $\|S\|_{\mathrm{cb}} \|S^{-1}\| \leq \beta$. Let $X \subset Y$ be operator spaces and $T : X \to \tilde{Z}$ be a completely bounded map. Now consider the diagram:

$$\begin{tikzcd} Y \arrow[ddrr, "V"] & & \\ \cup & & \\ X \arrow[r, "T"] & \tilde{Z} \arrow[r, "S"] & Z \end{tikzcd} \quad .$$

In the case where Z is λ-mixed injective, choose V a linear operator completing this diagram with

$$\|V\| \leq \|S \circ T\|_{\mathrm{cb}} \leq \lambda \|T\|_{\mathrm{cb}} \|S\|_{\mathrm{cb}} \ . \tag{77}$$

Then letting $\tilde{T} = S^{-1}V$, we obtain that $\tilde{T} : Y \to \tilde{Z}$ is an extension of T with

$$\begin{aligned} \|\tilde{T}\| &\leq \lambda \|S\|_{\mathrm{cb}} \|S^{-1}\| \, \|T\|_{\mathrm{cb}} \qquad \text{by (77)} \\ &\leq \lambda\beta \|T\|_{\mathrm{cb}} \ . \end{aligned} \tag{78}$$

Of course (78) yields $\tilde{Z}$ is $\lambda\beta$-mixed injective.

In the case where Z has the λ-MSEP, simply assume that Y is in addition separable, to obtain the desired conclusion. $\square$

Proposition 3.9 has the immediate consequence: If X is completely semi-isomorphic to a space with the CSEP, X has the MSEP. This suggests the following question.

Problem 3.2. *Let X be a separable operator space with the* MSEP*. Is X completely semi-isomorphic to a space with the* CSEP*?*

We do not know if $\mathbf{K}$ is semi-isomorphic to a space with the CSEP, although we suspect this is not the case. Let us note, however, that the presently known examples of separable spaces with the MSEP do have the property specified in this problem, e.g., $\mathcal{S}p(\infty)$ is completely semi-isomorphic to R.

The 1-mixed injective finite-dimensional spaces are completely classified up to Banach isometry, based in part on deep work of E. Cartan [C] for which there seems to be no decent modern exposition.

Theorem A. *Let X be a finite-dimensional isometrically injective operator space. Then X is Banach isometric to a (finite) ℓ^∞-direct sum of spaces E each of the following form for some $m, n \in \mathbb{N}$*

I. $E = \mathcal{M}_{n,m}$
II. $E = \mathcal{S}_n$
III. $E = \mathcal{AS}_n$
IV. $E = \mathcal{S}p(n)$

Spaces of the form I–IV are known as *Cartan* factors of types I–IV. Of course any ℓ^∞ finite direct sum of Cartan factors of types I–IV *is* isometrically mixed injective (in its natural operator space structure). Now Proposition 3.8 coupled with the spaces listed in Theorem A, yields a rather vast supply of finite-dimensional 1-mixed injectives (e.g., $R_n \cap C_n$ is of this form). Are these the only ones?

Problem 3.3. *Is every finite dimensional 1-mixed injective completely semi-isometric to an ℓ^∞ direct sum of Cartan factors of types I–IV?*

The work in [AF] is certainly related to this problem, especially if the answer is negative! Problem 3.2 ought to be solved in this century! The next problem, on the other hand, seems quite intractable at this time (although a negative answer need not be). An affirmative solution would imply an affirmative solution to the famous "finite-dimensional $\mathcal{P}_\lambda$ problem" in the commutative theory.

Problem 3.4. *Given $\lambda > 1$, is there a β so that every λ-mixed injective finite-dimensional space is β-completely semi-isomorphic to a 1-mixed injective?*

A remarkable factorization theorem due to M. Junge [J] yields that a purely local formulation of the classification problem for finite-dimensional mixed injectives. We are indebted to M. Junge for the proof of this result, which yields that the finite-dimensional β-mixed injectives are essentially, up to complete semi-isomorphism, the β-complemented subspaces of $\mathcal{M}_n$'s.

Proposition 3.10. *Let X be a finite-dimensional operator space and $\lambda \geq 1$. The following are equivalent:*

(1) *X is λ-mixed injective.*
(2) *For all $\varepsilon > 0$, there exist an n and linear maps $U : X \to \mathcal{M}_n$ and $V : \mathcal{M}_n \to X$ so that $I_X = VU$ and $\|V\|\,\|U\|_{\mathrm{cb}} < \lambda + \varepsilon$. That is, we have the diagram*

$$\begin{tikzcd} & \mathcal{M}_n \arrow[dr, "V"] & \\ X \arrow[ur, "U"] \arrow[rr, "I"] & & X \end{tikzcd} .$$

Corollary 3.11. *If X is finite-dimensional and λ-mixed injective, then for all $\varepsilon > 0$, there is a subspace Y of $\mathcal{M}_n$ so that X is $(\lambda+\varepsilon)$-completely semi-isomorphic to Y and Y is $(\lambda+\varepsilon)$-mixed injective.*

Remark. Of course the conclusion of the Corollary implies that Y is $(\lambda+\varepsilon)$-Banach complemented in $\mathcal{M}_n$.

Proof of 3.11. Set $Y = U(X)$, where U, V are chosen as in (2) of 3.10. It follows that $U : X \to Y$ is a semi-isomorphism with $U^{-1} = V|_Y$, hence $d_s(X,Y) \leq \|V|_Y\|\,\|U\|_{\mathrm{cb}} < \lambda + \varepsilon$. Setting $P = UV$, then P is a projection from $\mathcal{M}_n$ onto Y, and $\|P\| \leq \|U\|\,\|V\| < \lambda + \varepsilon$, as desired. $\square$

Proof of Proposition 3.10. (1) $\Rightarrow$ (2): Assume without loss of generality that $X \subset B(H)$. Then there exists a surjective linear projection $P : B(H) \to X$ with $\|P\| \leq \lambda$. Let $Y = (X, \mathrm{MIN})$, and let $T : X \to Y$ be the formal identity map and $i : X \to B(H)$ be the identity injection. Thus T completely factors through $B(H)$, $T = Pi$, and $\|P\|_{\mathrm{cb}}\|i\|_{\mathrm{cb}} \leq \lambda$. Hence by a basic factorization theorem in [J] (reproved as Theorem 7.6 in [EJR]; see also Remark 3.6 in [JM]), we may choose n and linear maps $U : X \to \mathcal{M}_n$, $\tilde{V} : \mathcal{M}_n \to Y$ with $T = \tilde{V}U$ and $\|\tilde{V}\|_{\mathrm{cb}}\|U\|_{\mathrm{cb}} < \|P\|_{\mathrm{cb}}\|i\|_{\mathrm{cb}} + \varepsilon \leq \lambda + \varepsilon$. But now just letting $V = T^{-1}\tilde{V}$, we obtain (2) of 3.10, for trivially $\|V\| \leq \|T^{-1}\|\,\|\tilde{V}\|_{\mathrm{cb}} = \|\tilde{V}\|_{\mathrm{cb}}$.

(2) $\Rightarrow$ (1): Let $Y \subset Z$ be separable operator spaces and $T : Y \to X$ be a given linear map. Let $\varepsilon > 0$ and choose $\mathcal{M}_n$ and U, V as in (2). Now since $\mathcal{M}_n$ is 1-injective, choose $S : Z \to \mathcal{M}_n$ a linear map with $\|S\|_{\mathrm{cb}} = \|UT\|_{\mathrm{cb}} \leq \|U\|_{\mathrm{cb}}\|T\|_{\mathrm{cb}}$. Thus we have the diagram

$$\begin{tikzcd} & & \mathcal{M}_n \arrow[dr, "V"] & \\ Y \arrow[urr, "S"] \arrow[r, "T"] & X \arrow[ur, "U"] \arrow[rr, "I"] & & X \end{tikzcd}$$

Then $\tilde{T}_\varepsilon \overset{\mathrm{df}}{=} VS$ extends T and $\|\tilde{T}_\varepsilon\|_{\mathrm{cb}} < \lambda+\varepsilon$. Since X is finite-dimensional, we may choose a sequence (ε_n) tending to zero and an operator $\tilde{T} : Z \to X$ so that $\tilde{T}_{\varepsilon_n} \to \tilde{T}$ in the strong operator topology. It follows that $\tilde{T}$ extends T and $\|\tilde{T}\|_{\mathrm{cb}} \leq \lambda$. Thus X has the λ-MSEP, so by Proposition 3.4, X is λ-mixed injective. $\square$

We briefly indicate the remarkable connection of Theorem A with a rather vast domain of modern research. A closed linear subspace X of $B(H)$ is called a (concrete) JC^*-triple if

$$TT^*T \in X \quad \text{whenever} \quad T \in X . \tag{79}$$

It then follows by polarization that

$$2\{A,B,C\} \stackrel{\text{df}}{=} AB^*C + CB^*A \text{ belongs to } X \text{ whenever } A,B,C \text{ do.} \tag{80}$$

$\{A,B,C\}$ is called the *triple product* of A, B, and C; an abstract generalization of this led to the theory of JB^* triples. In turn, this theory yields the following remarkable general result, which includes Theorem A.

Theorem B. *Let X be a finite-dimensional Banach space. The following are equivalent.*

1. *X is isometric to a contractively complemented subspace of some C^*-algebra.*
2. *X is isometric to a JC^*-triple.*
3. *The open unit ball of X is biholomorphically transitive, and X contains no contractively complemented subspace isometric to a Cartan factor of type V.*
4. *X is of the form specified in Theorem A.*

Several of the implications in Theorem B hold in infinite-dimensions as well. In fact, it is known that $1 \Rightarrow 2 \Rightarrow 3$ in general and finally if X satisfies 3. and is isometric to a dual space, then $3 \Rightarrow 1$. $1 \Rightarrow 2$ is due to Y. Friedman and B. Russo [FR] and $2 \Rightarrow 3$ (without the assertion concerning the type V factor, which came later) is due to L. Harris [H]; see also [K2]. For $3 \Rightarrow 1$ for X a dual, see C.-H. Chu and B. Iochum [CI]. As far as we know, the following are open questions in general: Does $3 \Rightarrow 1$? Does $2 \Rightarrow 1$? The profound result of Cartan's which underlies this: *the unit ball of a finite-dimensional Banach space is biholomorphically transitive iff the space is an ℓ^∞-direct sum of Cartan factors*. In addition to the factors of types I–IV, there are two more, types V and VI; the type VI factor consists of the 3×3 Hermitian matrices over the complex octonions, and is 27-dimensional. The type V factor embeds in this one; as a Banach space, it may however be explicitly identified as follows: Let $e_0, e_1, \dots, e_7$ be the usual basis for the complex octonions O (with e_0 the identity). For $a \in O$, $a = \sum_{i=0}^7 a_i e_i$ with the a_i's complex scalars, set $|a| = (\sum |a_i|^2)^{1/2}$ and $n(a) = \sum a_i^2$; also set $\tilde{a} = a_0 e_0 - \sum_{i=1}^7 a_i e_i$. Now the Cartan factor of type V may be identified with $X = O \times O$ where, if $x = (a,b)$, then

$$\|x\|^2 = |a|^2 + |b|^2 + \sqrt{(|a|^2+|b|^2)^2 - |n(a)|^2 + |n(b)|^2 + 2|\tilde{a}\,b|^2}\ . \tag{81}$$

Note that if a and b have only real coefficients, $|\tilde{a}\,b| = |\tilde{a}|\,|b| = |a|\,|b|$ by a fundamental property of the real octonions, whence

$\|x\|^2 = |a|^2 + |b|^2$, the ordinary Euclidean norm of the vector x.

(The industrious reader may dig the proof of Theorem B out of the references [C], [FR], [K1], [K2]. See also [H] for important earlier structure results on JC^*-triples. Also see [CI] and [LO], [D], and finally [Ru] for a comprehensive survey on JB^*-triples. Also, although (81) is a simple deduction from known work, this explicit expression for the actual *norm* on the type V Cartan factor, seems to be new.)

To further penetrate the fundamental question of whether $\mathbf{K}$ has the MSEP, we introduce the following new concept in pure Banach space theory.

Definition. A Banach space X is called *Extendably Locally Reflexive* (ELR) if there exists a $\lambda \geq 1$ so that for all finite dimensional subspaces F and G of X^* and X^{**} respectively and all $\varepsilon > 0$, there exists an operator $T : X^{**} \to X^{**}$ with

$$\begin{cases} \text{(i)} & TG \subset X \\ \text{(ii)} & \langle Tg, f\rangle = \langle g, f\rangle \text{ for all } g \in G,\ f \in F \\ \text{(iii)} & \|T\| < \lambda + \varepsilon\ . \end{cases} \tag{82}$$

In case λ works, we say X is λ-ELR.

The terminology is motivated as follows: by the Local Reflexivity Principle due jointly to J. Lindenstrauss and the second author of this paper [LR2] (see also [JRZ]); *for all X, and F,G as above, $\varepsilon > 0$, there exists an operator $T : G \to X$ with $\|T\| < 1+\varepsilon$ and satisfying* (82)(ii). Then X is ELR precisely when there exist such operators which admit uniformly bounded extensions $\tilde{T}$ to all of X^{**}, i.e., we have

$$\begin{array}{ccc} X^{**} & \xrightarrow{\tilde{T}} & X^{**} \\ \bigcup & & \bigcup \\ G & \xrightarrow{T} & X \end{array} \quad \text{with } \|\tilde{T}\| \le C$$

for some absolute constant C.

The next result yields several equivalences for Extendable Local Reflexivity.

Proposition 3.12. *Let $\lambda \ge 1$, X a given Banach space. The following are equivalent:*

(i) *X is λ-ELR.*

(ii) *there exists a net (T_α) of linear operators on X^{**} with $\|T_\alpha\| \le \lambda$ for all α, so that for all $x^{**} \in X^{**}$*

   (a) *$T_\alpha x^{**} \to x^{**}$ weak**

and

   (b) *$T_\alpha x^{**}$ is ultimately in X.*

(iii) *same as (ii), with the addition*

   (c) *$T_\alpha x \to x$ in norm, for all $x \in X$.*

(iv) *for all F,G finite-dimensional subspaces of X^* and X^{**} respectively, there is an operator $T : X^{**} \to X^{**}$ satisfying* (82)(i)–(iii) *and in addition*

$$Tg = g \text{ for all } g \in G \cap X \ . \tag{83}$$

Proof. (i) $\Rightarrow$ (ii). Let $\mathcal{D} = \{F,G,\varepsilon : F,G$ are finite-dimensional subspaces of X^* and X^{**} respectively, and $0 < \varepsilon < 1$. Direct $\mathcal{D}$ by:

$(F,G,\varepsilon) \le (F',G',\varepsilon')$ if $F \subset F'$, $G \subset G'$, and $\varepsilon' \le \varepsilon$.

Given $\alpha = (F,G,\varepsilon)$ in $\mathcal{D}$, choose $T\alpha = T$ satisfying (82), and set $T_\alpha = \frac{\lambda}{\lambda+\varepsilon} T_\alpha$. Then $(T_\alpha)_{\alpha\in\mathcal{D}}$ has the desired property.

(ii) $\Rightarrow$ (iii). Let $\mathcal{D}$ be the directed set given in the above proof, and also suppose the net satisfying (ii) is given by $(T_\alpha)_{\alpha\in\mathcal{G}}$. Now given $d = (F,G,\varepsilon)$ in $\mathcal{D}$, choose $\beta \in \mathcal{G}$ so that for all $\alpha \ge \beta$,

$$\begin{aligned} &\text{(i)} \quad T_\alpha G \subset X \\ &\text{(ii)} \quad |\langle T_\alpha g, f\rangle - \langle g,f\rangle| \le \varepsilon \|g\|\,\|f\| \text{ for all } g \in G,\ f \in F \ . \end{aligned} \tag{84}$$

Now if $x \in G \cap X$, then $T_\alpha x \to x$ weakly; hence certain far out convex combinations converge in norm. But then, thanks to the finite-dimensionality of G, we may choose a convex combination S_d of $\{T_\alpha : \alpha \ge \beta\}$ so that

$$\|S_d x - x\| \le \varepsilon \|x\| \text{ for all } x \in G \cap X \ . \tag{85}$$

Now S_d still satisfies (84)(i) (replacing "T_α" by "S_d" there) and of course $\|S_d\| \le \lambda$ also, hence it follows that the net $(S_d)_{d\in\mathcal{D}}$, satisfies the conclusion of (iii).

(iv) $\Rightarrow$ (i) – trivial.

(iii) $\Rightarrow$ (iv). Let $\varepsilon > 0$ and fix F,G finite-dimensional subspaces of X^*, X^{**} respectively, choose $f_1,\dots,f_n$ a basis for F, and choose $x_1,\dots,x_n$ in X with $f_i(x_j) = \delta_{ij}$ for all i and j. Now assuming (T_α) satisfies (iii), we may choose α so that (84)(i) holds and also

$$\begin{aligned} &\text{(i)} \quad |\langle T_\alpha g - g), f_i\rangle| \le \delta \|g\| \text{ for all } g \in G \\ &\text{(ii)} \quad \|T_\alpha g - g\| \le \varepsilon \|g\| \text{ for all } g \in G \cap X \ . \end{aligned} \tag{86}$$

Now also choose Y a linear subspace of G with

$$Y \oplus (G \cap X) = G \ ; \tag{87}$$

then choose P, Q linear projections onto Y, $G \cap X$ respectively so that

$$G \cap X \subset \ker P \text{ and } Y \subset \ker Q \ . \tag{88}$$

Of course then by (87) and (88), $QP = PQ = 0$, and

$$P|G \text{ is a projection onto } Y \text{ with kernel } G \cap X, \quad \text{and } Q|G = (I - P)|G \ . \tag{89}$$

Let $\delta > 0$, to be decided later. Assuming (T_α) satisfies (iii) of the Theorem, choose α so that (84)(i) holds and also

$$\begin{aligned} &\text{(i)} \quad |\langle (T_\alpha g - g), f_i \rangle| \leq \delta \|g\| \text{ for all } g \in G \\ &\text{(ii)} \quad \|T_\alpha g - g\| \leq \delta \|g\| \text{ for all } g \in G \cap X \ . \end{aligned} \tag{90}$$

Then define $T : X^{**} \to X^{**}$ by

$$Tz = \sum_{j=1}^{n} \langle Pz - T_\alpha Pz, f_j \rangle x_j + T_\alpha Pz + Qz + T_\alpha Rz \tag{91}$$

for all $z \in X^{**}$. Now if $z \in G \cap X$, then $P(z) = R(z) = 0$ and $Q(z) = z$, so $Tz = z$; hence (83) holds, and so

$$\langle Tz, f \rangle = \langle z, f \rangle \text{ for all } f \in F \ . \tag{92}$$

If $z \in Y$, then $Q(z) = R(z) = 0$ and $P(z) = z$; whence

$$Tz = \sum_{j=1}^{n} \langle z, f_j \rangle x_j - \sum_{j=1}^{n} \langle T_\alpha z, f_j \rangle x_j + T_\alpha z \ . \tag{93}$$

But then for each j,

$$\langle Tz, f_j \rangle = \langle z, f_j \rangle - \langle T_\alpha x, f_j \rangle + \langle T_\alpha z, f_j \rangle = \langle z, f_j \rangle \ . \tag{94}$$

Since the f_j's are a basis for F, (92) holds. But then since (93) holds for $z \in G \cap X$ and $z \in Y$, (82)(ii) holds. Finally, we estimate the norm of T. Now fixing z in X^{**}, $z = Pz + Qz + Rz$. Hence

$$(T - T_\alpha)z = \sum_{j=1}^{n} \langle Pz - T_\alpha Pz, f_j \rangle x_j + Qz - T_\alpha Qz \ . \tag{95}$$

Thus, we obtain by (90) that

$$\|(T - T_\alpha)z\| = \delta \|P\| \sum_{j=1}^{n} \|x_j\| \, \|z\| + \delta \|Q\| \, \|z\| \ . \tag{96}$$

Hence, simply choosing δ so small that

$$\delta \biggl(\|P\| \sum_{j=1}^{n} \|x_j\| + \|Q\| \biggr) < \varepsilon \ , \tag{97}$$

we obtain that the finite-rank perturbation $T - T_\alpha$ of T has norm smaller than α, whence

$$\|T\| < \|T_\alpha\| + \varepsilon \leq \lambda + \varepsilon \ . \tag{98}$$

completing the proof of Proposition 3.12. $\square$

Although not evident from the definition of the Extendable Local Reflexivity, there is an astonishing connection between this property and the bounded approximation property (the bap).

Theorem 3.13. *Let X be a given Banach space. The following assertions are equivalent.*

(i) *X is* ELR *and has the* bap.

(ii) *There exists a uniformly bounded net (T_α) of finite rank operators on X^{**} with $T_\alpha x^{**} \to x^{**}$ weak* for all $x^{**} \in X^{**}$.*

(iii) *X^* has the* bap.

Remarks.

1. The second author of this paper discovered the ELR concept as well as the implication (iii) $\Rightarrow$ (i) during a research visit to Odense University, November 1997. (Of course X^* has bap $\Rightarrow X$ has bap is an old standard result.) The implication (i) $\Rightarrow$ (iii) was discovered by the authors of [JO] shortly after an initial draft of the present paper was prepared.

2. Our proof yields that one can choose a net (T_α) satisfying (ii) with $\|T_\alpha\| \leq \lambda$ for all α iff X^* has the λ-bap. On the other hand, if X is λ-ELR and has the β-bap, X^* has the $\lambda\beta$-bap (as also obtained in [JO]).

3. It follows immediately from Theorem 3.13 and a deep result of T. Szankowski [S] that C_1 *fails* to be ELR (C_1 the trace class operators on Hilbert space). This and other examples of Banach spaces failing to be ELR are given in [JO].

Proof of Theorem 3.13. (i) $\Rightarrow$ (ii). Suppose X is λ-ELR and has the β-bap. Let $\mathcal{D}$ be the directed set given in the proof of (i) $\Rightarrow$ (ii) of the preceding Proposition. Given $\alpha = (F, G, \varepsilon)$ in $\mathcal{D}$, first choose $T : X^{**} \to X^{**}$ satisfying (82). Now choose $S : X \to X$ a finite rank operator with

$$\|S\| < \beta + \varepsilon \text{ and } Sx = x \text{ for all } x \in TG\ . \tag{99}$$

Hence

$$\langle S^{**}Tg, f\rangle = \langle STg, f\rangle = \langle g, f\rangle \text{ for all } g \in G,\ f \in F,\ \text{ by (82) and (99)}\ . \tag{100}$$

Finally, let $U_\alpha = S^{**}T$. Then $U_\alpha x^{**} \to x^{**}$ ω^* for all $x^{**} \in X^{**}$,

$$\|U_\alpha\| \leq (\lambda + \varepsilon)(\beta + \varepsilon)$$

for all α, and moreover $\overline{\lim}_{\alpha\in\mathcal{D}} \|U_\alpha\| \leq \lambda\beta$. So if we let $T_\alpha = \frac{\lambda\beta}{(\alpha+\varepsilon)(\beta+\varepsilon)} U_\alpha$, then (T_α) satisfies (ii) with $\|T_\alpha\| \leq \lambda\beta$ for all α (as claimed in Remark 1 above).

(ii) $\Rightarrow$ (iii). Let (T_α) be a net of finite rank operators satisfying (ii), and suppose $\|T_\alpha\| \leq \lambda$ for all α. Now since the T_α's are finite rank, it follows that we may assume the T_α's are *weak*-continuous*. To see this, again let $\mathcal{D}$ be the directed set given above. For $\beta \in \mathcal{D}$, $\beta = (F, G, \varepsilon)$, choose α so that

$$|\langle T_\alpha g, f\rangle - \langle g, f\rangle| \leq (1 + \varepsilon)\|g\|\,\|f\| \tag{101}$$

for all $g \in G$ and $f \in F$.
Then applying the local reflexivity principle (see Lemma 3.1 of [JRZ]), choose $\tilde{T}_\beta$ a weak* continuous finite rank operator on X^{**} so that

$$\|\tilde{T}_\beta\| < \lambda + \varepsilon \text{ and } \tilde{T}_\beta | G = T_\alpha | G\ . \tag{102}$$

It then follows that $\tilde{T}_\beta \to I$ weak* on X^{**}, and finally so does the net $(\frac{\lambda}{\lambda+\varepsilon}\tilde{T}_\beta)_{\beta\in\mathcal{D}}$.

Now choose for all α, S_α a linear operator on X^* with $S_\alpha^* = T_\alpha$, $\|S_\alpha\| \leq \lambda$. But then it follows immediately from (ii) that

$$S_\alpha f \to f \quad \text{weakly for all} \quad f \in X^*\ . \tag{103}$$

But then there exists a net (V_α) of convex combinations of the S_α's so that $V_\alpha f \to f$ in norm for all f in X^*. Hence X^* has the λ-bap.

(iii) $\Rightarrow$ (i). Suppose X^* has the λ-bap. Then it is a standard fact that X has the λ-bap. (Actually, our proof that in Proposition 3.12, (ii) $\Rightarrow$ (iii), already gives the argument.) Now let $\varepsilon > 0$, and let F and G be finite dimensional subspaces of X and X^* respectively.

Then choose S a finite rank operator on X^* so that

$$S|F = I|F \quad \text{and} \quad \|S\| < \lambda + \varepsilon \ . \tag{104}$$

Next, let $H = S^* X^{**}$. H is finite dimensional so by the local reflexivity principle, we may choose $U : H \to X$ with

$$\|U\| < 1 + \varepsilon \quad \text{and} \quad \langle Uh, f\rangle = \langle h, f\rangle \text{ for all } h \in H \text{ and } f \in F \ . \tag{105}$$

Finally, let $T = US^*$. Then for all $g \in G$ and $f \in F$,

$$\begin{aligned} \langle Tg, f\rangle = \langle US^* g, f\rangle &= \langle S^* g, f\rangle \quad \text{by (105)} \\ &= \langle g, Sf\rangle = \langle g, Sf\rangle \quad \text{by (104).} \end{aligned} \tag{106}$$

Of course $\|T\| < (1+\varepsilon)(\lambda+\varepsilon)$, whence since $\lambda > 0$ is arbitrary, X is λ-ELR. $\square$

Remark. Extendable Local Reflexivity may easily be extended to the complete category, and then the quantized versions of our results are valid. Thus, we *define* an operator space X to be Completely Extendably Locally Reflexive (CELR) if there is a $\lambda \geq 1$ so that for all $\varepsilon > 0$ and finite-dimensional subspaces F and G of X^* and X^{**}, (82) holds, except that we replace "$\|T\|$" by "$\|T\|_{\mathrm{cb}}$" in (82)(iii). In case λ works, we say X is λ-CELR. We then obtain that appropriate quantized versions of Proposition 3.12 and Theorem 3.13 are valid. Thus in Proposition 3.12, we replace "Banach" by "operator", "λ-ELR" by "λ-CELR" and "$\|T_\alpha\|$" by "$\|T_\alpha\|_{\mathrm{cb}}$".

A quantized version of Theorem 3.13 goes as follows:

Theorem 3.13′. *Let X be a given operator space. Then the following are equivalent.*

(i) *X is* CELR *and has the* cbap.
(ii) *There exists a uniformly completely bounded net (T_α) of weak*-continuous finite-rank operators from X^{**} to X with $T_\alpha x^{**} \to x^{**}$ weak* for all $x^{**} \in X^{**}$.*
(ii)′ *X is locally reflexive and X satisfies (ii) without assuming the T_α's are weak*-continuous.*
(iii) *X^* has the* cbap *and X is locally reflexive.*

Also the quantitative statements go through; if X^* has the λ-cbap and X is β-locally reflexive, then X is $\lambda\beta$-CELR and has the $\lambda\beta$-cbap, while if X is λ-CELR and has the β-cbap, then X^* has the $\lambda^2\beta$-cbap. Moreover if (T_α) satisfies (ii) with $\|T_\alpha\|_{\mathrm{cb}} \leq \lambda$ for all α, then X and X^* both have the λ-cbap and X is λ-CELR. It then follows that nuclear C^*-algebras are 1-CELR, for it is known that such are 1-locally reflexive with duals having the 1 cbap which are also 1-locally reflexive [EJR].

We are indebted to N. Ozawa for pointing out that the implication (iii) $\Rightarrow$ (i) is false without the assumption that X is locally reflexive. Actually, we construct a non-reflexive operator space Y in Corollary 4.9 so that $\mathbf{K}_0 \subset Y \subset \mathbf{K}_0^{**}$ with $Y/\mathbf{K}_0$ completely isometric to c_0. As pointed out to us by N. Ozawa, since $\mathbf{K}_0$ is a complete M-ideal $\mathbf{K}_0^{**}$, Y^* is completely isometric to $C_1 \oplus \ell^1$ (ℓ^1-direct sum), whence Y^* has the cmap and moreover Y^{**} is isometrically injective. It can also be seen (using arguments similar to those for Theorem 4.7 below), that Y fails the cbap, thus answering a question of Ozawa's.

The next result yields an unusual connection between Extendable Local Reflexivity and the CSCP.

Theorem 3.14. *Let $X \subset Y$ be separable operator spaces so that X has the* CSCP *and X^{**} is isomorphically mixed injective. Suppose there exists an operator space Z which is (Banach)* ELR *and $X \subset Y \subset Z$. Then X is complemented in Y.*

First, an immediate consequence.

Corollary 3.15. *Suppose $B(H)$ or every separable C^* algebra is* ELR*, and let X be an operator space with the* CSCP *so that X^{**} is mixed injective. Then X has the* MSEP*; in particular, K has the* MSEP*.*

Proof. Let X have the CSCP and suppose $X \subset Y \subset B(H)$ with Y separable. If $B(H)$ is ELR, X is complemented in Y by Theorem 3.14. Letting $\mathcal{A}$ be the C^*-algebra generated by Y, $\mathcal{A}$ is separable, so again X is complemented in $\mathcal{A}$ and hence in Y by Theorem 3.14. □

Theorem 3.14 is a simple consequence (via known results) of the crucial

Lemma 3.16. *Let $X \subset Y$ be operator spaces. Assume the following:*

(i) *X is locally reflexive*
(ii) *X^{**} is complemented in Y^{**}*
(iii) *Y is* ELR.

Then there is a NEW *operator space structure on Y, agreeing (isometrically) with the given one on X, so that (Y, NEW) is locally reflexive.*

Remark. Our proof yields that if X is λ-locally reflexive, X^{**} is β-cocomplemented in Y^{**}, and Y is γ-ELR, then (Y, NEW) is $(\gamma\beta + \lambda\beta + \lambda)$-locally reflexive.

Proof of Theorem 3.14. Since X^{**} is mixed injective, X^{**} is complemented in Z^{**}. By the Lemma, choose a NEW operator structure on Z which agrees with that on X so that (Z, NEW) is locally reflexive. But then (Y, NEW) is also locally reflexive, (see [ER]). Hence X is completely complemented in (Y, NEW), which of course gives that X is complemented in Y. □

We now proceed with the proof of Lemma 3.16. The idea goes as follows. By a standard Banach space construction (which we give), $X^{\perp\perp}$ is in fact weak*-complemented in X^{**}. Now letting Z be a weak* complement, (Y, NEW) is defined in such a way that (Y^{**}, NEW) *coincides* on $X^{\perp\perp}$ with the given operator space structure, while it is *equivalent* to MAX on Z. The hypothesis that Y is ELR then allows us to obtain a "local reflexivity operator" $T_1 : G_1 \to Y$, for given $G_1 \subset Z$ finite-dimensional, with T_1 uniformly completely bounded, and also given $G_2 \subset X^{**}$, $T_2 : G_2 \to X$ is found by the local reflexivity of X. Then if $G = G_1 \oplus G_2$, $T = T_1 \oplus T_2$ is the desired local reflexivity operator.

Proof of Lemma 3.16. We identify X^{**} with $X^{\perp\perp}$, $X^{\perp}$ with $(Y/X)^*$, and, as usual, Z with its canonical embedding in Z^{**}, (for any Banach space Z). We first have (the standard result) that the hypotheses are equivalent to: X^{**} is weak*-complemented in Y^{**}. In fact, fix $\beta \geq 1$.

Fact. X^{**} is β-cocomplemented in Y^{**} iff $X^{\perp}$ is β-complemented in Y^*.

Proof. Let $L : Y^{***} \to Y^*$ be the canonical projection defined by

$$\langle Ly^{***}, y\rangle = \langle y^{***}, y\rangle \quad \text{for all} \quad y^{***} \in Y^{***},\ y \in Y \ . \tag{107}$$

Now suppose first that $P : Y^{**} \to Y^{**}$ is a projection with $\ker P = X^{\perp\perp}$ and $\|P\| \leq \beta$. Define Q by

$$Q = L \circ (P^* | Y^*) \ . \tag{108}$$

Now we claim that Q is a projection on Y^*, onto $X^{\perp}$; of course it's trivial that $\|Q\| \leq \beta$.

By definition, we have for all $y^* \in Y^*$ and $y \in Y$, that

$$\langle Qy^*, y\rangle = \langle LP^* y^*, y\rangle = \langle P^* y^*, y\rangle = \langle y^*, Py\rangle \ . \tag{109}$$

Now suppose first $y^* \in X^{\perp}$. Then for $y \in Y$,

$$\begin{aligned} \langle Qy^*, y\rangle &= \langle y^*, Py\rangle \quad \text{by (109)}, \\ &= \langle y^*, y\rangle \quad \text{since} \quad y - Py \in X^{\perp\perp} \ . \end{aligned} \tag{110}$$

Hence $Qy^* = y^*$. On the other hand, if y^* is arbitrary and $x \in X$, then

$$\langle Qy^*, x\rangle = \langle y^*, Px\rangle = 0 \tag{111}$$

since $x \in X^{\perp\perp}$, (109) and (111) prove our claim. Of course conversely if $Q : Y^* \to X^\perp$ is a projection with $\|Q\| \le \beta$, $Q^* \overset{\mathrm{df}}{=} P$ is a projection on X^{**} with kernel $X^{\perp\perp}$.

Next, for Z an arbitrary operator space, let $\|\cdot\|_{\mathrm{op}(Z)}$ denote the given norm on $\mathbf{K}\otimes Z$; thus also $\|\cdot\|_{\mathrm{op}(Z^*)}$ is then the induced norm on $\mathbf{K}\otimes Z^*$, given by the expression

$$\|T\|_{\mathrm{op}(Z^*)} = \sup\{\|\langle T,S\rangle\| : S \in \mathbf{K}\otimes Z, \|S\|_{\mathrm{op}(Z)} \le 1\} \tag{112}$$

where for $T = \sum K_i \otimes z_i^*$ in $\mathbf{K}\otimes Z^*$, $S = \sum L_j \otimes z_j$ in $K \otimes z$,

$$\langle T,S\rangle = \sum_{i,j} z_i^*(z_j) K_i \otimes L_j \tag{113}$$

(regarded as an operator on $\ell^2 \otimes \ell^2$). Recall also, for T as above,

$$\|T\|_{\mathrm{MIN}} = \sup\left\{\Big\| \sum_i z_i^*(z) K_i \Big\| : z \in Ba(Z)\right\} . \tag{114}$$

Now we first define a new operator structure on $K \otimes Y^*$, and then let $\|\cdot\|_{\mathrm{NEW}}$ on $K\otimes Y$ be the one induced by this.

Definition. For $T = \sum K_i \otimes y_i^*$ in $\mathbf{K}\otimes Y^*$, set

$$\|T\|_{N^*} = \max\left\{\|T\|_{\mathrm{MIN}}, \sup\{\|\langle T,S\rangle\| : S \in \mathbf{K}\otimes X,\ \|S\|_{\mathrm{op}(X)} \le 1\right\} . \tag{115}$$

Now it is easily verified that $\|\cdot\|_{N^*}$ on $\mathbf{K}\otimes Y^{**}$ satisfies Ruan's axioms (cf. [ER]), here Y^* is indeed an operator space in this new structure. Next, we observe that $\|\cdot\|_{N^*}$ is induced by a NEW operator structure on Y. It suffices to prove that given n, $T = \sum_{i=1}^n K_i \otimes y_i^*$, and a net (T_α) with $T_\alpha = \sum_{i=1}^n K_i \otimes y_{i,\alpha}^*$ for all α, then if $y_{i,\alpha}^* \to y_i^*$ ω^* for all i and $\|T_\alpha\|_{N^*} \le 1$ for all α, also $\|T\|_{N^*} \le 1$. But it is evident that then given $y \in Y$, $\|y\| \le 1$, that

$$\sum_i y_{i,\alpha}^*(y) K_i \to \sum y_i^*(y) K_i \ \text{ in norm,} \tag{116}$$

and moreover given $S \in Ba(K\otimes X)$, that

$$\langle T_\alpha, S\rangle \to \langle T,S\rangle \ \text{ in norm.} \tag{117}$$

Hence $\|\sum y_i^*(y) K_i\| \le 1$, so $\|T\|_{\mathrm{MIN}} \le 1$, and also $\|\langle T,S\rangle\| \le 1$, thus $\|T\|_{N^*} \le 1$ as desired.

Now let $\|\cdot\|_{\mathrm{NEW}}$ be the operator space structure induced on $K\otimes Y$ by $\|\cdot\|_{N^*}$; we have thus by duality that

$$\|\cdot\|_{\mathrm{NEW}^*} = \|\cdot\|_{N^*} . \tag{118}$$

Next, we show that $\|\cdot\|_{\mathrm{NEW}}$ equals $\|\cdot\|_{\mathrm{op}(X)}$ on $K\otimes X$. Now first note that

$$\|\cdot\|_{\mathrm{NEW}} \ge \|\cdot\|_{\mathrm{op}(Y)} . \tag{119}$$

Indeed, (119) follows immediately by duality, since $\|T\|_{N^*} \leq \|T\|_{\mathrm{op}(Y^*)}$ for all $T \in \mathbf{K} \otimes Y^*$. but if $S \in K \otimes Y$ and $\|S\|_{\mathrm{op}(X)} \leq 1$, then by definition, $\|\langle T, S\rangle \leq 1$ for all $T \in \mathbf{K} \otimes Y^*$ with $\|T\|_{N^*} \leq 1$, hence $\|S\|_{\mathrm{NEW}} \leq 1$; proving

$$\|\cdot\|_{\mathrm{NEW}} \leq \|\cdot\|_{\mathrm{op}(X)} \ \text{ on } \ K \otimes Y \ . \tag{120}$$

We now assume that X^{**} is β-cocomplemented in Y^{**}, i.e., $X^\perp$ is β-complemented in Y^*, by the Fact. Now choose $P : Y^* \to X^\perp$ a projection with $\|P\| \leq \beta$, and let $E = \ker P$. We next claim that $\|\cdot\|_{\mathrm{NEW}^{**}}$ is *equivalent* to $\|\cdot\|_{\mathrm{MAX}}$ on $K \otimes E^\perp$. Now it follows immediately from the definition that

$$\|T\|_{\mathrm{NEW}^*} = \|T\|_{\mathrm{MIN}} \ \text{ for all } \ T \in K \otimes X^\perp \ . \tag{121}$$

By duality, we have that for any $S \in K \otimes Y^{**}$,

$$\|S\|_{\mathrm{MAX}} = \sup\{\|\langle T, S\rangle\| : T \in K \otimes Y^*,\ \|T\|_{\mathrm{MIN}} \leq 1\} \ . \tag{122}$$

But if $T \in K \otimes Y^*$, say $T = \sum K_i \otimes y_i^*$, then letting $\tilde{P}T = \sum K_i \otimes Py_i^*$, we have that

$$\|\tilde{P}T\|_{\mathrm{MIN}} \leq \beta \|T\|_{\mathrm{MIN}} \tag{123}$$

and of course $\tilde{P}T \in K \otimes X^\perp$. Hence we obtain that if $S \in K \otimes E^\perp$, then for any $T \in K \otimes Y^*$

$$\langle S, T\rangle = \langle S, \tilde{P}T\rangle \tag{124}$$

whence

$$\begin{aligned} \|\langle S, T\rangle\| = \|\langle S, \tilde{P}T\rangle\| &\leq \|S\|_{\mathrm{NEW}^{**}} \|\tilde{P}T\|_{\mathrm{NEW}^*} \\ &= \|S\|_{\mathrm{NEW}^{**}} \|\tilde{P}T\|_{\mathrm{MIN}} \ . \end{aligned} \tag{125}$$

Thus

$$\|S\|_{\mathrm{MAX}} \leq \beta \|S\|_{\mathrm{NEW}^*} \tag{126}$$

as desired. Finally, we show that (Y, NEW) is locally reflexive. Assume then that X is λ-locally reflexive and now suppose Y is γ-ELR. Let F, G be finite-dimensional spaces with $F \subset Y^*$, $G \subset Y^{**}$. Now we may assume without loss of generality, by simply enlarging G and F if necessary, that

$$G = G_1 \oplus G_2 \ \text{ and } \ F = F_1 \oplus F_2 \tag{127}$$

with

$$G_1 \subset E^\perp,\ G_2 \subset X^{\perp\perp},\ F_1 \subset X^\perp,\ F_2 \subset E \ . \tag{128}$$

Let $\varepsilon > 0$. Since X is λ-locally reflexive, choose $T_2 : G_2 \to X$ with

$$\begin{aligned} &\text{(i)} \quad \|T_2\|_{\mathrm{cb}} < \lambda + \varepsilon \\ &\text{(ii)} \quad \langle T_2 y, f\rangle = \langle g, f\rangle \text{ for all } g \in G_2,\ f \in E \ . \end{aligned} \tag{129}$$

Since Y is λ-ELR, we may choose $T_1 : E^\perp \to Y^{**}$ with

$$\begin{aligned} &\text{(i)} \quad \|T_1\| < \gamma + \varepsilon \\ &\text{(ii)} \quad T_1 G_1 \subset Y \\ &\text{(iii)} \quad \langle T_1 g, f\rangle = \langle g, f\rangle \ \text{ for all } \ g \in G_1,\ f \in F_1 \ . \end{aligned} \tag{130}$$

Now it follows that

$$\|T_1\|_{\mathrm{cb}} < (\gamma+\varepsilon)\beta \ . \tag{131}$$

(Here, we are computing the cb norm with respect to $(E^\perp, \mathrm{NEW}^{**})$.) Indeed, if I denotes $I|K$, then if $S \in K \otimes E^\perp$,

$$\|(I \otimes T_1)(S)\| \le \|T\| \, \|S\|_{\mathrm{MAX}} < (\gamma+\varepsilon)\beta\|S\|_{\mathrm{NEW}^{**}} \ . \tag{132}$$

Finally, we define $T : G \to Y$ by

$$T = T_1|G_1 \oplus T_2 \ . \tag{133}$$

Then by (129)(ii), (130)(iii), and (127),

$$\langle Tg, f\rangle = \langle g, f\rangle \ \text{ for all } \ g \in G, \ f \in F \ . \tag{134}$$

Now letting $R = P^*|G$, we have

$$\|R\|_{\mathrm{cb}} \le \|P^*\|_{\mathrm{cb}} = \|P\|_{\mathrm{cb}} \le \|P\| \le \beta \tag{135}$$

since $X^\perp$ has the MIN operator structure by (121). Now $T = T_1R + T_2(I-R)$ by (128), hence

$$\begin{aligned} \|T\|_{\mathrm{cb}} &\le \|T_1R\|_{\mathrm{cb}} + \|T_2(I-R)\|_{\mathrm{cb}} \\ &\le (\gamma+\varepsilon)\beta + (\lambda+\varepsilon)(1+\beta) \end{aligned} \tag{136}$$

by (129), (131) and (135).

Since $TG \subset Y$ and (122) holds, we have established that (Y, NEW) is $(\gamma+\lambda)\beta+\lambda$-locally reflexive. $\square$

Remarks. 1. The alert reader may notice that the ELR assumption on Y is used only at the very end. Thus, without this, we still obtain that (Y, NEW) coincides on X with the original operator space structure, and (Y^*, NEW) is still the MAX structure on $E^\perp$ (to a constant), and the given structure on X^{**}. However if $G \subset E^\perp$ is finite-dimensional, we cannot insure that a Banach local reflexivity operator $T : G \to Y$ is uniformly completely bounded, since G may not have MAX as its induced operator structure. The synthesis of the ELR concept occurred precisely to overcome this (apparently insurmountable) difficulty.

2. There is really no reason to assume that Y is an operator space at all. We really make no essential use of the given operator space structure on Y; the inequality (110) can instead be easily established directly (replacing "X" in its statement). We also obtain that X^{**} is completely complemented in (Y^{**}, NEW) (in fact if X^{**} is β-cocomplemented in Y^{**}, it is completely β-cocomplemented in (Y^{**}, NEW)).

3. It is an open question *if maximal operator spaces are locally reflexive*. If the answer to this question is affirmative, the conclusion of Lemma 3.16 would hold *without* the assumption that Y is ELR; consequently Theorem 3.14 would hold *without* the assumption of the existence of the ELR Z in its statement, and it would follow that $\mathbf{K}$ has the MSEP. (Moreover here, we would just require that separable maximal operator spaces are locally reflexive.) Indeed, the NEW operator space structure on Y is defined so that the induced structure on $E^\perp \subset Y^{**}$ is equivalent to MAX there. ELR of Y is used solely to insure the existence of the "local reflexivity" of $T_1|G_1$ with controlled cb-norm. Now suppose (Y, MAX) is locally reflexive. But then we could choose $T_1 : G_1 \to Y$ satisfying (130)(iii) with $\|T_1\|_{\mathrm{cb}} \le \tau$, where τ is a constant depending only on the local reflexivity constant of (Y, MAX) and on β (as defined in the proof). We note concerning this open question that it is equivalent (in general to the problem: *is* $(B(H), \mathrm{MAX})$ *locally reflexive*? Indeed, fixing a maximal operator space Y, choose a Hilbert space H so that $Y \subset B(H)$. But *then* the induced operator structure on Y via $(B(H), \mathrm{MAX})$ coincides with the given maximal structure, thanks to the injectivity of $B(H)$. Thus if $(B(H), \mathrm{MAX})$ is locally reflexive, so is Y.

Section 4

$\mathbf{K}_0$ fails the CSEP: a new proof and generalizations

To formulate the main result of this section, we first recall a concept introduced in [R2].

Definition. A family $\mathcal{Z}$ of operator spaces is said to be *finite matrix type* if there is a $C \geq 1$ so that for any finite-dimensional operator space E, there is an $n = \mathbf{n}(E)$ so that

$$\|T\|_{\mathrm{cb}} \leq C\|T\|_n \text{ for all linear operators } T : E \to Z \text{ and all } Z \in \mathcal{Z} .$$

If C works, we say that $\mathcal{Z}$ is of C-finite matrix type, or briefly, that $\mathcal{Z}$ is C-finite. Finally, we say that an operator space Z is of finite matrix type provided $\{Z\}$ has this property.

It is established in [Ro2], Proposition 2.15, that *if for some λ, $Z_1, Z_2, \dots$ are separable λ-injective operator spaces with $\{Z_1, Z_2, \dots\}$ of finite matrix type, then $(Z_1 \oplus Z_2 \oplus \cdots)_{c_0}$ has the* CSEP. Our main result in this section establishes the converse.

Theorem 4.1. *Let $Z_1, Z_2, \dots$ be operator spaces so that $\{Z_1, Z_2, \dots\}$ is not of finite matrix type. If all of the Z_i's have finite matrix type, let $Z = (Z_1 \oplus Z_2 \oplus \cdots)_{c_0}$. Otherwise, choose i so that Z_i is not of finite matrix type, and set $Z = c_0(Z_i)$. Then there exists an operator space Y with Y/Z separable such that Z is not completely complemented in Y.*

We then easily obtain a converse to the result from [Ro2] mentioned above, in view of the fact that separable injective operator spaces are necessarily injective (Corollary 2.9 of [Ro2]).

Corollary 4.2. *Let $Z_1, Z_2, \dots$ be reflexive separable operator spaces so that $(Z_1 \oplus Z_2 \oplus \cdots)_{c_0}$ has the* CSEP*, and assume that Z_i is of finite matrix type for all i. Then there is a λ so that Z_j is λ-injective for all j, and $\{Z_1, Z_2, \dots\}$ is of finite matrix type.*

Remark. We conjecture that the last hypothesis is superfluous; see the Conjecture following Corollary 4.3.

Proof of 4.2. By the results of [Ro2], there exists a λ so that $(Z_1 \oplus Z_2 \oplus \cdots)_{c_0}$ has the λ-CSEP. Hence for each j, Z_j has the λ-CSEP.

Since Z_j is reflexive, Z_j is λ-injective by Proposition 2.10 of [Ro2]. Of course Theorem 4.1 then yields that $\{Z_1, Z_2, \dots\}$ is of finite matrix type. $\square$

Now standard results yield that $\{\mathcal{M}_1, \mathcal{M}_2, \dots\}$ is *not* of finite matrix type, where for all n, $\mathcal{M}_n$ denotes the operator space of $n \times n$ matrices. (We give a quantitative refinement of this fact below.) Thus we obtain the result of E. Kirchberg [Ki1] (see also [W]):

Corollary 4.3. *$\mathcal{K}_0$ fails the* CSEP.

Conjecture. If a separable operator space has the CSEP, it is of finite matrix type.

The next immediate consequence of 4.1 supports this conjecture.

Corollary 4.4. *Let Z be a separable operator space which is not of finite matrix type. Then $c_0(Z)$ fails the* CSEP*. Hence if $c_0(Z)$ is completely isomorphic to Z, then Z fails the* CSEP.

We now proceed with the proof of Theorem 4.1. The following construction gives the crucial tool.

Lemma 4.5. *Let $(Z_1, Z_2, \dots)$ be a given sequence of operator spaces, k a positive integer, $C > 1$, and E an m-dimensional operator space. Assume there exists a sequence $1 = n_0 < n_1 < n_2 < \cdots$ of positive integers and for all $k \geq 1$, a linear map $U_k : E \to Z_k$ so that*

$$\|U_k\|_{\mathrm{cb}} \leq 1 \tag{137i}$$

$$\|U_k\|_{n_k} > 1 - \frac{1}{k} \tag{137ii}$$

$$\|U_k\|_{n_{k-1}} \leq \frac{1}{C}\ \frac{k}{k-1} \text{ if } k > 1. \tag{137iii}$$

Then setting $Z = (Z_1 \oplus Z_2 \oplus \cdots)_{c_0}$, there exists an operator space $Y \supset Z$ with $\dim Y/Z \leq m$ so that $\|P\|_{\mathrm{cb}} \geq C$ for any surjective linear projection $P : Y \to Z$.

Proof. In this discussion, we let I_j denote the identity map on $\mathcal{M}_j$. We construct Y as a subspace of $W \overset{\mathrm{df}}{=} (Z_1 \oplus Z_2 \oplus \cdots)_\infty$. Define $U : E \to W$ by

$$U(x) = (U_1(x), U_2(x), \dots) \quad \text{for} \quad x \in E \tag{138}$$

and let $F = U(E)$, $Y = Z + F$. Let $P : Y \to W$ be a linear projection. Now given $k_0 > 1$, by making a small perturbation if necessary, we may without loss of generality assume that there is a $k > k_0$ with

$$P(F) \subset Z_1 \oplus \cdots \oplus Z_{k-1} \ . \tag{139}$$

Let Q_j be the coordinate projection from W onto Z_j, for all j. Now by (137ii), choose $\tau \in E \otimes \mathcal{M}_{n_k}$ with $\|\tau\| = 1$ and

$$\|U_k \otimes I_{n_k}(\tau)\| > 1 - \frac{1}{k} \ . \tag{140}$$

Then letting $\beta = (U \otimes I_{n_k})(\tau)$, we have by (137i) and (140) that

$$1 \ge \|\beta\| \ge \|Q_k \otimes I_{n_k}(\beta)\| > 1 - \frac{1}{k} \ , \tag{141}$$

and by (137iii) that

$$\|Q_\ell \otimes I_{n_k}(\beta)\| \le \frac{1}{C}\ \frac{k}{k-1} \text{ for all } \ell > k \ . \tag{142}$$

Finally, let $\gamma = \beta - \sum_{j=1}^{k}(Q_j \otimes I_{n_k})(\beta)$. Then

$$\|\gamma\| = \sup_{j>k} \|Q_j \otimes I_{n_k}(\beta)\| \le \frac{1}{C}\ \frac{k}{k-1} \le \frac{1}{C}\ \frac{k_0}{k_0-1} \ . \tag{143}$$

However we have that $(Q_k \otimes I_{n_k})(P \otimes I_{n_k})(\gamma) = -Q_k \otimes I_{n_k}(\beta)$ by (134), hence

$$\begin{aligned} \|P \otimes I_{n_k}(\gamma)\| &\ge \|(Q_k \otimes I_{n_k})(P \otimes I_{n_k})(\gamma)\| \\ &= \|Q_k \otimes I_{n_k}(\beta)\| > 1 - \frac{1}{k} \text{ by (141)} \\ &\ge 1 - \frac{1}{k_0} \ . \end{aligned} \tag{144}$$

Since $k_0 > 1$ is arbitrary, (143) and (144) yield that $\|P\|_{\mathrm{cb}} \ge C$, as desired. $\square$

The next quantitative result easily yields Theorem 4.1.

Lemma 4.6. *Let $C > 1$ and let $\mathcal{Z}$ be a family of operator spaces which is not C-finite. There exist $Z_1, Z_2, \ldots$ in $\mathcal{Z}$ and an operator space $Y \supset Z \overset{\mathrm{df}}{=} (Z_1 \oplus Z_2 \oplus \cdots)_{c_0}$ with Y/Z finite-dimensional so that $\|P\|_{\mathrm{cb}} \ge C$ for any linear surjective projection $P : Y \to Z$.*

Proof. Choose E a finite dimensional operator space so that for all $n \in \mathbb{N}$, there exists a $Z \in \mathcal{Z}$ and a linear operator $U : E \to Z$ with

$$\|U\|_{\mathrm{cb}} = 1 \quad \text{and} \quad \|U\|_n < \frac{1}{C} \ . \tag{145}$$

Also note, that for any completely bounded map T between operator spaces,

$$\|T\|_{\mathrm{cb}} = \sup_n \|T\|_n \ . \tag{146}$$

Using (145), choose $Z_1 \in \mathcal{Z}$ and a linear operator $U_1 : E \to Z_1$ with $\|U_1\|_{\mathrm{cb}} = 1$ and $\|U_1\|_1 < \frac{1}{C}$. Choose $n_1 > 1$ with $\|U_1\|_{n_1} > 0$. Suppose $k > 1$ and n_{k-1} has been chosen. By (145), we may choose $Z_k \in \mathcal{Z}$ and a linear operator $U_k : E \to Z_k$ with $\|U_k\|_{\mathrm{cb}} = 1$ and $\|U_k\|_{n_{k-1}} < \frac{1}{C}$. Then using (146), choose $n_k > n_{k-1}$ with $\|U_k\|_{n_k} > 1 - \frac{1}{k}$.

This completes the inductive construction. Then (137) holds for all k, so the U_k's satisfy the hypotheses of 4.5, which thus yields Lemma 4.6. $\square$

We are now prepared for the

Proof of Theorem 4.1. Suppose first that X is an operator space which is not of finite matrix type, and let $Z = c_0(X)$. Then by Lemma 4.6, for each $n \in \mathbb{N}$ we may choose Y_n an operator space with $Y_n \supset Z$ so that Z is not n-completely complemented in Y_n and Y_n/Z is finite dimensional. Let $Y = (Y_1 \oplus Y_2 \oplus \cdots)_{c_0}$ and $\tilde{Z} = (Z \oplus Z \oplus \cdots)_{c_0}$. Then $\tilde{Z}$ is canonically completely isometric to Z, $Y/\tilde{Z}$ is separable, and $\tilde{Z}$ is not completely complemented in Y.

Now let $\{Z_1, Z_2, \dots\}$ be as in the statement of 4.1. If i is such that Z_i is not of finite matrix type, the above argument establishes the conclusion of 4.1. Otherwise, Lemma 4.6 and its proof yield that we may choose infinite pairwise disjoint subsets $M_1, M_2, \dots$ of $\mathbb{N}$ so that for each j, letting $W_j = (\bigoplus_{i \in M_j} Z_i)_{c_0}$, there exists an operator space $Y_j \supset W_j$ with Y_j/W_j finite-dimensional and W_j not j-completely complemented in Y_j.

Indeed, it follows from the definition of families of finite matrix type that there then exist $\ell_1 < \ell_2 < \cdots$ so that Z_{ℓ_j} is *not* j-finite for all j. Then let $\tilde{M}_1, \tilde{M}_2, \dots$ be infinite pairwise disjoint sets so that $\bigcup_{j=1}^{\infty} \tilde{M}_j = \{\ell_1, \ell_2, \dots\}$. Now it follows that letting $\mathcal{Z}_j = \{Z_m : m \in \tilde{M}_j\}$, then $\mathcal{Z}_j$ is not of finite matrix type for all j; now Lemma 4.6 yields an appropriate infinite $M_j \subset \tilde{M}_j$, for all j, satisfying the above.

Now letting $\tilde{Y} = (Y_1 \oplus Y_2 \oplus \cdots)_{c_0}$ and $\tilde{W} = (W_1 \oplus W_2 \oplus \cdots)_{c_0}$, then $\tilde{Y}/\tilde{W}$ is separable and $\tilde{W}$ is not completely complemented in Y. Finally, let $M_0 = W \sim \bigcup_{j=1}^{\infty} M_j$ and $\tilde{Z} = (\bigoplus_{i \in M_0} Z_i)_{c_0}$. Then let $Y = \tilde{Y} \oplus \tilde{Z}$. Z is canonically isometric to $\tilde{W} \oplus \tilde{Z}$, and of course $\tilde{W} \oplus \tilde{Z}$ is uncomplement in Y and $Y/(\tilde{W} \oplus \tilde{Z}) = \tilde{Y}/\tilde{W}$. $\square$

We now give a "tight" quantitative version of Corollary 4.3 (which is one of the main motivating results of this section). Recall that for a finite-dimensional operator space X, the exactness constant of X, denoted $\mathrm{Ex}(X)$, is defined by

$$\begin{aligned} \mathrm{Ex}(X) &= \inf\{d_{\mathrm{cb}}(X, F) : F \subset \mathbf{K}\} \\ &\overset{\text{Fact}}{=} \inf\{d_{\mathrm{cb}}(X, F) : F \subset \mathcal{M}_n \text{ for some } n\} \ . \end{aligned}$$

Theorem 4.7. *Let E be a finite-dimensional operator space, and let $C = \mathrm{Ex}(E^*)$. There exists an operator space Y containing $\mathbf{K}_0$ and a finite-dimensional subspace F of Y so that*

(i) *$Y/\mathbf{K}_0$ is completely isometric to E.*
(ii) *$\mathbf{K}_0$ is Banach $(1+\varepsilon)$ co-completely in Y for every $\varepsilon > 0$.*
(iii) *$\|P\|_{\mathrm{cb}} \ge C$ for any surjective linear projection $P : Y \to \mathbf{K}_0$.*

We first require a lemma, which really yields a precise local, quantitative version of the fact that $\{\mathcal{M}_n : n = 1, 2, \dots\}$ is *not* of finite matrix type.

Lemma 4.8. *Let E be a finite-dimensional operator space, $\ell > 1$, $\varepsilon > 0$ and set $C = \mathrm{Ex}(E^*)$. There exist an m and a 1–1 operator $T : E \to \mathcal{M}_m$ satisfying the following:*

(i) $(1+\varepsilon)C > \|T\|_{\mathrm{cb}} > (1-\varepsilon)C$
(ii) $\frac{1}{1+\varepsilon}\|x\| \le \|T \otimes I_\ell(x)\| \le (1+\varepsilon)\|x\|$ *for all $x \in E \otimes \mathcal{M}_\ell$.*

Proof. We let $P_k : \mathcal{M}_\infty \to \mathcal{M}_k \subset \mathcal{M}_\infty$ be the natural truncation operator; i.e.,

$P_k(a_{ij}) = a_{ij}$ if $1 \le i, j \le k$
$P_k(a_{ij}) = 0$ otherwise.

Of course $\|P_k\|_{\mathrm{cb}} = 1$ and $P_k T \to T$ in the strong operator topology.

We first note that it suffices to find $T : E \to \mathcal{M}_\infty$ satisfying (i) and (ii). Indeed, if such a T has these properties, then for m large enough, (since E is finite-dimensional), $(1+\varepsilon)C > \|P_m T\|_{\mathrm{cb}} > (1-\varepsilon)C$ also and

$$\frac{1}{(1+\varepsilon)^2}\|x\| \le \|P_m T \otimes I_\ell(x)\| \le (1+\varepsilon)^2\|x\| \tag{147}$$

for all $x \in E \otimes \mathcal{M}_\ell$, hence $\tilde{T} \overset{\mathrm{df}}{=} P_m T$ has the desired property (for a little bigger ε).

Now we dualize; without loss of generality $E^* \subset \mathcal{M}_\infty$. Next we claim that for k sufficiently large,

$$\frac{1}{1+\varepsilon}\|x\| \le \|P_k \otimes I_\ell(x)\| \text{ for all } x \in E^* \otimes \mathcal{M}_\ell \tag{148}$$

and

$$\|(P_k|E^*)^{-1}\|_{\mathrm{cb}} < (1+\varepsilon)C\ . \tag{149}$$

(Note that by (148), we will have $P_k|E^*$ is 1–1; setting $G_k^* = P_k(E^*)$, $(P_k|E^*)^{-1}$ refers to the inverse of $E^* \xrightarrow{P_k|E^*} G_k^*$). Indeed, we may choose $n \ge \ell$ and $Y \subset \mathcal{M}_n$ with $d_{\mathrm{cb}}(E^*, Y) < (1+\varepsilon)C$. Hence we may choose $T : E^* \to Y$ a linear operator with

$$\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}} < (1+\varepsilon)^{1/2}C\ . \tag{150}$$

Next, since E^* is finite-dimensional, so is $E^* \otimes \mathcal{M}_\ell$, so we can in fact choose k so that

$$\frac{1}{(1+\varepsilon)^{1/2}}\|x\| \le \|P_k \otimes I_n(x)\| \text{ for all } x \in E^* \otimes \mathcal{M}_n\ . \tag{151}$$

which gives (149) immediately. But then we have that $T(P_k|E^*)^{-1} : G_k \to \mathcal{M}_n$, thus using a Lemma of Roger Smith (cf. [S], also see [Pi3]),

$$\begin{aligned} \|T(P_k|E^*)^{-1}\|_{\mathrm{cb}} &= \|T(P_k|E^*)^{-1}\|_n \\ &\le \|T\|_{\mathrm{cb}}\|(P_k|E^*)^{-1}\|_n \\ &\le (1+\varepsilon)^{1/2}\|T\|_{\mathrm{cb}} \text{ (by (151)}\ . \end{aligned} \tag{152}$$

But then

$$\begin{aligned} \|(P_k|E^*)^{-1}\|_{\mathrm{cb}} &= \|T^{-1}T(P_k|E^*)^{-1}\|_{\mathrm{cb}} \\ &\le \|T^{-1}\|_{\mathrm{cb}}\|(P_k|E^*)^{-1}\|_{\mathrm{cb}} \\ &\le (1+\varepsilon)C \text{ by (151) and (152)),} \end{aligned} \tag{153}$$

proving (149).

Finally, set $G_k^* = P_k(E^*)$, let $S = (P_k|E^*)^{-1} : G_k^* \to E^*$, and let $T = S^* : E \to G_k$. Then since $\|S^{-1}\|_{\mathrm{cb}} = \|P_k|E^*|_{\mathrm{cb}} \le 1$, and $E^* \subset \mathcal{M}_n$, $\|S\|_{\mathrm{cb}} \ge C$; hence in fact

$$C \le \|T\|_{\mathrm{cb}} \tag{154}$$

and, by (148),

$$\frac{1}{1+\varepsilon}\|x\| \le \|T \otimes I_\ell(x)\| \le \|x\| \text{ for all } x \in E \otimes \mathcal{M}_\ell\ . \tag{155}$$

(Also $\|T^{-1}\|_{\mathrm{cb}} = \|S^{-1}\|_{\mathrm{cb}} = \|P_k|E^*\|_{\mathrm{cb}} \le 1$, but we don't use this.) Thus T satisfies (i) and (ii) (regarding $T(E) \subset \mathcal{M}_\infty$), so at last we obtain the desired operator by our initial observations. □

Remark. Buried in this proof, we have a rather remarkable fact: if X is a finite-dimensional subspace of $\mathcal{M}_\infty$, *then for k sufficiently large, $P_k|X$ is 1–1 and $\|P_k|X\|^{-1} \to$ Ex)(X) as $k \to \infty$.* That is, not only do we locate a specific $Y \subset \mathcal{M}_k$ with $d_{\mathrm{cb}}(Y|X)$ close to $\mathrm{Ex}(X)$, we also obtain that setting $Y = P_k|X$, $T = P_k|X : X \to Y$ satisfies $\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}}$ is almost equal to $\mathrm{Ex}(X)$. (This fact may also be found buried in the discussion in [Pi2].)

We are now prepared for the

Proof of Theorem 4.7. Let $0 < \eta < 1$ with $\frac{1+\eta}{1-\eta} < 1+\varepsilon$. Using Lemma 4.8, we choose $1 = n_0 < n_1 < n_2 < \cdots$ and for all k, linear maps $U_k : E \to \mathcal{M}_{n_k}$ as follows: First choose $n_1 > 1$ and an operator $T_1 : E \to \mathcal{M}_1$ so that (i) and (ii) of 4.8 hold for "T" $= T$, $\varepsilon = \frac{\eta}{2}$, $\ell = 1$.

Set $U_1 = T_1/\|T_1\|_{\mathrm{cb}}$. Suppose $k > 1$ and n_{k-1} has been defined. Choose $n_k > n_{k-1}$ and an operator $T_k : E \to \mathcal{M}_{n_k}$ so that (i) and (ii) of 4.8 hold for "T" $= T_k$, $\varepsilon = \frac{\eta}{2^k}$, $\ell = n_{k-1}$. Then set $U_k = T_k/\|T_k\|_{\mathrm{cb}}$.

This completes the inductive construction of the U_k's. We then have for all k, letting $\tau_k = (1+\frac{\eta}{2^k})|(1-\frac{\eta}{2^k})$ and noting that $1-\varepsilon < \frac{1}{1+\varepsilon}$ if $\varepsilon < 1$, that

$$\|U_k\|_{\mathrm{cb}} = \|U_k\|_{n_k} = 1 \tag{155i)}$$

$$\|U_k\|_{n_{k-1}} = \|U_k \otimes I_{n_{k-1}}\| \le \tau_k/C \tag{155ii)}$$

$$\|(U_k \otimes I_{n_{k-1}})^{-1}\| \le \tau_k C\ . \tag{155iii)}$$

Setting $Z_k = \mathcal{M}_{n_k}$, C and the U_k's fulfill the hypotheses of Lemma 4.5, so let $\tilde{Y}$ be the space given in that construction and simply let $Y = \tilde{Y} \oplus (\mathcal{M}_{i_1} \oplus \mathcal{M}_{i_2} \oplus \cdots)_{c_0}$ where $i_1 < i_2 < \cdots$ is an increasing enumerator of $\mathbb{N} \sim \{n_1, n_2, \dots\}$. Now it is immediate that Y satisfies (iii) of 4.7; let us verify the other assertions of 4.7 (which immediately reduce to considering $\tilde{Y}$ instead).

Let U and Z be as in the proof of 4.5. Let $\pi : \tilde{Y} \to \tilde{X}/Z$ be the quotient map. Then we have for $\ell \ge 1$ and $x \in E \otimes \mathcal{M}_\ell$, that for all $k > \ell$, since then $n_{k-1} \ge \ell$,

$$\frac{1}{\tau_k C}\|x\| \le \|U_k \otimes I_\ell(x)\| \le (\tau_k/C)\|x\|\ . \tag{156}$$

But then for any $w \in Z \otimes \mathcal{M}_\ell$,

$$\lim_{k\to\infty} \|Q_k \otimes I_\ell(U \otimes I_\ell(x) - w)\| = \frac{\|x\|}{C}\ . \tag{157}$$

Since ℓ is arbitrary, this shows that

$$\|(\pi U) \otimes I_\ell(x)\| = \frac{\|x\|}{C}\ . \tag{158}$$

That is, $C\pi U$ is a complete isometry, proving (i) of Theorem 4.7.

Finally, let $\varepsilon > 0$, choose k_0 so that $\tau_k < 1+\varepsilon$ if $k > k_0$, and define $V : E \to \tilde{Y}$ by

$$\begin{aligned} Q_j V(e) &= 0 \text{ if } j < k_0\ , \\ Q_j V(e) &= Q U_k(e) \text{ if } j \ge k_0\ . \end{aligned} \tag{159}$$

Then setting $F = V(E)$, we have that $F \oplus \mathbf{K}_0 = \tilde{Y}$ and $f \in F$ and $z \in \mathbf{K}_0$ imply

$$\begin{aligned} \|f+z\| &\ge \lim_{k\to\infty} \|Q_k(f+z)\| \\ &= \lim_{k\to\infty} \|Q_k(f)\| \\ &\ge \frac{1}{1+\varepsilon}\|f\| \end{aligned} \tag{160}$$

by (156) (for $\ell = 1$), showing that Z is $(1+\varepsilon)$-cocomplemented in $\tilde{Y}$, completing the proof. □

We now draw some immediate consequences of Theorem 4.7 and previously known results.

Corollary 4.9. (a) *For all n, there exists an operator space Y_n containing $\mathbf{K}_0$ so that $Y_n/\mathbf{K}_0$ is completely isometric to ℓ_n^∞ and $\|P\|_{\mathrm{cb}} \geq \sqrt{n}/2$ for any surjective linear projection $P : Y_n \to \mathbf{K}_0$.*

(b) *There exists an operator space Y containing $\mathbf{K}_0$ so that $Y/\mathbf{K}_0$ is completely isometric to c_0 and $\mathbf{K}_0$ is completely uncomplemented in Y.*

Proof. (a) Set $E = \ell_n^\infty$ in Theorem 4.7. Then $E^* = (\ell_n^1, \mathrm{MAX})$ and it is known that $\mathrm{Ex}(\ell_n^1, \mathrm{MAX}) \geq \sqrt{n}/2$ [Pi2].

(b) Let $Y = (\oplus Y_n)_{c_0}$, and $\tilde{\mathbf{K}}_0 = c_0(\mathbf{K}_0)$. Of course $\tilde{\mathbf{K}}_0$ is isometric to $\mathbf{K}_0$, $\tilde{\mathbf{K}}_0$ is completely uncomplemented in Y by (a), and $Y/\tilde{\mathbf{K}}_0$ is completely isometric to $(\oplus \ell_n^\infty)_{c_0}$ which is completely isometric to c_0. $\square$

Remarks. 1. By a standard result, there exists a linear projection $P : Y_n \to \mathbf{K}_0$ with $\|P\|_{\mathrm{cb}} \leq \sqrt{n}+1$. Thus the order of magnitude result in (a) is best possible. Our construction yields that $\mathbf{K}_0$ is Banach $(1+\varepsilon)$-co-complemented in Y_n and Y, for any $\varepsilon > 0$.

2. Actually, in part (a), we may replace ℓ_n^∞ by any n-dimensional Banach space E endowed with the minimal operator space structure. Then by a result of M. Junge and G. Pisier, $Ex(E^*, \mathrm{MAX}) \geq \sqrt{n}/4$ [JP]. Hence we obtain an operator space Y_n containing $\mathbf{K}_0$ so that $Y_n/\mathbf{K}_0$ is completely isometric to E and $\mathbf{K}_0$ is not λ-completely complemented in Y_n if $\lambda < \sqrt{n}/4$.

3. A separable operator space X is defined to be *nuclear* if there exists a sequence (T_n) of finite rank operators on X with $T_n \to I_X$ in the strong operator topology, so that for all n, there exist ℓ_n and complete contractions $U_n : X \to \mathcal{M}_{\ell_n}$ and $V_n : \mathcal{M}_{\ell_n} \to X$ with $T_n = V_n U_n$. Thus, a separable C^*-algebra is nuclear precisely when it is a nuclear operator space. It follows from the results of E. Kirchberg in [Ki2] that *the space Y in* (b) *is not nuclear; however $\mathbf{K}_0$ and $Y/\mathbf{K}_0$ are obviously nuclear*. This is in marked contrast with the algebraic case (in fact since $\mathbf{K}_0$ *is* already an ideal in $\mathbf{K}_0^{**}$, if $\mathbf{K}_0 \subset \mathcal{A} \subset \mathbf{K}_0^{**}$ with $\mathcal{A}$ a C^*-algebra, then $\mathcal{A}/\mathbf{K}_0$ nuclear *implies* $\mathcal{A}$ is nuclear). Indeed, the work in [Ki2] yields that were Y nuclear, Y would be 1-locally reflexive, whence $\mathbf{K}_0$ would be completely complemented in Y since $\mathbf{K}_0$ has the CSCP ([Ro2]), contradicting Corollary 4.9(b).

Corollary 4.10. *Let Y_n be as in part (a). Then Y_n is not λ-locally reflexive for $\lambda \leq (\sqrt{n}/2) - 3$.*

Remark. Of course Y_n is locally reflexive; in fact just because $\dim Y_n/\mathbf{K}_0 = n$, there is an absolute constant c so that Y_n is $c\sqrt{n}$ locally reflexive.

Proof. Suppose that Y_n is C-locally reflexive. By Sublemma 3.11 of [Ro2], since $K_0^* = B(H)$ is isometrically injective, $\mathbf{K}_0$ is $C+3+\varepsilon$-completely complemented in Y_n for all $\varepsilon > 0$. Hence by Corollary 4.9, $C+3+\varepsilon \geq \sqrt{n}/2$ for all such $\varepsilon > 0$, so $C \geq (\sqrt{n}/2) - 3$. $\square$

Our next (and final) application of the arguments for Theorem 4.7 yields that every descending sequence of 1-exact Banach isometric finite-dimensional spaces is bounded below.

Proposition 4.11. *Let (λ_k) be a sequence of real numbers with $\lambda_k \geq 1$ for all k and $\prod_{k=1}^\infty \lambda_k < \infty$. Let (E_j) be a sequence of 1-exact finite dimensional operator spaces so that E_k is λ_k-semi-isometric to E_{k+1} for all k. Then $\overline{\lim}_{k,n\to\infty} d_{\mathrm{cb}}(E_k, E_n) \leq 4$.*

Proof. For each k, choose $J_k : E_k \to E_{k+1}$ a linear map with

$$\|J_k^{-1}\| \leq \lambda_k \quad \text{and} \quad \|J_k\|_{\mathrm{cb}} = 1 \quad \text{for all} \quad k\ . \tag{161}$$

Suppose the conclusion were false; then by passing to a subsequence if necessary, we may assume that for some $C > 4$,

$$\|J_k^{-1}\|_{\mathrm{cb}} \geq C \quad \text{for all} \quad k\ . \tag{162}$$

(Note that if $n_1 < n_2 < \cdots$ is given, then letting $\tilde{J}_k = (J_{n_{k+1}-1}) \cdots J_{n_k+1} J_{n_k}$, then (E_{n_j}) satisfies the same hypotheses as (E_ℓ), replacing "λ_k" by $\tilde{\lambda}_k = \prod_{i=n_k}^{n_{k+1}-1} \lambda_i$ for all k.)

Now by a "small perturbation" argument, we may also assume that there are $\ell_1 < \ell_2 < \cdots$ so that $E_k \subset \mathcal{M}_{\ell_k} \overset{\text{df}}{=} Z_k$ for all k. Now let $Z = (Z_1 \oplus Z_2 \oplus \cdots)_{c_0}$ and $F \subset (Z_1 \oplus Z_2 \oplus \cdots)_{\ell^\infty}$ be defined by

$$F = \Big\{ \{e, J_1 e, J_2 J_1 e, \dots\} : e \in E_1 \Big\} \ . \tag{163}$$

Then setting $Y = Z + F \subset (Z_1 \oplus Z_2 \oplus \cdots)_{\ell^\infty}$, Y is a 1-exact operator space (since $\lambda_k \to 1$).

Now results of E. Kirchberg and standard techniques yield that Y is 1-locally reflexive. Indeed, a standard argument yields that any 1-exact operator space embeds in a nuclear operator space; the results in [Ki2] yield in turn that nuclear operator spaces are 1-locally reflexive. (See also the last paragraph of [KR].) Thus by Lemma 3.9 of [Ro2] (and the remark following it), Z is 4-completely complemented in Y. On the other hand, the argument for Lemma 4.5 yields that if $P : Y \to Z$ is a linear projection, then $\|P\|_{\text{cb}} \geq C$, a contradiction. $\square$

We next show that $\mathbf{K}_0$ (and hence $\mathbf{K}$) *fails* to admit completely bounded extensions from certain subspaces of particular separable locally reflexive operator spaces.

Proposition 4.12. *There exists an operator space $\tilde{Y}$ which is separable 1-locally reflexive, a closed linear subspace $\tilde{X}$, and a completely bounded map $T : \tilde{X} \to \mathbf{K}_0$ so that T has no completely bounded extension to $\tilde{Y}$.*

This result follows from our work above, known results, and the following elementary tool.

Lemma 4.13. *Let X, Y and $\tilde{Y}$ be operator spaces with $X \subset Y$, and let $q : \tilde{Y} \to Y$ be a complete metric surjection; set $\tilde{X} = q^{-1}(X)$ and let $T = q|\tilde{X}$. Then if T has a completely bounded (resp. bounded) extension $\tilde{T} : \tilde{Y} \to X$, X is completely complemented (resp. complemented) in Y.*

Proof. Let $W = \ker q$; then

$$W \subset X \ . \tag{164}$$

Now suppose $\tilde{T}$ is a completely bounded (resp. bounded) extension, and let $\Pi : \tilde{Y} \to \tilde{Y}/\tilde{W}$ be the quotient map and $S : \tilde{Y}/W \to Y$ the canonical complete surjective isomorphism so that

$$q = S\Pi \ . \tag{165}$$

By (164), we may define a map $U : \tilde{Y}/W \to X$ by

$$\tilde{T} = U\Pi \ . \tag{166}$$

Indeed, for $f \in \tilde{Y}$, set $U(\Pi f) = \tilde{T}(f)$. If $f \in W$, $f \in X$, hence $\tilde{T}(f) = T(f) = q(f) = 0$; this shows U is well defined, and we also obtain that U is completely bounded (resp. bounded) with $\|U\|_{\text{cb}} = \|\tilde{T}\|_{\text{cb}}$ (resp. $\|U\| = \|\tilde{T}\|$).

Now define $P : Y \to X$ by

$$P = US^{-1} \ . \tag{167}$$

Since T is a surjective quotient map from $\tilde{X}$ into X, if we let $x \in X$ and choose $\tilde{x} \in \tilde{X}$ with $T\tilde{x} = x$, we have that

$$\begin{aligned} P(x) &= US^{-1}q(\tilde{x}) = U\Pi(\tilde{x}) \ \text{ by (165)} \\ &= \tilde{T}(\tilde{x}) \ \text{ by (166)} \\ &= T(\tilde{x}) = X \ . \end{aligned} \tag{168}$$

Thus P is a completely bounded (resp. bounded) surjective projection. $\square$

Proposition 4.12 follows immediately from Corollary 4.3 and the next result.

Proposition 4.14. *There exists a* 1*-locally reflexive separable operator space* $\tilde{Y}$ *with the following property: Given a separable operator space* X*, if every completely bounded (resp. bounded) linear map from a subspace* $\tilde{X}$ *of* $\tilde{Y}$ *to* X *admits a completely bounded (resp. bounded) linear extension to* $\tilde{Y}$*, then* X *has the* CSEP *(resp the* MSEP*).*

Proof. Let $\tilde{Y} = C_1$ or $(\oplus C_1^n)_{\ell^1}$, where C_1 is the space of trace-class operators (resp. C_1^n is the n-dimensional trace-class), endowed with its dual structure via $C_1 = \mathbf{K}^*$ (resp. $(\oplus C_1^n)_{\ell^1} = \mathbf{K}_0^*$). A remarkable result of M. Junge yields that $\tilde{Y}$ is 1-locally reflexive ([J]; see also [EJR] and [JM]). But *every* separable operator space is completely isometric to a quotient space of $\tilde{Y}$ [B]. Proposition 4.14 now follows immediately from Lemma 4.13. □

Remark. We do not know if $\tilde{Y}$ in Proposition 4.12 may be chosen so that $\tilde{Y}^*$ is separable, or so that $\tilde{Y}^*$ has the CMAP.

References

[AF] J. Arazy and Y. Friedman, *Contractive projections in C_1 and C_∞*, Memoirs Amer. Math. Soc. **13** no.200 (1978).

[B] D.P. Blecher, *The standard dual of an operator space*, Pacific J. Math. **153** (1992), 15–30.

[BCLT] J. Bourgain, P.G. Casazza, J. Lindenstrauss and L. Tzafriri, *Banach spaces with a unique unconditional basis up to permutation*, Memoirs Amer. Math. Socl., No.322, 1985.

[C] E. Cartan, *Sur les domaines bornés homogènes de l'espace de n variables complexes*, Abh. Math. Sem. Univ. Hamburg **11** (1935), 116–162.

[CI] Cho-Ho Chu and Bruno Iochum, *Complementation of Jordan Triples in von-Neumann algebras*, Proc. Amer. Math. Soc. **108** (1990), 19–24.

[D] D. Drucker, *Exceptional Lie Algebras and the structure of hermitian symmetric spaces*, AMS Memoirs **16** no.208 (1978).

[EJR] E. Effros, M. Junge and Z.J. Ruan, *Integral mappings and the principal of local reflexivity for non-commutative L^1-spaces* (to appear).

[FR] Y. Friedman and B. Russo, *Solution of the contractive projection problem*, J. Funct. Anal. **60** (1985), 56–79.

[H] U. Haagerup, *Injectivity and decomposition of completely bounded maps*, in Operator Algebras and their connection with topology and ergodic theory, SLN **1132** (1985), 170–222.

[Ha] L.A. Harris, *Bounded symmetric homogeneous domains in infinite dimensional spaces*, SLN **364** (1974), 13–40.

[JO] W.B. Johnson and T. Oikberg, *Separable lifting property and extensions of local reflexivity* (to appear).

[JRZ] W.B. Johnson, H.P. Rosenthal and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9** (1971), 488–506.

[J] M. Junge, *Factorization theory for spaces of operators*, Habilitatieresschrift, University of Kiel (1996).

[JM] M. Junge and Ch. Le Merdy, *Factorization through matrix spaces for finite rank operators between C^*-algebras* (to appear).

[JP] M. Junge and G. Pisier, *Bilinear forms on exact operator spaces and $B(H)\otimes B(H)$*, Geometric and Functional Analysis **5** (1995), 329–363.

[K1] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. **183** (1983), 503–529.

[K2] ———, *Contractive projections on Jordan C^*-algebras and generalizations*, Math. Scand. **54** (1984), 95–100.

[Ki1] E. Kirchberg, *On non-semisplit extensions, tensor products and exactness of group C^*-algebras*, Invent. Math. **112** (1993), 449–489.

[Ki2] E. Kirchberg, *On subalgebras of the CAR-algebra*, J. Funct. Anal. **129** (1995), 35–63.

[KR] S-H. Kye and Z-J. Ruan, *On local lifting property for operator spaces*, preprint.

[LR1] J. Lindenstrauss and H.P. Rosenthal, *Automorphisms in c_0, ℓ_1, and m*, Israel J. Math. **7** (1969), 222–239.

[LR2] J. Lindenstrauss and H.P. Rosenthal, *The $\mathcal{L}_p$ spaces*, Israel J. Math. **7** (1969), 325–349.

[Lo] O. Loos, *Bounded symmetric domains and Jordan pairs*, Lecture Notes, University of California at Irvine, 1977.

[Pa] V. Paulsen, *Representations of functions algebras, abstract operator spaces, and Banach space geometry*, J. Funct. Anal. **109** (1992), 113–129.

[Pe] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. **29** (1969), 209–227.

[Pf] H. Pfitzner, *Weak compactness in the dual of a C^*-algebra is determined commutatively*, Math. Ann. **298** (1994), 349–371.

[Pi1] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, CBMS No.60, Amer. Math. Soc., Providence, 1986.

[Pi2] G. Pisier, *Exact operator spaces Colloque sur les algèbres d'operateurs*, in Recent Advances in Operator Algebras, Asterisque, Soc. Math. France **232** (1995), 159–186.

[Pi3] G. Pisier, *An introduction to the theory of operator spaces*, preprint.

[R] A. Guyan Robertson, *Injective matricial Hilbert spaces*, Math. Proc. Camb. Phil. Soc. **110** (1991), 183–190.

[Ro1] H.P. Rosenthal, *On complemented and quasi-complemented subspaces of quotients of $C(S)$ for Stonian S*, Proc. Natl. Acad. Sci. **60** (1968), 1165–1169.

[Ro2] H.P. Rosenthal, *The complete separable extension property*, J. Operator Theory (to appear).

[Ru] B. Russo, *Structure of JB^*-triples*, Jordan Algebras, Walter de Gruyter, Berlin, 1994, pp. 209–280.

[S] R.R. Smith, *Completely bounded maps between C^*-algebras*, J. London Math. Soc (2) **27** (1983), 157–166.

[So] A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. **47** (1941), 938–947, MR3-205.

[W] S. Wassermann, *Exact C^*-algebras and related topics*, Seoul National University Lecture Notes Series, No. 19, 1994.

[Z] M. Zippin, *The separable extension problem*, Israel J. Math. **26** (1977), 372–387.

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