The Banach spaces $C(K)$

Haskell P. Rosenthal*

Department of Mathematics
The University of Texas at Austin
Austin, TX 78712-1082, USA
Email: rosenthl@math.utexas.edu

Table of Contents

1 Introduction

A $C(K)$-space is just the space of scalar valued continuous functions on a compact Hausdorff space K. We focus here mainly on the case where K is *metrizable* i.e., the case of *separable* $C(K)$-spaces. Our main aim is to present the most striking discoveries about the Banach space structure of $C(K)$-spaces, and at the same time to describe the beautiful, deep intuitions which underlie these discoveries. At times, we go to some length to describe the form and picture of an argument, without giving the full technical discussion. We have also chosen to present proofs which seem the most illuminating, in favor of more advanced

* Research partially supported by NSF Grant DMS-0070547.

and sophisticated but (to us) less intuitive arguments. The following is a summary of our exposition.

Section 2 deals with the by now classical isomorphic classification of the separable $C(K)$-spaces, dating from the 50's and 60's. It begins with Milutin's remarkable discovery: *$C(K)$ is isomorphic to $C([0,1])$ if K is an uncountable compact metric space.* We give a fully detailed proof, modulo some standard basic facts (summarized in Lemma 2.5), which follows an argument due to S. Ditor. This yields that *every separable $C(K)$ space is isometric to a contractively complemented subspace of* $C(K)$, $\mathbf{D}$ the Cantor discontinuum, (Theorem‘2.4), through a natural inverse limit argument, given in Lemma 2.11 below. The way inverse limits work (in the metrizable setting) is given in Lemma 2.12, and Theorem 2.4 is deduced after this. The isomorphic classification of the $C(K)$-spaces with separable duals, due to Bessaga and Pełczyński, occupies the balance of this section. Their remarkable result: *The spaces $C(\omega^{\omega^\alpha}+)$ form a complete set of representatives of the isomorphism classes, over all countable ordinals* (Theorem 2.14). We do give a detailed proof that $C(K)$ is isomorphic to one of these spaces, for all countable compact K (of course, we deal only with *infinite-dimensional* $C(K)$-spaces here). This is achieved through Theorem 2.24 and Lemma 2.26. We do not give the full proof that these spaces are all isomorphically distinct, although we spend considerable time discussing the fundamental invariant which accomplishes this, the Szlenk index, and the remarkable result of C. Samuel: $Sz(C(\omega^{\omega^\alpha}+) = \omega^{\alpha+1}$ for all countable ordinals α (Theorem 2.15). We give a variation of Szlenk's original formulation following 2.15, and show it is essentially the same as his in Proposition 2.17. We then summarize the invariant properties of this index in Proposition 2.18, and give the relatively easy proof that $Sz(C(\omega^{\omega^\alpha}+) \geq \omega^{\alpha+1}$ in Corollary 2.21. We also show in Section 2 how the entire family of spaces $C(\alpha+)$ (up to algebraic isometry) arises from a natural Banach space construction: simply start with c_0, then take the smallest family of commutative C^*-algebras containing this, and closed under unitizations and c_0-sums. (This is the family $(Y_\alpha)_{1\leq\alpha<\omega_1}$, given at the beginning of part B of Section 1.) The *isomorphic* description, however, is achieved through taking tensor products at successive ordinals and c_0-sums and unitizations at limit ordinals. (This is the transfinite family $(X_\alpha)_{\alpha<\omega_1}$ given following Definition 2.22.)

Section 3 deals with three unrelated structural properties. The first, due to Pełczyński, is that *every separable $C(K)$ space X is weakly injective, that is, any isomorph of X in a separable Banach space Y, contains a subspace isomorphic to X and complemented in Y* (Theorem 3.1). The second one, due to Bessaga and Pełczyński, is that *every $C(K)$-space with separable dual is c_0-saturated* (Proposition 3.6). This follows quickly from our first transfinite description of these spaces mentioned above. It is not a difficult result, but is certainly fundamental for the structure of these spaces. We also briefly note the rather long standing open problem: is every subspace of a quotient of $C(\alpha+)$ c_0-saturated, for countable ordinals α? We note also: it is unknown if ℓ^2 is isomorphic to a subspace of a quotient of $C(\omega^\omega+)$.

The third result in Section 3 deals with Amir's Theorem: *$C(\omega^\omega+)$ fails to be separably injective.* We also give a fully detailed proof of Milutin's classical discovery; *the Cantor map of $\{0,1\}^{\mathbf{N}}$ onto $[0,1]$ induces an uncomplemented isometric embedding of $C([0,1])$ in* $C(\mathbf{D})$. We give a unified account of both of these results through the space $\mathrm{rcl}([0,1])$ of functions on $[0,1]$ wich are right continuous with left limits (also called *cadlag* by French probabilists). For any countable compact subset K of $[0,1]$, we let $\mathrm{rcl}(K)$ be the analogous function space, just defined on K. *If K has enough two-sided cluster points, then $C(K)$ is uncomplemented in* $\mathrm{rcl}(K)$. Similarly, *if D is any countable dense subset of* $[0,1]$*, then* $C([0,1])$ *is uncomplemented in* $\mathrm{rcl}([0,1],D)$. (The latter is simply the space of all functions in $[0,1]$ continuous at all $x \neq D$, right continuous with left limits at all $x \in D$.) These results are proved in Theorem 3.14. Section 3 concludes with the proof that if D is the dyadic rationals, then the embedding of $C([0,1])$ into $C(\mathbf{D})$ via the Cantor map is essentially just the identity injection of $C([0,1])$ in $\mathrm{rcl}([0,1],D)$, and so is uncomplemented (Proposition 3.18).

Section 4 deals mainly with several deep fixing results for operators on $C(K)$-spaces, all of which heavily bear on the famous long standing problem discussed in the final section of this article. An operator T between Banach spaces is said to *fix* a Banach space Z if there is an isometric copy Z' of Z in the domain with $T|Z'$ an isomorphism. In the present context, it turns out there are isometric copies Z' of Z which are fixed. The first of these is Pełczyński's Theorem that *non-weakly compact operators on $C(K)$-spaces fix c_0*, Theorem 4.5. We show this follows quite naturally from Grothendieck's classical description of weakly compact sets in $C(K)^*$ (Theorem 4.29), and a relative disjointness result on families of measures, due to the author (Proposition 4.30). Next, we take up characterizations of operators fixing $C(\omega^\omega+)$. Our main aim is to give an intuitive picture of the isometric copy of $C_0(\omega^\omega)$ which is actually fixed. Theorem 4.25 itself states Alspach's remarkable equivalences, which in particular yield that *an operator on a separable $C(K)$ space fixes $C(\omega^\omega+)$ if and only if its ε-Szlenk index is at least ω for all $\varepsilon > 0$*. We follow Bourgain's approach here, stating his deep extension of this result to arbitrary countable ordinals in Theorem 4.17. Bourgain achieves his results on totally disconnected spaces K, obtaining the fixed copy as the span of the characteristic functions of a regular family of clopen sets (Definition 4.15). In turn, the direct Banach space description of $C(\alpha+)$ spaces is given by Bourgain's formulation in terms of trees (Definition 4.13); these yield an intuitive direct description of monotone bases for such spaces, which are actually the clopen sets mentioned above, in the needed concrete realization of these spaces (formulated in Proposition 4.15). We discuss in considerable detail Bourgain's remarkable result (which rests on 4.25): *an operator on a $C(K)$ space fixes $C(\omega^\omega+)$ if and only if it is a non Banach-Saks operator* (see Definition 4.8). We first give Schreier's proof that $C(\omega^\omega+)$ fails the Banach-Saks property, in Propositions 3.8 and 3.9. Next we recall the author's dichotomy: *a weakly null sequence in an arbitrary Banach space either has a subsequence whose arithmetic averages converge to zero in norm, or a subsequence which generates a spreading model isomorphic to ℓ^1* (Theorem 4.23). We then use this to deduce Bourgain's non-Banach-Saks characterization (Theorem 4.22).

The final result discussed in Secton 4 is the author's result: *An operator on a separable $C(K)$-space fixes $C([0,1])$ if its adjoint has non-separable range.* We formulate three basic steps in the proof, Lemma 4.25, Lemma 4.29, and Proposition 4.30. These are then put together to outline the proof, and finally the "almost isometric" Lemma 4.25 is explained somewhat, via Lemma 4.31, to give a picture of the actual isometric copy of $C(\mathbf{D})$ which is finally fixed.

Section 5 is purely expository; only obvious deductions are given. The remarkable partial progress on the Complemented Subspace Problem (CSP) illustrates the deep penetration into the structure of $C(K)$ spaces that has been achieved. The problem itself and especially certain unresolved special cases show there is still much to be understood about their structure (see Problems 1–4 in Section 5). $\mathcal{L}_\infty$ spaces and $L^1(\mu)$ preduals are briefly discussed. Zippin's fundamental lemma is presented in the context of the CSP as Lemma 5.11. Possibly the most striking of the known results on the CSP, due to Benyamini, rests on Lemma 5.11. These assert that *every complemented subspace of a separable $C(K)$ space is either isomorphic to c_0 or contains a subspace isomorphic to $C(\omega^\omega+)$. Moreover every complemented subspace X of a separable $C(K)$ space with X^* separable is isomorphic to a quotient space of $C(\alpha+)$ for some countable ordinal α* (Theorems 5.9 and 5.15). To prove this, Benyamini also establishes an extension result for general separable Banach spaces which actually yields a new proof of Milutin's Theorem (Theorem 5.12). Section 5 concludes with a brief discussion of the positive solution to the CSP in the isometric setting: *every contractively complemened subspace of a separable $C(K)$-space is isomorphic to a $C(K)$-space.*

An exciting new research development deals with many of the issues discussed here in the context of C^*-algebras. Neither time nor space was available to discuss this developement here. We shall only briefly allude to two discoveries. The first is Kirchberg's non-commutative analogue of Milutin's theorem: *Every separable non-type I nuclear C^*-algebra is completely isomorphic to the CAR algebra* [Ki]. The second concerns quantized formulations of the separable extension property, due to the author [Ro7], and the joint theorem of T. Oikhberg and the author: the space of compact operators on separable Hilbert space has the Complete Separable Complentation Property [OR]. For a recent survey and perspective on these developments, see [Ro8].

2 The isomorphic classification of separable $C(K)$ spaces

A. Milutin's Theorem

Our first main objective is the following remarkable result due to A. Milutin [M].

Theorem 2.1 *Let K be an uncountable compact metric space. Then $C(K)$ is isomorphic to $C([0,1])$.*

Although this is *not* an isometric result, its proof is based on isometric considerations. Let us introduce the following definitions and notations.

Definition 2.2 *Let X and Y be given Banach spaces.*

1) *$X \hookrightarrow Y$ means that X is isomorphic to a subspace of Y.*
2) *$X \overset{\mathbf{c}}{\hookrightarrow} Y$ means that X is isomorphic to a complemented subspace of Y.*
3) *$X \overset{cc}{\hookrightarrow} Y$ means that X is isometric to a contractively complemented subspace of Y.*
4) *$X \overset{c}{\sim} Y$ means that $X \overset{c}{\hookrightarrow} Y$ and $Y \overset{c}{\hookrightarrow} X$. We say X is complementably equivalent to*

Y.

5) $X \overset{\mathbf{cc}}{\approx} Y$ *means that* $X \overset{cc}{\hookrightarrow} Y$ *and* $Y \overset{cc}{\hookrightarrow} X$*. We say* X *is contractively complementably equivalent to* Y*.*

6) $X \sim Y$ *means that* X *is isomorphic to* Y*.*

Of course one has that the first three relations are a kind of partial order on Banach spaces; e.g., one easily has that

$$X \overset{\mathbf{cc}}{\hookrightarrow} Y \text{ and } Y \overset{\mathbf{cc}}{\hookrightarrow} Z \text{ implies } X \overset{\mathbf{cc}}{\hookrightarrow} Z\ . \tag{2.1}$$

(The relation $\overset{c}{\hookrightarrow}$ was crystallized by D. Alspach in some unpublished work.) Of course the relation $\overset{c}{\hookrightarrow}$ is implicit in the *decomposition method* given on page 14 of [JL], which was developed by A. Pełczyński [Pe1]. The proof in [JL] (as well as that in [Pe1]) yields the following result.

Proposition 2.3 *Let* X,Y *be Banach spaces. Then* $X \sim (X \oplus X \oplus \cdots)_{c_0}$ *and* $X \overset{c}{\sim} Y$ *implies* $X \sim Y$*.*

Milutin's Theorem now easily reduces to the following fundamental result (known as Milutin's Lemma).

Theorem 2.4 *Let* K *be a compact metric space. Then* $C(K) \overset{cc}{\hookrightarrow} C(\mathbf{D})$ *where* $\mathbf{D}$ *denotes the Cantor discontinuum.*

We give a proof due to S. Ditor [D1]. We first summarize some standard needed results.

Lemma 2.5 *Let* K *be a given infinite compact metric space.*

a) $\mathbf{D}$ *is homeomorphic to a subset of* K *if* K *is uncountable.*

b) Let L *be a compact subset of* K*. Then there exists a linear isometry* $T : C(L) \to C(K)$ *such that* $T1_L = 1_K$ *and* $T(f\mid_K) = f$ *for all* $f \in C(K)$*.*

c) D *is homeomorphic to* K *if* K *is perfect and totally disconnected.*

d) c_0 *is isometric to a subspace of* $C(K)$*.*

e) $C(K) \sim C_0(K,k_0)$ *where* $k_0 \in K$ *and* $C_0(K,k_0) = \{f \in C(K) : f(k_0) = 0\}$*.*

In fact, there is an absolute constant γ so that $d(C(K), C_0(k,k_0)) \le \gamma$.

Remark 2.6 $d(X,Y)$ denotes the multiplicative Banach-Mazur distance between Banach spaces X and Y.

Proof. (b), the *linear* form of the Tietze-extension theorem, is due to Borsuk [B]. (a) and (c) are standard topological results. To see (d) (which holds for any infinite compact Hausdorff space), let $U_1, U_2, \ldots$ be disjoint non-empty open subsets of K, and for each j, choose $0 \le \varphi_j \le 1$ in $C(K)$ with $\|\varphi_j\| = 1$ and support $\varphi_j \subset U_j$ for all j. One has immediately that then (φ_j) is isometrically equivalent to the usual c_0-basis. To obtain (e), let X be a subspace of $C(K)$ isometric to c_0. In fact, our argument shows that we may choose $X \subset C_0(K,k_0)$.

Since K is separable, X is complemented in $C(K)$ by Sobczyk's theorem [Sob] Thus in fact we may choose Y a closed linear subspace of $C_0(K,k_0)$ with

$$C_0(K,k_0) = X \oplus Y \ . \tag{2.2}$$

Thus we have that

$$\begin{aligned} C(K) &= [\mathbf{1}] \oplus C_0(K,k_0) = [\mathbf{1}] \oplus X \oplus Y \\ &\sim [\mathbf{1}] \oplus c_0 \oplus Y \sim c_0 \oplus Y \sim X \oplus Y = C_0(K,k_0) \ . \end{aligned} \tag{2.3}$$

(Note that $[\mathbf{1}]$ is simply Φ the 1-dimensional space of scalars.)

Now the existence of γ may be obtained by tracing through this argument quantitatively, using Sobczyk's result that in fact there is a projection of $C(K)$ onto X of norm at least two. Indeed, suppose Z is a separable Banach space containing a subspace X isometric to c_0. By Sobczyk's theorem, there is a subspace Y of Z with $d(Z, c_0 \oplus Y) \le 6$ (where we take direct sums in the ℓ^∞-norm). But also if Z_0 is any co-dimension 1 subspace of Z, then by the Hahn-Banach theorem, $d(Z_0, Z \oplus \Phi) \le 6$. Hence since $d(Z \oplus \Phi), c_0 \oplus \Phi \oplus Y) \le 6$, and $c_0 \oplus \Phi$ is isometric to c_0, it follows that $d(Z, Z_0) \le 36$. $\square$

Remark 2.7 It is unknown if 2.5(e) holds for non-metrizable compact Hausdorff spaces K.

Proof of Milutin's Theorem (modulo 2.4)

We first note that for any $k_0 \in \mathbf{D}$,

$$(C(\mathbf{D}) \oplus C(\mathbf{D}) \oplus \cdots)_{c_0} \cong C_0(\mathbf{D}, k_0) \sim C(\mathbf{D}) \tag{2.4}$$

(where $X \cong Y$ means X is isometric to Y). Indeed this follows from Lemma 2.5 (c).

Thus, using Theorem 2.4 and Proposition 2.3, it suffices to show that given K uncountable compact metric, then

$$C(\mathbf{D}) \overset{cc}{\hookrightarrow} C(K) \ . \tag{2.5}$$

Indeed, we then obtain that $C(K) \sim C(\mathbf{D})$. So of course also $C(K) \sim C([0,1])$.

But 2.5 follows immediately from Lemma 2.5 (a) and (b). Indeed, choose L a compact subset of K homeomorphic to $\mathbf{D}$ and choose T as in 2.5 (b); set $X = T(C(L))$. Then X is isometric to $C(\mathbf{D})$ and is contractively complemented in $C(K)$ via the map: $P(f) = T(f \mid L)$ $\square$

Remark. Actually, the above argument and the proof of Milutin's Lemma gives even more isometric information; namely one has

Theorem 2.8 *Let K and L be compact metric spaces with K uncountable. Then there is a unital isometry from $C(L)$ onto a subspace X of $C(K)$, which is contractively complemented* *in $C(K)$. The unital isometry and contractive projection are thus positive maps.*

We now present the proof of Theorem 2.4. We first formulate the $\overset{cc}{\hookrightarrow}$ order as follows, leaving the simple proof to the reader.

Proposition 2.9 *Let X and Y be given Banach spaces. The following are equivalent.*

(a) $X \overset{cc}{\hookrightarrow} Y$.
(b) There exist linear contractions $U : X \to Y$ and $V : Y \to X$ so that $I_X = V \circ U$ That is, the following diagram holds.

$$\begin{tikzcd} & Y \arrow[dr, "V"] & \\ X \arrow[ur, "U"] \arrow[rr, "I"] & & X \end{tikzcd} \quad . \tag{2.6}$$

Definition 2.10 *Let L and K be compact metric spaces and $\varphi : L \to K$ be a continuous surjection.*

(a) $\varphi^0 : C(K) \to C(L)$ *denotes the map*

$$\varphi^0 f = f \circ \varphi \text{ for all } f \in C(K) \ .$$

(b) A linear map $T : C(L) \to C(K)$ is called a regular averaging operator for φ *if* $\|T\| = 1$ *and $(T\varphi^0)f = f$ for all $f \in C(K)$, i.e., (2.6) holds with $X = C(K)$, $U = \varphi^0$, $V = T$.*

Note that $\varphi^0(C(K))$ is in fact a unital subalgebra of $C(L)$ isometric to $C(K)$. It is easily seen (via the argument for Proposition 2.9) that $\varphi^0(C(K))$ is contractively complemented in $C(L)$ iff φ admits a regular averaging operator T. Thus Milutin's Lemma means one can choose a continuous surjection $\varphi : D \to K$ which admits a regular averaging operator. Milutin did this by an explicit construction, while Ditor's argument proceeds conceptually, but indirectly. (For further results, see [D2], and especially [AA] for recent comprehensive work on regular averaging operators in both the metric and non-metrizable setting.)

We first deal with the basic ingredient in the proof. Given $X_1, \dots, X_n$ topological spaces, $X_1 \oplus \cdots \oplus X_n$ denotes their topological disjoint sum. Of course if these are compact metric spaces, so is $X_1 \oplus \cdots \oplus X_n$; we may formally identify $X_1 \oplus \cdots \oplus X_n$ with the metric space

$$\bigcup_{i=1}^n X_i \times \{i\} \qquad \text{where} \quad \operatorname{dist}((x,i),(y,j)) = 1 \quad \text{if} \quad i \neq j$$
$$\text{and} \quad \operatorname{dist}((x,i),(y,i)) = d_{X_i}(x,y) \ .$$

(where d_{X_i} is the metric on X_i).

Lemma 2.11 *Let K be a compact metric space, and $K_1, \dots, K_n$ be non-empty compact* *subsets such that*

$$K = \bigcup_{i=1}^{n} \operatorname{int} K_i \ . \tag{2.7}$$

Let $\tau : K_1 \oplus \cdots \oplus K_n \to K$ *be the map defined by*

$$\tau(k,i) = k \textit{ for all } i \textit{ and } k \in K_i \ . \tag{2.8}$$

Then τ *admits a regular averaging operator.*

Remark. We obviously may assume that $\operatorname{int} K_i = \text{interior } K_i \neq \emptyset$ for all i. However we do not insist that the K_i's are distinct; in fact we may need repetitions in our application of 2.11. It is also clear that τ is a continuous surjection of $\oplus_{i=1}^n K_i$ onto K.

Proof of Lemma 2.11 We may choose $\varphi_1, \ldots, \varphi_n$ a partition of unity fitting the open cover of K, $(\operatorname{int} K_i)_{i=1}^n$. That is, the φ_j's are in $C(K)$ and satisfy

$$0 \leq \varphi_j \leq 1 \text{ for all } j \ , \tag{2.9}$$

$$\operatorname{supp} \varphi_j \subset \operatorname{int} K_j \text{ for all } j \ , \tag{2.10}$$

$$\sum_{j=1}^{n} \varphi_j \equiv 1 \ . \tag{2.11}$$

$$(\operatorname{supp} \varphi = \{x : |\varphi(x)| > 0\}) \ .$$

Next, fix i, and for $f \in C(\oplus_{j=1}^n K_j)$, define f_i on K by

$$f_i(k) = f(k,i) \text{ if } k \in K_i \tag{2.12}$$
$$f_i(k) = 0 \text{ if } k \notin K_i \ . \tag{2.13}$$

Then

$$f_i \cdot \varphi_i \text{ is continuous.} \tag{2.14}$$

Indeed, since τ maps $K_i \times \{i\}$ homeomorphically into K_i, it follows that $f_i|K_i$ is continuous, and so of course $(f_i \cdot \varphi_i)|K_i$ is also continuous. Since $\varphi_i(x) = 0$ for all $x \notin K_i$ and φ_i is continuous on K, it follows that if (x_n) is a sequence in $K \sim K_i$ such that $x_n \to x$ with $x \in K_i$, then $(f_i \cdot \varphi_i)(x_n) = 0$ for all n and also $(f_i \cdot \varphi_i)(x) = 0$, proving (2.14).

Finally, define $T : C(\oplus_{i=1}^n K_i) \to C(K)$ by

$$Tf = \sum_{i=1}^{n} f_i \varphi_i \text{ for all } f \in C(\oplus_{i=1}^n K_i) \ . \tag{2.15}$$

Then fixing $f \in C(\oplus_{i=1}^n K_i) \to C(K)$, we have that indeed $Tf \in C(K)$ by (2.14) Moreover for any $k \in K$

$$
\begin{aligned}
|Tf(k)| &\leq \sum_{i=1}^{n} |f_i(k)| \varphi_i(k) \\
&\leq \max_i |f_i(k)| \sum \varphi_i(k) \qquad (2.16) \\
&\leq \|f\|_\infty \ .
\end{aligned}
$$

Thus $\|T\| = 1$ and of course T is linear.

Finally, if $f \in C(K)$, then for any k,

$$
\begin{aligned}
T(\tau^0 f)(k) &= \sum_{i=1}^{n} (\tau^0 f)_i(k) \varphi_i(k) \\
&= \sum f(k) \varphi_i(k) \qquad (2.17) \\
&= f(k) \sum_{i=1}^{n} \varphi_i(k) = f(k) \ ,
\end{aligned}
$$

completing the proof. $\square$

We need one more tool; inverse limit systems of topological spaces. We just formulate the special case needed here (see Lemma 2 of [D1] for the general situation).

Lemma 2.12 *Let $(K_n)_{n=1}^\infty$ be a sequence of compact metric spaces, and for each n, let $\varphi_n : K_{n+1} \to K_n$ be a given continuous surjection. There exists a compact metrizable space K_∞ satisfying the following for all n:*

$$\text{There exists a continuous surjection } \tilde{\varphi}_n : \mathcal{K}_\infty \to K_n \ . \qquad (2.18)$$

$$\varphi_n \tilde{\varphi}_{n+1} = \tilde{\varphi}_n \ . \qquad (2.19)$$

Letting $Y_n = \tilde{\varphi}_n^0(C(K_n))$, then

$$\bigcup_{n=1}^{\infty} Y_n \text{ is dense in } C(K) \ . \qquad (2.20)$$

If moreover φ_n admits a regular averaging operator for each n, then $\tilde{\varphi}_1$ admits a regular averaging operator.

Remark. The space K_∞ is essentially determined by (2.16) and (2.17) and is called the *inverse limit* of the system $(K_n, \varphi_n)_{n=1}^\infty$.

PROOF. Let K_∞ be the subset of $\prod_{n=1}^\infty K_n$ defined by

$$(k_j) \in K_\infty \text{ iff } k_j = \varphi_j(k_{j+1}) \text{ for all } j \ . \qquad (2.21)$$

Of course the axiom of choice yields that K_∞ is not only non-empty, but for all n, $\tilde{\varphi}_n : K_\infty \to K_n$ is a surjection, where

$$\tilde{\varphi}_n((k_j)) = k_n \text{ for any } (k_j) \in K_\infty \ . \tag{2.22}$$

K is also a closed subset of $\prod_{n=1}^\infty K_n$, where the latter is endowed with the Tychonoff topology, It also follows immediately that fixing n, then (2.17) holds. But this implies that

$$Y_n \subset Y_{n+1} \ . \tag{2.23}$$

Indeed, say $y \in Y_n$ and let $y = \tilde{\varphi}_n^0(f)$, for a (unique) $f \in C(K_n)$. But

$$\tilde{\varphi}_n^0(f) = f \circ \tilde{\varphi}_n = f \circ \varphi_n \circ \tilde{\varphi}_{n+1} = \tilde{\varphi}_{n+1}^0(\varphi_n^0 f) \in Y_{n+1} \ . \tag{2.24}$$

Now it follows that $\cup_{n=1}^\infty Y_n$ is a unital subalgebra of $C(K_\infty)$ separating its points, hence this is dense in $C(K_\infty)$ by the Stone-Weirstrauss theorem. Finally, if each φ_n admits a regular averaging operator, then for all n there exists a contractive linear projection $P_n : Y_{n+1} \to Y_n$ from Y_{n+1} *onto* Y_n. It follows that there exists a unique contractive linear projection $P : \cup_{j=1}^\infty Y_j \to Y_1$ such that for all n and $y \in Y_{n+1}$,

$$P(y) = P_1 P_2 \cdots P_{n-1} P_n(y) \ . \tag{2.25}$$

But then P uniquely extends to a unique contractive projection from $C(K_\infty)$ onto $P(Y_1)$ by (2.17), completing the proof via Proposition 2.9.

Finally, we give the

Proof of Theorem 2.4 Let $K_1 = K$. We inductively define $K_2, K_3, \ldots, K_N, \ldots$ satisfying the hypotheses of Lemma 2.5.

Step 1. Choose $n_1 > 1$ and $W_1^1, \ldots, W_{n_1}^1$ compact subsets of K_1, each with non-empty interior and diameter less than one, such that

$$K = \bigcup_{j=1}^{n_1} \operatorname{int} W_j^1 \ . \tag{2.26}$$

Set $K_1 = \oplus_{j=1}^{n_1} W_j^1$; *endow* K_1 with the metric described in the comments preceding Lemma 2.12, and let $\varphi_1 : K_1 \to K$ be the continuous surjection given by Lemma 2.11. Thus $\varphi_j(W_j^1 \times \{j\}) = W_j^1$ for all j and φ_j admits a regular averaging operator by Lemma 2.11.

Step m. Assume that $K_m = \oplus_{j=1}^{n_m} W_j^m$ has been defined, and fix j, $1 \leq j \leq n_m$. Thus $W_j W_j^m \times \{j\}$ is a *natural* clopen subset of K_m. Choose $k_j > 1$ and $W_{j,1}, \ldots, W_{j,k_j}$ compact subsets of W_j, each with non-empty interior, and diameter less than $1/(m+1)$, with

$$W_j = \bigcup_{i=1}^{k_j} \operatorname{int} W_{j,i} \ . \tag{2.27}$$

Next, let $n_{m+1} = \sum_{j=1}^{n_m} k_j$ and

$$W_i^{n+1} = W_{\psi(i)} \text{ for } 1 \leq i \leq n_{m+1} \ , \tag{2.28}$$

where $\psi : \{1, \ldots, n_{m+1}\} \to \{(j,i) : 1 \leq i \leq k_j, 1 \leq j \leq n_m\}$ is a bijection.

Set $K_{n+1} = \oplus_{j=1}^{n_{m+1}} W_j^{m+1}$, endow K_{m+1} with the metric described preceding Lemma 2.11 and choose $\varphi_m : K_{m+1} \to K_m$ the continuous surjection admitting a regular averaging operator given by Lemma 2.11. Thus in fact $\varphi_m(W_{j,\ell} \times \{\ell\}) = W_{j,i}$, where $\psi(\ell) = (j,i)$ for all j and i and moreover

$$K_m = \bigcup_{j=1}^{n_{m+1}} \operatorname{int} w_j^{m+1} \tag{2.29}$$

This completes the inductive construction of the K_m's and φ_m's. Lemma 2.12 now yields the existence of a continuous surjection $\varphi : K_\infty \to K$ admitting a regular averaging operator, where K_∞ satisfies the conclusion of Lemma 2.12. It remains only to check that K_∞ is perfect and totally disconnected, hence homeomorphic to D. The details for this quite naturally involve the further structure of inverse systems. For each $1 \leq j \leq n$, define the map $\varphi_{j,n} : K_{n+1} \to K_j$ by $\varphi_{j,n} = \varphi_j \circ \varphi_n$.

Now in our particular construction, we have that for all m and $1 \leq j \leq n_m$,

$$\varphi_m \text{ maps } W_j^m \times \{j\} \text{ isometrically onto } W_j^m \ . \tag{2.30}$$

But then it follows that

$$\varphi_{j,m} \mid W_j^m \times \{j\} \text{ is an isometry for all } i \leq m \text{ and } 1 \leq j \leq n_m \ . \tag{2.31}$$

Let us endow K_∞ with the metric

$$d((x_j),(y_j)) = \sum_{j=1}^{\infty} \frac{d_j(x_j,y_j)}{2^j} \tag{2.32}$$

where d_j is the metric on K_j.

Then it follows that for all m and $1 \leq j \leq n_m$;

$$\operatorname{diam} \tilde{W}_j^m < \frac{2}{m} \text{ where } \tilde{W}_j^m = \tilde{\varphi}_j(W_j^m \times \{j\}) \ . \tag{2.33}$$

Indeed, suppose (x_k) and (y_k) belong to $\tilde{W}_j^m$. But then there are points x and y in $W_j^m \times \{j\}$ such that $x_i = \varphi_{i,m}(x)$ and $y_i = \varphi_{i,m}(y)$ for all $1 \leq i \leq m$, and have by (2.31),

$$d(x_i,y_i) \leq \left(\sum_{j=1}^{m} \frac{1}{2^j} \right) d_m(x_i,y_i) + \sum_{j=m+1}^{\infty} \frac{1}{2^j} \leq \frac{1}{m} + \frac{1}{2^m} \tag{2.34}$$

since $\operatorname{diam} W_j^m < \frac{1}{m}$.

It then follows that the family of clopen subsets $\{\tilde{W}_j^m : 1 \le j \le n_m; m = 1, 2, \ldots\}$ is a base for the topology of K_∞. Hence K_∞ is totally disconnected. Finally, our insistence that at each stage m we "label" at least 2 sets contained in $W_j^m \times \{j\}$ (i.e., $k_j > 1$), insures that $\tilde{W}_j^m$ contains at least two points for all m and $1 \le j \le n_m$, whence K_∞ is indeed perfect. □

B. $C(K)$ **spaces with separable dual via the Szlenk index.**

Of course $C(K)^*$ is separable if and only if K is infinite countable compact metric. It is a standard result in topology that every such K is homeomorphic to $C(\alpha+)$ for some countable ordinal $\alpha \ge \omega$. We use standard facts about ordinal numbers. An ordinal α denotes the set of ordinals β with $\beta < \alpha$; $\alpha+$ denotes $\alpha + 1$. Finally, for α a limit ordinal, $C_0(\alpha)$ denotes the space of continuous functions on α vanishing at infinity, which of course can be identified with $\{f \in C(\alpha+) : f(\alpha) = 0\}$.

The Banach spaces $C(\omega^\alpha+)$ arise quite naturally upon applying a natural inductive construction to $c = C(\omega+)$ (the space of converging sequences). Indeed, for any locally compact Hausdorff spaces $X_1, X_2, \ldots$, we have that

$$(C_0(X_1) \oplus C_0(X_2) \oplus \cdots)_{c_0} \stackrel{\text{def}}{=} Y \tag{2.35}$$

is again algebraically isometric to $C_0(X)$ where $X = \oplus_{j=1}^\infty X_j$, the "direct sum" of the spaces $X_1, X_2, \ldots$.

Of course then Y has a unique "unitization" as the space of continuous functions on the one point compactification of X, which we'll denote by $Y \oplus [1]$. The norm here is quite explicitly given as

$$\|y \oplus c \cdot 1\| = \sup_j \sup_{\omega \in X_j} |y_j(\omega) + c| \tag{2.36}$$

where $y = (y_j) \in Y$.

Now define families of $C(K)$ spaces $(Y_\alpha)_{1 \le \alpha < \omega_1}$ as follows. Let $Y_1^0 = c_0$ and $Y_1 = c = Y_1^0 \oplus [1]$. Let $Y_2^0 = (c \oplus c \oplus \cdots)_{c_0}$ and $Y_2 = Y_2^0 \oplus [1]$. Suppose β is a countable ordinal and X_α, Y_α defined for all $\alpha < \beta$. If β is a successor, say $\beta = \alpha + 1$, set $Y_\beta^0 = (Y_\alpha \oplus Y_\alpha \oplus \cdots)_{c_0}$ and $Y_\beta = Y_\beta^0 \oplus [1]$. If β is a limit ordinal, choose $\alpha_n < \beta$ with $\alpha_n \nearrow \beta$ and set $Y_\beta^0 = (Y_{\alpha_1} \oplus Y_{\alpha_2} \oplus \cdots)_{c_0}$ and $Y_\beta = Y_\beta^0 \oplus [1]$. It is not difficult to see that then for all $1 \le \alpha < \omega_1$,

$$\begin{aligned} &Y_\alpha^0 \textit{ is algebraically isometric to } C_0(\omega^\alpha) \\ &\textit{and } Y_\alpha \textit{ is algebraically isometric to } C(\omega^\alpha+)\ . \end{aligned} \tag{2.37}$$

Now the topological classification of infinite countable compact metric spaces K is known; each such space is homeomorphic to exactly one of the ordinals $(\omega^\alpha+1)\cdot n$ for some $1 \le \alpha < \omega_1$ and positive integer n ([MS]). Indeed, one has that $\alpha = \mathrm{Ca}(K)$ and $n = \# K^{(\alpha)}$. (See the comments following 2.19 below for the definition of $\mathrm{Ca}(K)$). We thus obtain:

Proposition 2.13 *Let K be an infinite countable compact metric space. Then there exists a unique $1 \le \alpha < \omega_1$ and a unique n so that $C(K)$ is (algebraically) isometric to $Y_\alpha \underbrace{\oplus \cdots \oplus}_{n} Y_\alpha$, the direct sum taken in the* sup *norm.*

The isomorphic classification of these spaces is far more delicate. The result is as follows (C. Bessaga and A. Pełczyński [BP]).

Theorem 2.14 *Let K be an infinite countable compact metric space.*

(a) $C(K)$ is isomorphic to $C(\omega^{\omega^\alpha}+)$ for some countable ordinal $\alpha \ge 0$.
(b) If $0 \le \alpha < \beta < \omega_1$, then $C(\omega^{\omega^\alpha}+)$ is not isomorphic to $C(\omega^{\omega^\beta}+)$.

It turns out that the Szlenk index actually distinguishes these spaces. This index was introduced for Banach spaces with separable dual, by W. Szlenk [Sz], eight years after the seminal work of [BP]. For X with X^* separable, we denote its Szlenk index by $Sz(X)$. The following remarkable result was established by C. Samuel [Sa], based in part on work of D. E. Alspach and Y. Benyamini [AB2].

Theorem 2.15 *Let $0 \le \alpha < \omega_1$. Then $Sz(C(\omega^{\omega^\alpha}+) = \omega^{\alpha+1}$.*

We give a detailed proof of 2.14(a), but only indicate some of the ideas involved in 2.15, which of course yields 2.14(b). We first give an alternate derivation for the Szlenk index, similar to that indicated in Section 1.3 of [AGR].

Fix X a separable Banach space, and let $\varepsilon > 0$. We define a derivation d_ε on the ω^*-compact subsets K of X^* as follows:

Let $\delta_\varepsilon(K)$ denote the set of all $x^* \in K$ such that there exists a sequence (x_n^*) in K with

$$x_n^* \to x^* \ \omega^* \quad \text{and} \quad \|x_n^* - x^*\| \ge \varepsilon \text{ for all } n \ . \tag{2.38}$$

Now define:

$$d_\varepsilon(K) = \overline{\delta_\varepsilon(K)}^{\omega^*} \ . \tag{2.39}$$

Now define a transfinite descending family of sets $K_{\alpha,\varepsilon}$ for $0 \le \alpha < \omega_1$ as follows. Let $K_{0,\varepsilon} = K$ and $K_{1,\varepsilon} = d_\varepsilon(K)$. Let γ be a countable ordinal and suppose $K_{\alpha,\varepsilon}$ defined for all $\alpha < \gamma$. If γ is a successor, say $\gamma = \alpha + 1$, set

$$K_{\gamma,\varepsilon} = d_\varepsilon(K_{\alpha,\varepsilon}) \ . \tag{2.40}$$

If γ is a limit ordinal, choose (α_n) ordinals with $\alpha_n < \gamma$ for all n and $\alpha_n \to \gamma$; set

$$K_{\gamma,\varepsilon} = \bigcap_{n=1}^{\infty} K_{\alpha_{n,\varepsilon}} \ . \tag{2.41}$$

We may now define ordinal indices as follows.

Definition 2.16 *Let K be a ω^*-compact subset of X^*, with X a separable Banach space.*

(a) $\beta_\varepsilon(K) = \sup\{\alpha \le \omega_1 : K_{\alpha,\varepsilon} \ne \emptyset\}$
(b) $\beta(K) = \sup_{\varepsilon>0} \beta_\varepsilon(K)$.
(c) $Sz(X) = \beta(Ba\ X^*)$.

Now it is easily seen that in fact there is a (least) $\alpha < \omega_1$ with $K_{\alpha,\varepsilon} = K_{\alpha+1,\varepsilon}$. Moreover one has that then $K_{\alpha,\varepsilon} = \emptyset$ iff K is norm separable iff $\beta_\varepsilon(K) < \omega_1$, and then $\alpha = \beta_\varepsilon(K) + 1$. Thus one obtains that $Sz(X) < \omega_1$ iff X^* is norm-separable.

Szlenk's index was really only originally defined for Banach spaces with separable dual. In fact, however, one arrives at exactly the same final ordinal indices as he does, assuming that ℓ^1 is not isomorphic to a subspace of X, in virtue of the ℓ^1-Theorem [Ro4].

The derivation in [Sz] is given by: $K \to \tau_\varepsilon(K)$ where $\tau_\varepsilon(K)$ is the set of all x^* in K so that there is a sequence (x_n^*) in K and a weakly null sequence (x_n) in $Ba(X)$ such that $\overline{\lim}_{n\to\infty} |x_n^*(x_n)| \ge \varepsilon$. Now let $P_\alpha(\varepsilon, K)$ be the transfinite sequence of sets arising from this derivation as defined in [Sz]. Let also $\eta_\varepsilon(K)$, the "ε-Szlenk index of K," equal $\sup\{\alpha < \omega_1 : P_\alpha(\varepsilon, K) \ne \emptyset\}$ and $\eta(K) = \sup_{\varepsilon>0} \eta_\varepsilon(K)\}$.

The following result shows the close connection between Szlenk's derivation and ours; its routine proof (modulo the ℓ^1-theorem) is omitted.

Proposition 2.17 *Let X be a separable Banach space containing no isomorph of ℓ^1, and let K be a ω^*-compact subset of X^*. Then for all $\varepsilon > 0$ and countable ordinals α,*

$$P_\alpha\left(\frac{\varepsilon}{2}, K\right) \supset K_{\alpha,\varepsilon} \supset P_\alpha(2\varepsilon, K)\ . \tag{2.42}$$

Hence

$$\eta\left(\frac{\varepsilon}{2}, K\right) \ge \beta_\varepsilon(K) \ge \eta(2\varepsilon, K) \tag{2.43}$$

and thus $\eta(K) = \beta(K)$.

One may now easily deduce the following permanence properties.

Proposition 2.18 *Let X, Y be given Banach spaces and K, L weak* compact norm separable subsets of X^*.*

(a) *$L \subset K$ implies $\beta(L) \le \beta(K)$.*
(b) *If $T : Y \to X$ is a surjective isomorphism, then $\beta(T^*K) = \beta(K)$.*
(c) *If $Y \subset X$ and $\pi : X^* \to Y^*$ is the canonical quotient map, then $\beta(\pi K) \le \beta(K)$.*

In turn, this yields the following isomorphically invariant properties of the Szlenk index.

Corollary 2.19 *Let X and Y be given Banach spaces with norm-separable duals. Then if Y is isomorphic to a subspace of a quotient space of X, $Sz(Y) \leq Sz(X)$.*

Remark. Of course this yields that $Sz(X) = Sz(Y)$ if X and Y are of the same Kolomogroff dimension; i.e., each is isomorphic to a subspace of a quotient space of the other. This reveals at once both the *power* and the *limitation* of the Szlenk index.

For the next consequence of our permanence properties of the Szlenk index, recall the Cantor-Bendixon index $Ca(K)$ of a compact metrizable space K, defined by the cluster point derivation. For $W \subset K$, let W' denote the set of cluster points of W. Then define $K^{(\alpha)}$, the α^{th} derived set of K, by $K^{(0)} = K$, $K^{(\alpha+1)} = (K^{(\alpha)})'$, and $K^{(\beta)} = \bigcap_{\alpha<\beta} K^{(\alpha)}$ for countable limit ordinals α. Then define (for $K \neq \emptyset$)

$$Ca(K) = \sup\{0 \leq \alpha \leq \omega_1 : K^{(\alpha)} \neq \emptyset\} \ . \tag{2.44}$$

One has that K is countable iff $Ca(K) < \omega_1$, and of course if α is the least γ with $K_{(\alpha)} = K_{(\alpha+1)}$, then either $K^{(\alpha)} = \emptyset$ and $\alpha = Ca(K) + 1$, or $K^{(\alpha)}$ is perfect. (Note: we are unconventional here; the index of Cantor-Bendixon is traditionally defined as $Ca(K) + 1$.)

Corollary 2.20 (a) *Let X be a separable Banach space and K be a countable ω^*-compact subset of X^* so that for some $\delta > 0$, K is δ-separated, i.e.,*

$$\|k - k'\| \geq \delta \ \textit{ for all } \ k \neq k' \textit{ in } K \ . \tag{2.45}$$

Then

$$\beta_\delta(K) = \beta(K) = Ca(K) \ . \tag{2.46}$$

(b) *$Sz(C(\omega^\alpha + 1)) \geq \alpha$ for any countable ordinal α.*

PROOF. (a) We actually have that for any $0 < \varepsilon \leq \delta$ and any closed subset W of K,

$$\delta_\varepsilon(W) = d_\varepsilon(W) = W' \ . \tag{2.47}$$

But then for all countable ordinals α,

$$K_{\alpha,\varepsilon} = K^{(\alpha)} \tag{2.48}$$

which immediately yields (2.46). As for (b), we have that

$$Ca(\omega^\alpha +) = \alpha \ . \tag{2.49}$$

But $\omega^\alpha +$ is obviously ω^*-homeomorphic to a 2-separated subset K of $Ba(C(\omega^\alpha +))^*$. Thus we have

$$Sz(Ca(\omega^\alpha +)) = \beta(K) = \beta_2(K) = \alpha \ . \qquad \square \tag{2.50}$$

We may now obtain, via Theorem 2.14, the "easier" half of Theorem 2.15.

Corollary 2.21 *Let* $0 \leq \alpha < \omega$. *Then*

$$Sz(C(\omega^{\omega^\alpha}+)) \geq \omega^{\alpha+1} \ . \tag{2.51}$$

PROOF. For each positive integer n, we have that

$$Sz(C(\omega^{\omega^\alpha \cdot n}+)) \geq \omega^\alpha \cdot n \text{ by 2.20(b).} \tag{2.52}$$

But $C(\omega^{n\cdot\omega^\alpha}+)$ is isomorphic to $C(\omega^{\omega^\alpha}+)$ by Theorem 2.14, and hence since the Szlenk index is isomorphically invariant (by Corollary 2.19)

$$Sz(C(\omega^{\omega^\alpha}+)) \geq n \cdot \omega^\alpha \text{ for all integers } n \ , \tag{2.53}$$

which implies (2.51). $\square$

We next deal with (a) of Theorem 2.14. We first give a functional analytical presentation of the spaces $C(\omega^{\omega^\alpha})$, using injective tensor products. We first recall the definitions (see [DJP], specifically pp. 485–486).

Definition 2.22 *Given Banach spaces* X *and* Y, *the injective tensor norm,* $\|\cdot\|_\varepsilon$, *is defined on* $X \otimes Y$, *the algebraic tensor product of* X *and* Y, *by*

$$\|\sum_{k=1}^n x_k \otimes y_k\|_\varepsilon = \sup\left\{|\sum_{k=1}^n x^*(x_k)y^*(y_k)| : x^* \in \operatorname{Ba} X^*, y \in \operatorname{Ba} Y^*\right\} \tag{2.54}$$

for any n, $x_1,\ldots,x_k$ *in* X *and* $y_1,\ldots,y_k$ *in* Y. *The completion of* $X \otimes Y$ *under this norm is called the injective tensor product of* X *and* Y, *denoted* $X \overset{\vee}{\otimes} Y$.

When K and L are compact Hausdorff spaces, then we have that $C(K) \overset{\vee}{\otimes} C(L)$ is canonically isometric to $C(K \times L)$, where the elementary tensor $x \otimes y$ in $C(K) \otimes C(L)$ is simply identified with the function $(x \otimes y)(k,\ell) = x(k)y(\ell)$ for all $k, \ell \in K \times L$.

We then obtain the following natural tensor product construction of the spaces $C(\omega^{\omega^\alpha}+)$ (where we use the "unitization" given in (2.36).

Define a family $(X_\alpha)_{\alpha<\omega_1}$ as follows. Let $X_0 = c$ and also let $X_{0,1} = c_0$. Suppose X_α has been defined. Set $X_{\alpha,n} = X_\alpha \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} X_\alpha$ (n-times). Then set $X_{\alpha+1,0} = (X_{\alpha,1} \oplus X_{\alpha,2} \oplus \cdots \oplus X_{\alpha,n} \oplus \cdots)_{c_0}$, and $X_{\alpha+1} = X_{\alpha+1,0} \oplus [1]$. Finally, suppose β is a countable limit ordinal, and X_α has been defined for all $\alpha < \beta$. Choose (α_n) a sequence with $\alpha_n \nearrow \beta$, set $X_{\beta,0} = (X_{\alpha_1} \oplus \cdots \oplus X_{\alpha_n} \oplus \cdots)_{c_0}$; then set $X_\beta = X_{\beta,0} \oplus [1]$.

The following result now follows by transfinite induction.

Proposition 2.23 *Let $\alpha < \omega_1$ and $1 \leq n < \infty$. Then $X_{\alpha,0}$ is algebraically isometric to $C_0(\omega^{\omega^\alpha})$, and $X_{\alpha,n}$ is algebraically isometric to $C(\omega^{\omega^\alpha \cdot n}+)$. In particular, $X_\alpha = C(\omega^{\omega^\alpha}+)$.*

We next give the main step in the proof of Theorem 2.14(a).

Theorem 2.24 *There exists an absolute constant κ so that for any infinite countable compact metric space K,*

$$d(C(K),(C(K) \oplus C(K) \oplus \cdots)_{c_0}) \leq \kappa \ .$$

Remark 2.25 Of course, it follows immediately from (a) that also for any n, $d(C(K), \oplus_1^n C(K)) \leq \kappa$ where $\oplus_1^n X$ denotes the ℓ^∞-direct sum of n copies of X.

Proof It suffices to show this for all the spaces $C(\omega^\alpha+)$. Indeed, once this has been done, it follows from Proposition 2.13 that given K, there exists a unique α and n with $C(K)$ isometric to $\oplus_1^n C(\omega^\alpha+)$. But then $(C(K) \oplus C(K) \oplus \cdots)_{c_0}$ is isometric to $\oplus_1^n c(\omega^\alpha+) \oplus \cdots)_{c_0}$, which of course is isometric to $(C(\omega^\alpha+) \oplus \cdots)_{c_0}$; hence we obtain that also $d(C(K),(C(K) \oplus C(K) \oplus C(K) \oplus \cdots)_{c_0} \leq \kappa$.

Let $(X_\beta)_{\beta<\omega_1}$ and $(Y_\beta)_{\beta<\omega_1}$, be our transfinite presentation of these spaces, preceding 2.13. Surprisingly, this proof is *not* by transfinite induction. Suppose $\beta \geq 1$, and β is a successor, $\beta = \alpha + 1$. But then $X_\beta = (Y_\alpha \oplus Y_\alpha \oplus)_{c_0}$, and so X_β is isometric to its c_0 sum with itself. Then letting γ be the constant in Lemma 2.5 (e),

$$d(X_\beta \oplus [1], X_\beta) \leq \gamma \ . \tag{2.55}$$

Since also $Y_\beta = X_\beta \oplus [1]$,

$$d((Y_\beta \oplus Y_\beta \oplus \cdots)_{c_0}, X_\beta) \leq \gamma \ , \tag{2.56}$$

thus

$$d((Y_\beta \oplus Y_\beta \oplus \cdots)_{c_0}, Y_\beta) \leq \gamma^2 \ . \tag{2.57}$$

Now suppose β is a limit ordinal; choose (α_n) with $\alpha_n \nearrow \beta$. Then

$$\begin{aligned}
(X_\beta \oplus X_\beta \oplus \cdots) &= \left(\oplus_{j=1}^\infty (Y_{\alpha_j} \oplus Y_{\alpha_j} \oplus \cdots)_{c_0} \right)_{c_0} && (2.58)\\
&= (X_{\alpha_1+1} \oplus X_{\alpha_2+2} \oplus \cdots)_{c_0} && (2.59)\\
&\hookrightarrow (Y_{\alpha_1+1} \oplus Y_{\alpha_2+2} \oplus \cdots)_{c_0} && (2.60)\\
&\cong X_\beta && (2.61)
\end{aligned}$$

Here $\alpha_{n+1} < \beta$ too, and the final isometry follows from the invariance (isometric) in the definition of the X_β's, while the "2-complementation" follows trivially, since X_{α_n+1} is codimension 1 in Y_{α_n+1} for all n. That is, (2.58) yields that $(X_\beta \oplus X_\beta \oplus \cdots)_{c_0}$ is isometric to a 2 complemented subspace of X_β. But then a quantitative version of the Pełczyński decomposition method produces an absolute constant τ so that

$$d(X_\beta \oplus X_\beta \oplus \cdots, X_\beta) \leq \tau \ . \tag{2.62}$$

But then (2.55) yields immediately that

$$d((Y_\beta \oplus \cdots)_{c_0}, Y_\beta) \le \gamma^2 \tau \ . \tag{2.63}$$

This completes the proof.

We need one final ingredient, involving tensor products.

Lemma 2.26 *Let $\alpha = 1$ or α be a countable limit ordinal, and let n be a positive integer, and let $K = \omega^\alpha +$. Then*

$$C(K^n) \sim C(K) \ ,$$

Proof It really suffices to prove this for $n = 2$. For we obtain that setting $X = C(K)$, then

$$X \sim X \overset{\vee}{\otimes} X \ . \tag{2.64}$$

But now it follows immediately from induction that

$$X \sim \overset{n\vee}{\underset{1}{\otimes}} X \tag{2.65}$$

where $\overset{n\vee}{\underset{1}{\otimes}} X = X \overset{\vee}{\otimes} \cdots \overset{\vee}{\otimes} X$ n-times, and of course $\overset{n\vee}{\underset{1}{\otimes}} X \cong C(K^n)$.

We first note an immediate consequence of Lemma 2.5(b) (the Borsuk theorem). Let M be any compact metric space and L be a closed subset of M; let $C_0(K, L) = \{f \in C(K) : f(\ell) = 0 \text{ for all } \ell \in L\}$. Then

$$C(M) \sim C_0(M, L) \oplus C(L) \ . \tag{2.66}$$

Now suppose that $\alpha = 1$. We wish to show that

$$C(\omega+) \sim C(\omega^2+) = C((\omega+) \times (\omega+)) \ . \tag{2.67}$$

To avoid confusing notation, set $p = \omega$ and denote $\{p\}$ just by p. Set $M = (\omega+) \times (\omega+)$ and let $L = [(\omega+) \times p] \cup [p \times (\omega+)]$. Then $C_0(M, L)$ is isometric to c_0, hence (2.66) yields that

$$C(\omega^2+) \sim c \oplus c_0 \sim c = C(\omega+) \ . \tag{2.68}$$

Now let α be a countable limit ordinal, and choose (α_n) with $\alpha_n < \alpha$ for all n and $\alpha_n \to \alpha$. Let $M = K \times K$, set $p = \omega^\alpha$, and let $L = [K \times p] \cup [p \times K]$. Then it follows (via Theorem 2.24) that

$$C(L) \sim C(K) \ . \tag{2.69}$$

Now for each j, let $K_j = \omega^{\alpha_j} +$. Then

$$C_0(M, L) \cong \big(\oplus_{1 \le j, n < \infty} C(K_j \times K_n) \big)_{c_0} \tag{2.70}$$

$$\cong \Big(\oplus_{n=1}^{\infty} \big(\oplus_{n=j}^{\infty} C(K_j \times K_n)\big)_{c_0}\Big)_{c_0} . \tag{2.71}$$

But for each $j \leq n$,

$$C(K_j \times K_n) \overset{cc}{\hookrightarrow} C(K_n \times K_n) . \tag{2.72}$$

Hence

$$\begin{aligned} \big(\oplus_{n=j}^{\infty} C(K_j \times K_n)\big)_{c_0} &\overset{cc}{\hookrightarrow} \oplus_{n=j}^{\infty} C(K_n \times K_n) & (2.73)\\ &\cong C_0(K,p) . & (2.74)\end{aligned}$$

(Indeed, note that $C(K_n \times K_n) \cong C(\omega^{\alpha_n \cdot 2}+)$ and $\alpha_n \cdot 2 < \alpha$ for all n. Thus finally, by (2.70) and (2.73),

$$\begin{aligned} C_0(M,L) &\overset{cc}{\hookrightarrow} (C_(M,L) \oplus C_0(M,p) \oplus \cdots)_{c_0} & (2.75)\\ &\sim C_0(M,p) & (2.76)\end{aligned}$$

by Theorem 2.24.

Of course, $C_0(K,p) \overset{cc}{\hookrightarrow} C_0(M,L)$ for $C_0(K,p) \cong (\oplus_{n=1}^{\infty} C(1 \times K_n))_{c_0}$. Then by the Pełczyński decomposition method

$$C_0(K,P) \sim C_0(M,L) \tag{2.77}$$

and so finally by (2.66),

$$C(M) \sim C_0(K,p) \oplus C(K) \sim C(K) . \quad \square \tag{2.78}$$

At last, we give the

Proof of Theorem 2.14(a) Let K be as in its statement. We know there is an infinite countable ordinal β with $C(K) \cong C(\beta+)$. Define

$$\alpha = \sup\{\gamma : \omega^{\omega^\gamma} \leq \beta\} . \tag{2.79}$$

Then

$$\omega^{\omega^\alpha} \leq \beta < \omega^{\omega^{\alpha+1}} . \tag{2.80}$$

Since $\omega^{\omega^{\alpha+1}} = \lim_{n\to\infty} \omega^{\omega^\alpha \cdot n}$, there is a positive integer n with

$$\beta \leq \omega^{\omega^\alpha \cdot n} . \tag{2.81}$$

Then evidently from (2.80) and (2.81),

$$C(\omega^{\omega^\alpha}+) \overset{cc}{\hookrightarrow} C(\beta+) \quad \text{and} \quad C(\beta+) \overset{cc}{\hookrightarrow} C(\omega^{\omega^\alpha \cdot n}) . \tag{2.82}$$

Now if $\alpha = 0$, $\omega^\alpha = 1$; otherwise, ω^α is a limit ordinal. So setting $M = \omega^{\omega^\alpha}+$, $C(M) \sim C(M^n)$ by Lemma 2.26, and of course $M^n = \omega^{\omega^{\alpha} \cdot n}+$, thus

$$C(\omega^{\omega^{\alpha} \cdot n}+) \sim C(\omega^{\omega^\alpha}) \ , \tag{2.83}$$

whence $C(K)$ is isomorphic to $C(\omega^{\omega^\alpha}+)$ by the decomposition method. $\square$

We finally make some observations about the proof of the "harder" half of Theorem 2.14. We first give the first of several arguments, here, showing that $c = C(\omega+)$ is *not* isomorphic to $C(\omega^\omega+)$, via the Szlenk index.

Proposition 2.27 $S_z(c_0) = \omega$.

Of course then $Sz(c) = Sz(C(\omega^n+)) = \omega$ also,(for all $n < \infty$), while $Sz(C(\omega^\omega+)) \geq \omega^2$ by Corollary 2.21.

The following elementary result easily yields 2.21, since of course $Sz(c_0) \geq \omega$ by (2.24). We identify c_0^* with ℓ^1; of course then a sequence in $\mathrm{Ba}(\ell^1)$ converges ω^* precisely when it converges pointwise on $\mathbf{N}$.

Lemma 2.28 *Let $0 < \delta < 1$ and let f, (f_n) in $\mathrm{Ba}(\ell^1)$ so that $f_n \to f$ ω^* and $\|f_n - f\| \geq \delta$ for all n. Then $\|f\| \leq 1 - \delta$.*

Proof Let $0 < \varepsilon$, and choose M with

$$\sum_{j=M+1}^{\infty} |f(j)| < \varepsilon \ . \tag{2.84}$$

Then choose n so that

$$\sum_{j=1}^{M} |f_n(j) - f(j)| < \varepsilon \ . \tag{2.85}$$

But then

$$\delta - \varepsilon < \sum_{j=M+1}^{\infty} |f_n(j) - f(j)| < \sum_{j=M+1}^{\infty} |f_n(j)| + \sum_{j=M+1}^{\infty} |f(j)| \tag{2.86}$$

so by (2.83)

$$\sum_{j=M+1}^{\infty} |f_n(j)| > \delta - 2\varepsilon \ , \tag{2.87}$$

hence

$$\sum_{j=M+1}^{\infty} |f_n(j)| > 1 - \delta + 2\varepsilon \ , \tag{2.88}$$

and finally

$$\sum_{j=1}^{M} |f(j)| \leq (1-\delta) + 3\varepsilon \tag{2.89}$$

by (2.85) and (2.88). Hence

$$\sum_{j=1}^{\infty} |f(j)| < 1 - \delta + 4\varepsilon \ , \tag{2.90}$$

proving the Lemma since $\varepsilon > 0$ is arbitrary. $\square$

Corollary 2.29 *Let $K = \mathrm{Ba}(\ell^1, \omega^*)$, and $0 < \delta < 1$. Then if $f \in K_{n,\delta}$ (as defined following (2.38)), $\|f\| \leq (1-\delta)^n$.*

Proof By induction on n. $n = 1$ is the previous result. So, suppose proved for n, and $f \in K_{n+1,\delta}$. Choose (f_m) in $K_{n,\delta}$ with $f_m \to f \ \omega^*$ and $\|f_m - f\| \geq \delta$ for all m. But then by induction hypothesis $\|f_m/(1-\delta)^n\| \leq 1$ for all m and since $\|(f_m - f)/(1-\delta)^n\| \geq \delta/(1-\delta)^n$ for all m, $\|f/(1-\delta)^n\| \leq 1-\delta$ by Lemma 2.28. Hence $\|f\| \leq (1-\delta)^{n+1}$, proving 2.29. $\square$

Proof of Proposition 2.27 Again let $0 < \delta < 1$. It follows from the preceding result that if $(1-\delta)^n < \delta$, then $K_{n+1,\delta} = \emptyset$, hence (c.f. Definition 2.16) $\beta_\delta(K) \leq n$ and so $\beta(K) = Sz(c_0) \leq \omega$. $\square$

The proof of Theorem 2.13 in [Sa] uses rather delicate properties of the ordinal numbers and their reflection in properties of the spaces $C(\omega^{\omega^\alpha}+)$. In fact, it is *first* proved in [Sa] that if $\alpha < \beta$, then $C(\omega^{\omega^\beta}+)$ is not isomorphic to a quotient space of $C(\omega^{\omega^\alpha}+)$, and then the desired inequality about the Szlenk index is deduced, using in part a result in [AB2]. Of course this result is in turn a consequence of 2.13 via the natural properties of the Szlenk index developed above. The present author "believes" a direct functional analytical proof of 2.13 should be possible, in the spirit of the presentation of the spaces $C(\omega^{\omega^\alpha})$ given above.

3 Some Banach space properties of separable $C(K)$-spaces

A. Weak injectivity.

We first consider a separable weak injectivity result, due to A. Pełczyński [Pe4].

Theorem 3.1 *Let Y be a subspace of a separable Banach space X, with Y isomorphic to a separable $C(K)$-space. Then there exists a subspace Z of Y which is isomorphic to Y and complemented in X.*

We give a proof due to J. Hagler [H1], which yields nice quantitative information. We say that Banach spaces X and Y are λ-isomorphic if there is a surjective isomorphism $T : X \to Y$

with $\|T\|\|T^{-1}\| \leq \lambda$. If $X \subset Y$, we say that X is λ-complemented in Y if there is a linear projection P of Y onto X with $\|P\| \leq \lambda$. The reader may then easily establish the *if the diagram 2.6 holds with linear maps U and V satisfying $\|U\|\|V\| \leq \lambda$, then X is λ-isomorphic to a λ-complemented subspace of Y.*

The quantitative version of 3.1 given in [H1] then goes as follows:

Theorem 3.2 *Let K be an infinite compact metric space, X a separable Banach space, and Y a subspace of X λ-isomorphic to $C(K)$. If K is uncountable, let $\Omega = D$, the Cantor set. If K is countable, let $\Omega = K$. Then Y contains a subspace λ-complemented in X and λ-isomorphic to $C(\Omega)$.*

Of course, 3.1 follows from 3.2 and Milutin's Theorem (Theorem 2.1). We require a topological lemma (due to Kuratowski in the uncountable case, Pełczyński in the countable case).

Lemma 3.3 *Let M and L be compact metric spaces and $\tau : M \to L$ be a continuous surjection.*

(a) If L is uncountable, there is a subset Ω of M homeomorphic to $\mathbf{D}$ with $\tau|\Omega$ a homeomorphism of Ω with $\tau(\Omega)$.
(b) If L is countable, there is a subset Ω of M homeomorphic to L with $\tau|\Omega$ a homeomorphism of Ω with $\tau(\Omega)$.

Proof of Theorem 3.2 If W is a ω^*-compact subset of the dual B^* of a Banach space B, let $R_W : B \to C(W)$ be the continuous map $(R_W b)(w) = w(b)$ for all $b \in B$, $w \in W$. Now choose $T : C(K) \to Y$ a surjective isomorphism with

$$\|T\| = 1 \text{ and } \|T^{-1}\| \leq \lambda \ . \tag{3.1}$$

Let $i : Y \to X$ be the identity injection. Then

$$(iT)^*(\lambda \operatorname{Ba}(X^*)) \supset \operatorname{Ba}(C(K)^*) \ . \tag{3.2}$$

Then regarding K as canonically embedded in $C(K)^*$, we have that setting $M = [(iT)^*]^{-1}(K)]$ $\cap \lambda \operatorname{Ba}(X^*)$ then M is ω^* metrizable and setting $\tau = (iT)^*|M$, then

$$\tau : M \to K \quad \text{is a continuous surjection.} \tag{3.3}$$

If K is uncountable, choose $\Omega \subset M$ homeomorphic to $\mathbf{D}$ satisfying (a) of Lemma 2.5; if K is countable, choose $\Omega \subset M$ homeomorphic to K, satisfying (b) of 2.5. Set $\Omega' = \tau(\Omega)$. Finally, let $\beta = \tau^{-1}$. Recall (c.f. Definition 2.10) that $\beta^0 : C(\Omega) \to C(\Omega')$ is the canonical (algebraic) isometry induced by β; since β is a homeomorphism. β^0 is surjective. At last let $E : C(\Omega') \to C(K)$ be an isometric linear extension operator (as provided by Lemma 2.5(h).) We then have that the following diagram is commutative.

$$\begin{tikzcd} & Y \arrow[r, "i"] & X \arrow[r, "R_\Omega"] & C(\Omega') \arrow[dd, "\beta^0"] \\ C(K) \arrow[ur, "T"] & & & \\ C(\Omega') \arrow[u, "E"] \arrow[rrr, "I"] & & & C(\Omega')\ . \end{tikzcd}$$

(To check this, let $\omega' \in \Omega'$ and set $\beta(\omega') = \omega$. Thus $\omega' = \tau(\omega) = (iT)^*(\omega)$. Let $f \in C(\Omega')$. Then $\beta^0 R_\Omega iTEf(\omega') = R_\Omega iTEF(\beta\omega') = iTEf(\omega) = Ef(iT)^*(\omega) = (Ef)(\omega') = f(\omega')$). Finally, let $U = iTE$ and $V = \beta^0 R_\Omega$. Then $\|u\| \le 1$ and since $\Omega \subset \lambda \operatorname{Ba}(X^*)$, $\|V\| \le \lambda$. Thus $Z = U(C(\Omega'))$ is λ-isomorphic to $C(\Omega')$ and λ-complemented in X. Of course $Z \subset Y$, thus completing the proof. $\square$

Theorem 3.1 has an interesting consequence for $C(K)$ quotients of separable spaces.

Corollary 3.4 *Let K be a compact metric space, and X be a separable Banach space. If $C(K)$ is isomorphic to a quotient of a subspace of X, then $C(K)$ is isomorphic to a quotient space of X.*

Proof Let Y be a subspace of X such that $C(K)$ is isomorphic to a quotient space of Y and let Z be a subspace of ℓ^∞, isometric to $C(K)$. We may thus choose a bounded linear surjection $T : Y \to Z$. Since ℓ^∞ is injective, we may choose $\tilde{T} : X \to \ell^\infty$ a bounded linear operator extending T. But then $W \overset{\text{def}}{=} \overline{\tilde{T}(X)}$ is separable and of course $Z \subset W$. Hence we may choose $Z' \subset Z$ with Z' isomorphic to Z and a bounded linear projection P from W onto Z'. But then $P\tilde{T}$ maps X onto Z', completing the proof. $\square$

B. c_0-saturation of spaces with separable dual.

We next discuss a "thin" property of $C(K)$ spaces for K countable, due to Bessaga and Pełczyński.

Definition 3.5 *Let X be an infinite dimensional Banach space X. X is called c_0-saturated if c_0 embeds (isomorphically) into every (closed linear) infinite dimensional subspace.*

Proposition 3.6 [BP] *Let K be a countable infinite compact metric space. Then $C(K)$ is c_0-saturated.*

It is unknown if every quotient space of such a $C(K)$ space is c_0-saturated. Actually, to "play the devil's advocate", it is also unknown if ℓ^2 is isomorphic to a subspace of a quotient of $C(\omega^\omega+)$.

Proposition 3.6 is really an immediate consequence of our transfinite description of the $C(K)$-spaces and the following natural permanence property of c_0-saturated spaces.

Lemma 3.7 *Let $X_1, X_2, \dots$ be c_0 saturated Banach spaces. Then $X \overset{\text{def}}{=} (X_1 \oplus X_2 \oplus \cdots)_{c_0}$*

is c_0 saturated.

Proof We first observe that for all n, $X_1 \oplus \cdots \oplus X_n$ is c_0 saturated. Indeed, it suffices, using induction, to show this for $n = 2$. If Y is an infinite dimensional subspace of $X_1 \oplus X_2$ which is not isomorphic to a subspace of X_1, then letting P be the natural projection of $X_1 \oplus X_2$ onto X_1, we may choose a normalized basic sequence (y_n) in Y with $\sum \|Py_n\| < \infty$. It then follows that for some N, $(y_j)_{j=N}^{\infty}$ is isomorphic to a subspace of X_2, whence $[y_j]_{j=N}^{\infty}$ contains a subspace isomorphic to c_0. Now for each n, let P_n be the natural projection of X onto $X_1 \oplus \cdots \oplus X_n$, and let Y be an infinite dimensional subspace of X. Then if Y is isomorphic to a subspace of $X_1 \oplus \cdots \oplus X_n$, c_0 embeds in Y by what we proved initially. If this is false for all n, we may choose a normalized basic sequence (y_n) in Y so that $\sum \|P_n(y_n)| < \infty$. Well, a standard travelling hump argument now yields that there is a subsequence (y'_n) of (y_n) with $(y'_n$ equivalent (almost isometrically) to the c_0 basis. $\square$

Proof of Proposition 3.6 We may just use the transfinite description of the spaces $Y_\alpha = C(\omega^\alpha +)$ given preceding proposition 2.13. It suffices to prove these spaces are c_0 saturated, since for every infinite countable ordinal β, $C(\beta+)$ is isometric to the n-fold direct sum of one of these, for some n.

Of course c_0 is itself c_0 saturated, and so then trivially so is $c = Y_1$. Suppose $\beta > 1$ is a countable ordinal, and it is proved that Y_α is c_0-saturated for all $\alpha < \beta$. If β is a successor, say $\beta = \alpha+1$, then $Y_\beta^0 = (Y_\alpha \oplus Y_\alpha \oplus \cdots)_{c_0}$ is c_0-saturated by Lemma 3.7, and Y_β is isomorphic to Y_β^0 by Lemma 2.5(e). But if β is a limit ordinal, then choose (α_n) with $\alpha_n \nearrow \beta$. So then Y_β is again isomorphic to $X_\beta = (Y_{\alpha_1} \oplus Y_{\alpha_2} \oplus \cdots)_{c_0}$, which again is c_0-saturated by Lemma 3.7. $\square$

C. Uncomplemented embeddings of $C([0,1])$ and $C(\omega^\omega +)$ in themselves.

The last result we discuss in some depth in this section, is D. Amir's theorem: $C(\omega^\omega +)$ *is not separably injective* [A]. By the results of the preceding section, it follows that if K is an infinite compact metric space, then $C(K)$ is separably injective only if $C(K)$ is isomorphic to c_0 (as also pointed out in [A]). (See ([JL, pp.18–19] for a short proof of the theorem that c_0 is separably injective.) Of course it follows that $C([0,1])$ is not separably injective. A concrete witness of this result: let $\phi\{0,1\}^{\mathbf{N}} \to [0,1]$ be the Cantor map, $\phi((\varepsilon_j)) = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{2j}$. Then $\phi^0(C[0,1])$ is uncomplemented in $C(\{0,1\}^{\mathbf{N}})$ and of course $\{0,1\}^{\mathbf{N}}$ is homoemorphic to $\mathbf{D}$ the Cantor discontinuum. This uncomplementation result is due to Milutin [M].

We give a proof of both of these results, by using a classical space of discontinuous functions on $[0,1]$ which arises in probability theory; namely the space of all scalar-valued functions on $[0,1]$ which are *right continuous* with *left limits*, denoted by $\mathrm{rcl}([0,1])$. We may easily generalize this to arbitrary compact subsets of $[0,1]$.

Definition 3.8 *Let K be an infinite compact subset of K. Let $\mathrm{rcl}(K,D)$ denote the family of all scalar valued functions f on K so that f is continuous for each $k \in K \sim D$, and so that f is right continous with left limit at each point $d \in D$. In case $D = K$, let $\mathrm{rcl}(K) = \mathrm{rcl}(K,K)$.*

We shall show that if D is a countable dense subset of $[0,1]$, then $C([0,1])$ is an uncomplemented subspace of $\mathrm{rcl}([0,1],D)$; and that this yields Milutin's result concerning the Cantor map, for D the set of all dyadic intervals. Finally, we show that there is a subset K of $[0,1]$ homeomorphic to $\omega^\omega+$ such that $C(K)$ is uncomplemented in $\mathrm{rcl}(K)$.

We first need the following concept.

Definition 3.9 *Let K be a subset of $[0,1]$, and let $K_{(1)}$ denote the set of 2-sided cluster points of K. That is, $x \in K_{(1)}$ provided there exist sequences (y_j) and (z_j) in K with $y_j < x < z_j$ for all j and $\lim_{j\to\infty} y_j = x = \lim_{j\to\infty} z_j$. Then for $n \geq 1$, let $K_{(n+1)} = (K_{(n)})_{(1)}$. Finally let $K_{(\omega)} = \cap_{n=1}^\infty K_{(n)}$.*

Of course we could define $K_{(\alpha)}$ for arbitrary countable ordinals, but we have no need of this. Also, if K is not closed, we need not have that $K_{(1)} \subset K$, and moreover, even if K is closed, $K_{(1)}$ may not be; e.g. $[0,1]_{(1)} = (0,1)$. We do, however, have the following simple result.

Proposition 3.10 *Let K be a compact subset of $[0,1]$. Then $K_{(n+1)} \subset K_{(n)}$ for all n.*

Proof. Set, for convenience, $K_{(0)} = K$. So the result trivially holds for $n = 0$. Suppose proved for n, and let $x \in K_{(n+2)}$. Then choosing (y_j) and (z_j) as in 3.9 with (y_j), (z_j) in $K_{(n+1)}$, for all n, the y_j's and z_j's also belong to $K_{(n)}$ by induction, thus $x \in K_{(n+1)}$. $\square$

Now we dig into the *way* in which $C(K)$ is embedded in $\mathrm{rcl}(K)$, which after all, *is* algebraically isometric to $C(M)$ for some comact Hausdorff M.

Proposition 3.11 *Let K be a compact subset of $[0,1]$, and assume D is an infinite countable subset of $K_{(1)}$. Set $B = \mathrm{rcl}(K,D)$. Then B is an algebra of bounded functions and $B/C(K)$ is isometric to c_0.*

Remark 3.12 Without the countability assumption we still get that $B/C(K)$ is isometric to $c_0(D)$.

Proof. For each $f \in B$, $d \in D$, let $f(d-) = \lim_{x\uparrow d} f(x)$ (i.e., the left limit of f at d). Fix $f \in B$. Now it is easily seen that f is bounded. In fact, a classical elementary argument shows that for all $\varepsilon > 0$,

$$\{d \in D : |f(d) - f(d-)| > \varepsilon\} \quad \text{is finite.} \tag{3.4}$$

It then follows that defining $T : B \to \ell^\infty(D)$ by

$$(Tf)(d) = \frac{f(d) - f(d-)}{2} \qquad \text{for all } f \in B, d \in D \tag{3.5}$$

then T is a linear contraction valued in $c_0(D)$. Now for each $d \in D$, *define* $f_d \in B$ by

$$f_d(k) = 1 \quad \text{if} \quad k < d \ , f_d(k) = -1 \quad \text{if} \quad k \geq d \tag{3.6}$$

Now since $d \in K_{(1)}$, it follows easily that $\mathrm{dist}(f_d, C(K)) = 1$. In fact, letting $\pi : B \to B/C(K)$ be the quotient map, we have that for any n, k disjoint points $d_1, \ldots, d_n$, and arbitrary scalars

$c_1, \ldots, c_n$,

$$\|\pi(\sum c_j f_{d_j})\| = \max_j |c_j| = \|T(\sum c_j f_{d_j}\| \ . \tag{3.7}$$

This easily yields that in fact T is a quotient map, and moreover if $d_1, d_2, \ldots$ is an enumeration of D, then $(\prod f_{d_j})$ is isometrically equivalent to the usual c_0 basis and $[\prod f_{d_j}] = B/C(K)$. $\square$

The next lemma is the crucial tool for our non-complementation results.

Lemma 3.13 *Let K be a compact subset of $[0,1]$ so that $K_{(n)} \neq \emptyset$. If K is countable, let $D = K_{(1)}$. If $K = [0,1]$, let D be a countable dense subset of $(0,1)$ (the open unit interval). Assume for each $d \in D$, there is given $g_d \in C(K)$. Then given $\varepsilon > 0$, there exist $d_1, \ldots, d_n$ in D and v in K so that*

$$|(f_{d_j} + g_{d_j})(\nu)| > 1 - \varepsilon \quad \textit{for all } 1 \leq j \leq n. \tag{3.8}$$

Proof. For convenience, we assume real scalars. Note also that in the case $K = (0,1)$, $K_{(n)} = (0,1)$. Let $\varepsilon > 0$ be fixed. First choose $d_1 \in K_{(n)} \cap D$. Now choose $\delta_1 > 0$ so that letting $V_1 = (d_1 - \delta_1, d_1 + \delta_1) \cup K$, then

$$|g_{d_1}(d_1) - g_{d_1}(x)| < \varepsilon \quad \text{for all } x \in V_1 \ . \tag{3.9}$$

For simplicity in notation, set $a = g_{d_1}(d_1)$. Now if $x \in V_1$ and $x > d_1$,

$$(f_{d_1} + g_{d_1})(x) > -1 + a + \varepsilon \ . \tag{3.10}$$

If $x \in V_1$, $x < d_1$, then

$$(f_{d_1} + g_{d_1})(x) > 1 + a - \varepsilon \ . \tag{3.11}$$

But

$$\max\{|1 + a - \varepsilon|, |-1 + a + \varepsilon|\} = 1 - \varepsilon + |a| \geq 1 - \varepsilon \ . \tag{3.12}$$

Now since d_1 is a 2-sided cluster point of $K_{(n-1)}$, it follows that

$$V_1^1 \overset{\text{def}}{=} (d_1 - \delta_1, d_1) \cap K_{(n-1)} \cap D \neq \emptyset$$

and

$$V_1^2 \overset{\text{def}}{=} (d_1, d_1 + \delta_1) \cap K_{(n-1)} \cap D \neq \emptyset \ .$$

Hence if follows from (3.7) - (3.11) that we may set $\tilde{V}_1 = V_1^1$ or V_1^2, and then

$$\tilde{V}_1 \cap K_{(n-1)} \cap D \neq \emptyset \tag{3.13}$$

and

$$|(f_{d_1} + g_{d_1})(x)| > 1 - \varepsilon \quad \text{for all } x \in \tilde{V}_1 \ . \tag{3.14}$$

Now choose $d_2 \in \tilde{V}_1 \cap K_{(n-1)} \cap D$, and proceed in exactly the same way as in the first step. Thus, we first choose $V_2 \subset \tilde{V}_1$ an open neighborhood of d_2 so that

$$|g_{d_2}(d_2) - g_{d_2}(x)| < \varepsilon \quad \text{for all } x \in V_2 \ . \tag{3.15}$$

Since d_2 is a right and left cluster point of $K_{(n-2)}$ we again choose $\tilde{V}_2$ and open subset of V_2 such that

$$|(f_{d_2} + g_{d_2})(x)| > 1 - \varepsilon \quad \text{for all } x \in V_2^2 \tag{3.16}$$

and so that

$$\tilde{V}_2 \cap K_{(n-2)} \cap D \neq \emptyset \ . \tag{3.17}$$

Continuing by induction, we obtain $d_1, \ldots, d_n$ in D, $d_{n+1} \in K$, and open sets in K, $K = \tilde{V}_0 \supset \tilde{V}_1 \supset \tilde{V}_2 \supset \cdots \supset \tilde{V}_n$ so that for all $1 \le j \le n+1$, $d_j \in \tilde{V}_j$ and

$$|(f_{d_j} + g_{d_j})(x)| > 1 - \varepsilon \quad \text{for all } x \in \tilde{V}_j \ . \tag{3.18}$$

Evidently then $d_1, \ldots, d_n$ and $v = d_{n+1}$ satisfy the conclusion of the lemma $\square$

We are now prepared for our main non-complementation result.

Theorem 3.14 *Let $n > 1$ and let K and D be as in Lemma 3.13. Set $B = \mathrm{rcl}(K, D)$. Then if P is a bounded linear projection of B onto $C(K)$,*

$$\|P\| \ge n - 1 \ . \tag{3.19}$$

Hence if $K = [0,1]$ or if K is countable and $K_{(n)} \neq \emptyset$ for all n, $C(K)$ is an uncomplemented subspace of $\mathrm{rcl}(K, D)$.

Proof. Let $\lambda = \|P\|$, and set $B = \mathrm{rcl}(K)$; also let $\pi : B \to B/C(K)$ be the quotient map. Then letting $Y = \mathrm{kernel}\, P$, standard Banach space theory yields that $\pi(Y) = B/C(K)$ and

$$\|y\| \le (\lambda + 1)\|\pi y\| \quad \text{for all } y \in Y \ . \tag{3.20}$$

Now by Proposition 3.11 and its proof, $B/C(K)$ is isometric to c_0 and in fact $[\pi(f_d)]_{d \in D} = B/C(K)$ and (πf_d) is isometrically equivalent to the c_0-basis (for $c_0(D)$). But it follows from (3.20) that we may then choose (unique) y_d's in Y so that

$$\pi y_d = \pi f_d \quad \text{for all } d \tag{3.21}$$

and

$$\|\sum c_d y_d\| \le (\lambda + 1) \max |c_d| \tag{3.22}$$

for any choice of scalars c_d with $c_d \neq 0$ for all but finitely many d. But (3.21) yields that for each $d \in D$ there is a $g_d \in C(K)$ so that

$$y_d = f_d + g_d \quad \text{for all } d \ . \tag{3.23}$$

At last, given $\varepsilon > 0$, we choose $d_1, \ldots, d_n$ in D and $v \in K$ satisfying (3.13), i.e. the conclusion of Lemma 3.13. But then we may choose scalars $c_1, \ldots, c_n$ with $|c_j| = 1$ for all j, so that

$$c_j(f_{d_j} + g_{d_j})(v) > 1 - \varepsilon \quad \text{for all } 1 \leq j \leq n \ . \tag{3.24}$$

Hence

$$\| \sum_{j=1}^{n} c_j y_{d_j} \| > (\sum_{j=1}^{n} c_j (f_{d_j} + g_{d_j}))(v) > n - n\varepsilon \ . \tag{3.25}$$

Finally, (3.22) and (3.25) yield that

$$\lambda + 1 > n - n\varepsilon \ . \tag{3.26}$$

But $\varepsilon > 0$ was arbitrary, so the conclusion of the theorem follows. □

To complete the proof of Amir's theorem, we only need to exhibit a subset K of $[0,1]$ with K homeomorphic to $\omega^\omega +$ and $K_{(w)} \neq \emptyset$. This is easily done, in the next result.

Proposition 3.15 *Let $\alpha = \omega^n +$ for some $1 \leq n \leq \omega$. Then there is a subset K of $[0,1]$ which is homeomorphic to α, so that $K^{(j)} = K_{(j)}$ for all $j \leq \omega$. Moreover then* $\mathrm{rcl}(K)$ *is algebraically isometric to $C(K) \oplus C(K)$.*

Proof. Obviously, we may put K inside $[-1,1]$ or any particular interval $[a,b]$ instead. For $n = 1$ let $K = \{\frac{1}{n}, -\frac{1}{n}, 0 : n = 1, 2, \ldots\}$. Then evidently $K_{(1)} = K^{(1)} = \{0\}$, K is homeomorphic to $\omega+$, and $\mathrm{rcl}(K)$ is clearly algebraically isometric to $c \oplus c = C(K) \oplus C(K)$. Suppose $1 \leq n < \infty$, $\alpha = \omega^n +$, and $K = K^\alpha$ has been constructed satisfying the conclusion of the proposition.

Let now $\{K_j : j \in \mathbf{Z} \sim \{0\}\}$ be a family of "copies" of K^α, where for each $j \geq 1$,

$$K_j \subset \left(\frac{1}{j+1}, \frac{1}{j} \right) \tag{3.28(i)}$$

while if $f \leq -1$

$$K_j \subset \left(\frac{1}{j}, \frac{1}{j+1} \right) \tag{3.28(ii)}$$

Finally, let $K^{\alpha+1} = \cup_{j \in Z'} K_j \cup \{0\}$ where $Z' = \mathbf{Z} \sim \{0\}$.

Then for any $1 \leq i \leq n$,

$$\begin{aligned} K^{\alpha+1,(i)} &= \cup_{j \in Z'} K_j^{(i)} \cup \{0\} \\ &= \cup_{j \in Z'} K_{j,(i)} \cup \{0\} \\ &= K_{(i)}^{\alpha+1} \ . \end{aligned} \tag{3.27}$$

In particular, for any j, $K_j^{(n)}$ consists of a single point, x_j.

Thus,

$$K^{\alpha+1,(n)} = K_{(n)}^{\alpha+1} = \{x_j, x_{-j}, 0 : j \in \mathbf{N}\} \tag{3.28}$$

and of course as in the first step

$$K^{\alpha+1,(n+1)} = K_{\alpha+1,(n+1)} = \{0\} \ . \tag{3.29}$$

Now letting $X = \{f \in \operatorname{rcl} K^{\alpha+1} : f(x) = 0 \text{ for all } x \geq 0\}$ and $Y = \{f \in \operatorname{rcl} K^{\alpha+1} : f(x) = 0 \text{ for all } x < 0\}$, then

$$\operatorname{rcl} K^{\alpha+1} = X \oplus Y \tag{3.30}$$

(algebraically and isometrically, ℓ^∞ direct sum). But it follows easily from our induction hypothesis that X and Y are both algebraically isometric to $C(\omega^{n+1}+)$, whence the final statement of the proposition holds. This proves the result for all $n < \omega$.

Of course, for $\alpha = \omega$, we may now just repeat the entire procedure, this time placing inside each interval $(\frac{1}{n}, \frac{1}{n+1})$ and $(\frac{1}{-n+1}, \frac{1}{-n})$, a "copy" of K^{ω^n+} which we have constructed above, thus achieving the proof. $\square$

Corollary 3.16 *For each $n > 1$, there is a unital subalgebra A_n of $C(\omega^n \cdot 2+)$ with A_n algebraically isometric to $C(\omega^n+)$, which is not λ-complemented in $C\omega^n \cdot 2+)$ for any $\lambda < n-1$. There is also a unital subalgebra B of $C(\omega^\omega+)$ which is algebraically isometric to $C(\omega^\omega+)$ which is uncomplemented in $C(\omega^\omega+)$.*

Proof. The first assertion follows immediately from the preceding two results, for the final assertion, it follows that $B_0 \overset{\text{def}}{=} \left(\oplus_{j=1}^\infty A_j\right)_{c_0}$ is uncomplemented in $\left(\oplus_{n=1}^\infty C(\omega^n \cdot 2+)\right)_{c_0}$. But the second space is just $C_0(\omega^\omega)$, while B_0 is *also* algebraically isometric to $C_0(\omega^\omega)$. Hence just taking the unitizations of each, the result follows. $\square$

Remark 3.17 Of course, since (ω^n+) *is* isomorphic to c_0, it has the separable extension property. Thus, there exists λ_n so that for all separable Banach Spaces $X \subset Y$ and operators $T : X \to C(\omega^n+)$, there is an extension $\tilde{T}$ of T to Y with $\|\tilde{T}\| \leq \lambda_n \|T\|$. Our argument yields that $\lambda_n > n-1$. Actually, Amir proves in [A] that $\lambda_n = 2n+1$ for all $n = 1, 2, \ldots$.

We finally deduce Milutin's result that the Cantor map induces an uncomplemented embedding of $C([0,1])$ in $C(\mathbf{D})$. We identify $\mathbf{D}$ with $\{0,1\}^N$ as above, and let $\varphi : \mathbf{D} \to [0,1]$ be the Cantor map defined above.

Proposition 3.18 *Let D be the set of dyadic rationals in $(0,1)$; i.e., $D = \{\frac{k}{2^n} : 1 \le k < 2^n, n = 1, 2, \ldots\}$. There exists an algebraic surjective isometry $T : C(\{0,1\}^{\mathbf{N}}) \to \mathrm{rcl}([0,1], D)$ such that $T(\varphi^0(C[0,1])) = C([0,1])$. Thus $\varphi^0(C[0,1])$ is uncomplemented in $C(\{0,1\}^N)$ by Theorem 3.14.*

Proof. Define a "standard" partial inverse map $\tau : [0,1] \to \mathbf{D}$ as follows. If $x \in [0,1]$, $x \notin D$, there is a *unique* $y \in D$ with $\varphi(y) = x$, and define

$$\tau(x) = y \ . \tag{3.31}$$

If $x \in D$, then there is a unique $(\varepsilon_j) \in \mathbf{D}$ so that for a unique $n \ge 1$, $\varepsilon_j = 0$ for all $j > n$, $\varepsilon_n = 1$, and $\varphi((\varepsilon_j)) = x$, i.e. $x = \sum_{j=1}^n \frac{\varepsilon_j}{2^j}$. Now define

$$\tau(x) = ((\varepsilon_j)) \ . \tag{3.32}$$

Of course then

$$\varphi(\tau(x)) = x \quad \text{for all } x \in [0,1] \ . \tag{3.33}$$

Now define T by

$$(Tf)(x) = f(\tau(x)) \quad \text{for all } f \in C(\mathbf{D}), x \in [0,1] \ . \tag{3.34}$$

Now it easily follows that T is an algebraic isometry mapping $C(\mathbf{D})$ into $\ell^\infty[0,1]$. We now easily check that

$$T(\varphi^0 f) = f \quad \text{for all } f \in C([0,1]) \ . \tag{3.35}$$

Moreover, if $f \in C(\mathbf{D})$, then

$$Tf \text{ is continuous at } x \text{ for all } x \notin D. \tag{3.36}$$

Finally, let $x \in D$ and $(\varepsilon_j) = \tau(x)$ with n as given preceding (3.32). Let (y_m) be a sequence in $[0,1]$ with $y_m \to x$. Suppose first that

$$x < y_m \quad \text{for all } m. \tag{3.37}$$

Then it follows that for all m,

$$\tau(y_m) = (\varepsilon_1, \ldots, \varepsilon_n, \beta_{n+1}^{(m)}, \beta_{n+2}^{(m)}, \ldots)$$

and in fact then $\tau(y_m) \to \tau(x) = (\varepsilon_1, \ldots, \varepsilon_n, 0, \ldots)$. Hence

$$(Tf)(y_m) = f(\tau(y_m)) \to f(\tau(x) = (Tf)(x) \tag{3.38}$$

by continuity of f. Thus Tf is indeed right continuous at x. Suppose next that

$$y_m < x \quad \text{for all } m. \tag{3.39}$$

This time, it follows that

$$\tau(y_m) = (\varepsilon_1, \ldots, \varepsilon_{n-1}, 0, \beta_{n+1}^{(m)}, \beta_{n+2}^{(m)}, \ldots)$$

for all m, and in fact *now*

$$\tau(y_m) \to (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, 0, 1, 1, , 1 \ldots) \stackrel{\text{def}}{=} z \; .$$

Hence now,

$$(Tf)(y_m) = f(\tau(y_m)) \to f(z) \tag{3.40}$$

by continuity of f, showing that Tf *has* a left-limit. Thus we have indeed proved that

$$T(C(\mathbf{D})) \subset \mathrm{rcl}([0,1], D) \; .$$

We may check, however that conversely if $f \in \mathrm{rcl}([0,1], D)$, then *defining* $\tilde{f}$ on $\mathbf{D}$ by

$$\tilde{f}(\tau x) = f(x) \quad \text{for all } x \in [0,1] \tag{3.41}$$

$$\tilde{f}(y) = f(\varphi(y)-) \tag{3.42}$$

if $y \in \mathbf{D} \sim \tau([0,1])$, then $\tilde{f} \in C(\mathbf{D})$, and hence finally T satisfies the conclusion of 3.18, completing the proof.

4 Operators on $C(K)$-spaces

Throughout, K denotes a compact Hausdorff space. By an operator on $C(K)$ we mean a bounded linear operator from $C(K)$ to some Banach space X. Of course, a “$C(K)$-space” is just $C(K)$ for some K. We first recall the classical result of Dunford and Pettis.

Theorem 4.1 [DP] *A weakly compact operator on a $C(K)$ space maps weakly compact sets to compact sets.*

See [JL] page 62 for a proof.

We note the following immediate structural consequence.

Corollary 4.2 *Let $T : C(K) \to C(K)$ be a given weakly compact operator. Then T^2 is compact. Hence if T is a projection, its range is finite-dimensional.*

Evidently the final statement may be equivalently formulated: *every reflexive complemented subspace of a $C(K)$ space is finite dimensional.*

We are interested here in non-weakly compact operators. Before focusing on this, we note the following structural result due to the author [Ro3].

Theorem 4.3 *A reflexive quotient space of a $C(K)$ space is isomorphic to a quotient space of an $L^p(\mu)$-space for some $2 \le p < \infty$.*

Let us note that conversely, L^p is isometric to a quotient space of $C([0,1])$ for all $2 \le p < \infty$. (Throughout, for all $1 \le p < \infty$, L^p denotes $L^p(\mu)$, where μ is Lebesgue measure on $[0,1]$.) Theorem 4.3 is in reality the dual of the version of the main result in [Ro3]: *every reflexive subspace of L^1 is isomorphic to a subspace of L^p for some $1 < p \le 2$*

We now focus on the main setting of this section — "fixing" properties of various classes of non-weakly compact operators on $C(K)$-spaces.

Definition 4.4 *Let X, Y, Z be Banach spaces. An operator $T : X \to Y$ fixes Z if there exists a subspace Z' of X with Z' isomorphic to Z so that $T|Z'$ is an isomorphism.*

We now summarize the main results to be discussed here. The first result is due to A. Pełczyńksi [Pe2].

Theorem 4.5 *A non-weakly compact operator on a $C(K)$-space fixes c_0.*

To formulate the next result, we will need the notion of the Szlenk-index of an operator.

Definition 4.6 *Let X and Y be separable Banach spaces and $T : X \to Y$ be a given operator; let $\varepsilon > 0$. The ε-Szlenk index of T, $\beta_\varepsilon(T)$, is defined as $\beta_\varepsilon(T^*(\mathrm{Ba}(X^*)))$, where β_ε is as in Definition 2.16.* $\mathrm{Sz}(T)$*, the Szlenk index of T, is defined as:*

$$\mathrm{Sz}(T) = \sup_{\varepsilon>0} \beta_\varepsilon(T) = \beta(T^*(\mathrm{Ba}(X^*))) \ .$$

The results in Section 2 show that $\mathrm{Sz}(C(\omega^\omega+)) = \omega^2$, it was in fact rather easy to obtain that $\mathrm{Sz}(C(\omega^\omega+)) \ge \omega^2$. It follows easily that if an operator on a separable $C(K)$-space fixes $c(\omega^\omega+)$, its Szlenk index is at least ω^2. The converse to this, is due to D. Alspach.

Theorem 4.7 [A1] *Let K be a compact metric space, X be a Banach space, and $T : C(K) \to X$ a given operator. The following are equivalent.*

1) $\mathrm{Sz}(T) \ge \omega^2$.
2) $\beta_\varepsilon(T) \ge \omega$ *for all* $\varepsilon > 0$.
3) T *fixes* $C(\omega^\omega+)$.

We next give another characterization of operators fixing $C(\omega^\omega+)$, due to J. Bourgain.

Definition 4.8 *Let X, Y be Banach spaces and $T : X \to Y$ be a given operator. T is called a* Banach-Saks operator *if whenever (x_j) is a weakly null sequence in X, there is a subsequence (x'_j) of (x_j) so that $\frac{1}{n}\sum_{j=1}^{n} T(x'_j)$ converges in norm to zero. X has the Banach-Saks property if I_X is a Banach-Saks operator.*

It is easily seen that c_0 has the Banach-Saks property, It is a classical result, due to J. Schreier, that $C(\omega^\omega+)$ fails the Banach-Saks property [S]. Thus any operator on a $C(K)$-space fixing $C(\omega^\omega+)$, is not a Banach-Saks operator. Bourgain established the converse to this result in [Bo1].

Theorem 4.9 *A non-Banach-Saks operator on a $C(K)$-space fixes $C(\omega^\omega+)$.*

Bourgain also obtains "higher ordinal" generalizations of Theorem 4.7, which we will briefly discuss.

The final "fixing" result in this summary is due to the author.

Theorem 4.10 [Ro2] *Let K be a compact metric space, X be a Banach space, and $T : C(K) \to X$ be a given operator. Then if $T^*(X^*)$ is non-separable, T fixes $C([0,1])$.*

The proofs of these results involve properties of $L^1(\mu)$-spaces, for by the Riesz representation theorem, $C(K)^*$ may be identified with $M(K)$ the space of scalar-valued regular countably additive set functions on $\mathcal{B}(K)$ the Borel subsets of K.

A. Operators fixing c_0.

Theorem 4.5 follows quickly from the following two L^1 theorems, which we do not prove here. The first is due to A. Grothendieck (Théorème 2, page 146 of [Gr]).

Theorem 4.11 *Let W be a bounded subset of $M(K)$. Then W is relatively weakly compact is and only if for ever sequence $O_1, O_2, \ldots$ of pairwise disjoint open subsets of K,*

$$\mu(O_j) \to 0 \quad \text{as} \quad j \to \infty\ , \quad \text{uniformly for all } \mu \text{ in } W. \tag{4.1}$$

The second is a relative disjointness result due to the author.

Proposition 4.12 [Ro1] *Let $\mu_1, \mu_2, \ldots$ be a bounded sequence in $M(K)$, and let $E_1, E_2, \ldots$ be a sequence of pairwise disjoint Borel subsets of K. Then given $\varepsilon > 0$, there exist $n_1 < n_2 < \cdots$ so that*

$$\text{for all } j, \quad \sum_{j \neq i} |\mu_{n_j}|(E_{n_i}) < \varepsilon\ . \tag{4.2}$$

(For any sequence (f_j) in a Banach space, $[f_j]$ denotes its closed linear span.)

Proof of Theorem 4.5. Assume X is a Banach space and $T : C(K) \to X$ is not weakly compact. Then also $T^* : X^* \to C(K)^* = M(K)$ is non weakly compact, so in particular

$$W \overset{\text{def}}{=} T^*(\mathrm{Ba}(X^*)) \text{ is non-weakly compact.} \tag{4.3}$$

Of course then W is not relatively weakly compact, since it is closed. Thus by Gothendieck's theorem, we may choose $\eta > 0$, a sequence $O_1, O_2, \ldots$ of disjoint open sets in K, and a sequence $\mu_1, \mu_2, \ldots$ in W with

$$|\mu_j(O_j)| > \eta \quad \text{for all } j. \tag{4.4}$$

Let then $0 < \varepsilon < \eta$. By passing to a subsequence of the O_j's and μ_j's, we may also assume by Proposition 4.12 that

$$\sum_{j \neq i} |\mu_j|(O_i) < \varepsilon \quad \text{for all } j. \tag{4.5}$$

For each j, by (4.4) we may choose $f_j \in C(K)$ of norm 1 with $0 \leq f_j \leq 1$ and f_j supported in O_j, so that

$$\Big| \int f_j d\mu_j \Big| > \eta \ . \tag{4.6}$$

Set $Z = [f_j]$. It is immediate that Z is isometric to c_0; in fact (f_j) is isometrically equivalent to the c_0 basis. Thus we have that given n and scalars $c_1, \ldots, c_n$,

$$\Big\| T\Big(\sum_{j=1}^n c_j f_j \Big) \Big\| \leq \|T\| \max_j |c_j| \ . \tag{4.7}$$

But for each j

$$\begin{aligned} \Big\| T\Big(\sum_{i=1}^n c_i f_i \Big) \Big\| &\geq \sup_{x^* \in \mathrm{Ba}\, X^*} \Big| (T^* x^*) \Big(\sum_{i=1}^n c_i f_i \Big) \Big| \\ &\geq \Big| \int \Big(\sum_{i=1}^n c_i f_i \Big) d\mu_j \Big| \\ &\geq |c_j| \Big| \int f_j d\mu_j \Big| - \sum_{i \neq j} |c_i| \int |f_i| d|\mu_j| \\ &\geq |c_j| \eta - \max |c_i| \sum_{i \neq j} |\mu_j|(O_i) \\ &\geq |c_j| \eta - \max |c_i| \varepsilon \ . \end{aligned} \tag{4.8}$$

But then taking the max over all j, we get that

$$\Big\| T\Big(\sum_{i=1}^n c_i f_i \Big) \Big\| \geq (\eta - \varepsilon) \max |c_i| \ . \tag{4.9}$$

(4.6) and (4.9) yield that $T|Z$ is an isomorphism, completing the proof. $\square$

B. Operators fixing $C(\omega^\omega+)$.

We will not prove 4.7. However, we will give the description of the isometric copy of $C(\omega^\omega+)$ which is fixed, in Bourgain's proof of this result. We first indicate yet another important description of the $C(\alpha+)$ spaces, due to Bourgain, which is fundamental in his approach.

Definition 4.13 *Let $\mathcal{T}_\infty$ be the infinitely branching tree consisting of all finite sequences of positive integers. For α, β in $\mathcal{T}_\infty$, define $\alpha \le \beta$ if α is an initial segment of β; i.e., if $\alpha = (j_1, \cdots, j_k)$ and $\beta = (m_1, \ldots, m_\ell)$, then $\ell \ge k$ and $j_i = m_i$ for all $1 \le i \le k$. Also, let $\ell(\alpha) = k$, the length of α. The empty sequence $\emptyset$ is the "top" note of $\mathcal{T}_\infty$. A set $\mathcal{T} \subset \mathcal{T}_\infty$ will be called a* tree *if whenever $\beta \in \mathcal{T}$ and $\alpha \in \mathcal{T}_\infty$, $\alpha \le \beta$ and $\alpha \ne \emptyset$, then $\alpha \in \mathcal{T}$. Finally, a tree $\mathcal{T}$ is called* well-founded *if it contains no strictly increasing sequence of elements of $\mathcal{T}_\infty$.*

We now define Banach spaces associated to trees $\mathcal{T}$, denoted $X_\mathcal{T}$, as follows.

Definition 4.14 *Let $\mathcal{T}$ be a well founded tree, and let $c_{00}(\mathcal{T})$ denote all systems $(c_\alpha)_{\alpha\in\mathcal{T}}$ of scalars, with finitely many c_α's non-zero. Define a norm $\|\cdot\|_\mathcal{T}$ on $c_{00}(\mathcal{T})$ by*

$$\|(c_\alpha)\|_\mathcal{T} = \max_{\alpha\in\mathcal{T}} \left| \sum_{\gamma\le\alpha} c_\gamma \right| . \tag{4.10}$$

Let $X_\mathcal{T}$ denote the completion of $c_{00}(\mathcal{T})$ under $\|\cdot\|_\mathcal{T}$.

Proposition 4.15 *Let $\mathcal{T}$ be an infinite well founded tree. Then there exists a countable limit cardinal α so that $X_\mathcal{T}$ is either isometric to $C_0(\alpha)$ (if $\mathcal{T}$ has infinitely many elements of length 1 and $\phi \notin \mathcal{T}$) or to $C(\alpha+)$. Conversely, given any such ordinal α, there exists a tree $\mathcal{T}$ with $X_\mathcal{T}$ isometric to $C_0(\alpha)$ or to $C(\alpha+)$. Moreover, let $\mathcal{T}$ be a given infinite tree, and for $\alpha \in \mathcal{T}$ let b_α be the natural element of $c_{00}(\mathcal{T})$: $(b_\alpha)(\beta) = \delta_{\alpha\beta}$. Let $\tau : \mathbf{N} \to \mathcal{T}$ be a bijection (i.e., an enumeration) so that if $\tau(i) < \tau(j)$, then $i < j$. Then $(b_{\tau(j)})_{j=1}^\infty$ is a monotone basis for $X_\mathcal{T}$.*

All of the above assertions and developments are due to Bourgain [Bo1], except for the basis assertion, which is due to E. Odell. The author is most grateful to Professor Odell for his personal explanations of these results.

We next indicate the trees $\mathcal{T}_n$ corresponding to the spaces $C(\omega^n+)$ for $0 \le n \le \omega$. For n finite, simply let $\mathcal{T}_n$ be all finite sequences of positive integers of length at most n; also let $\mathcal{T}_n^0 = \mathcal{T}_n \sim \{\emptyset\}$. Finally, let $\mathcal{T}_\omega^0 = \cup_{n=1}^\infty \{(n,\alpha) : \alpha \in \mathcal{T}_{n-1}\}$, and let $\mathcal{T}_\omega = \mathcal{T}_\omega^0 \cup \{\emptyset\}$. The reader should have no difficulty in establishing the assertions of 4.15 in this special case. In particular, *for all $1 \le n \le \omega$, $X_{\mathcal{T}_n^0}$ is isometric to $C_0(\omega^n)$ and $X_{\mathcal{T}_n}$ is isometric to $C(\omega^n+)$.* There still remains the intuitive issue: what is the picture for a subspace of $C(K)$ which is isomorphic to $C(\omega^\omega+)$ and fixed by an operator T satisfying the hypotheses of 4.7? The following elegant description gives Bourgain's answer.

Definition 4.16 *Let $\mathcal{F}$ be a family of non-empty clopen subsets of a totally disconnected infinite compact metric space K. $\mathcal{F}$ is called* regular *if*

(a) any two elements of $\mathcal{F}$ are either disjoint or one is contained in the other.
(b) $\mathcal{F}$ has no infinite totally ordered subsets, under the order $A \le B$ if $A \supset B$.

We again leave the proof of the following motivating result to the reader.

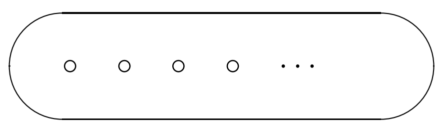
Proposition 4.17 *Let $\mathcal{F}$ be an infinite regular family of clopen subsets of K. There is a well founded tree $\mathcal{T}$ and an order preserving bijection $\tau : \mathcal{F} \to \mathcal{T}$. $[\mathcal{F}]$ is a subalgebra of $C(K)$ which is algebraically isometric to $C(\alpha+)$ or $C_0(\alpha)$ for some countable ordinal α. Moreover, identifying $\tau(A)$ with the basis elements $b_{\tau(A)}$ of Proposition 4.15, then τ extends to a linear isometry of $[\mathcal{F}]$ with $X_{\mathcal{T}}$. Conversely, given any well-founded tree $\mathcal{T}$, then there exists a regular family $\mathcal{F}$ (for a suitable K) with $\mathcal{F}$ order isomorphic to $\mathcal{T}$.*

Let us just indicate pictures for the regular families corresponding to $\mathcal{T}_n^0$ and $\mathcal{T}_n$, and thus to $C_0(\omega^n)$ and $C(\omega^n+)$. Of course, a sequence of disjoint clopen sets spans c_0 isometrically.

$\mathcal{T}_1^0$ $\qquad \circ \quad \circ \quad \circ \quad \circ \quad \cdots$

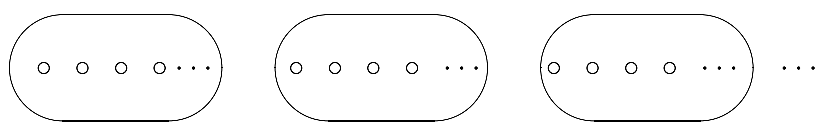
We get $\mathcal{T}_1$ by putting all these inside one clopen set, which actually corresponds then to the function $\mathbf{1}$ in $C(\omega+) = c$.

$\mathcal{T}_1$



Now we can get $\mathcal{T}_2^0$ by repeating $\mathcal{T}_1$ infinitely many times.

$\mathcal{T}_1$



Of course, we then put all these inside one clopen set, to obtain $\mathcal{T}_2$.

Finally, we obtain $\mathcal{T}_\omega^0$ by choosing a sequence of disjoint clopen sets $O_1, O_2, \ldots$ and *inside* O_n, we put the regular system corresponding to $\mathcal{T}_n^0$.

Bourgain proves Theorem 4.7 by establishing the following general result.

Theorem 4.18 *Let K a totally disconnected compact metric space, X a Banach space, and $T : C(K) \to X$ a bounded linear operator be given such that for some $\varepsilon > 0$ and countable ordinal α*

$$\beta_\varepsilon(T) \ge \omega^\alpha \ .$$

Then there is a regular system $\mathcal{F}$ of clopen subsets of K with $Y \overset{\text{def}}{=} [\mathcal{F}]$ isometric to $C_0(\omega^{\omega \cdot \alpha})$

such that $T|Y$ is an isomorphism.

The whole point of our exposition: one must *choose* a regular family of clopen sets to achieve the desired copy of $C(\omega^\omega+)$; this *requires* the above concepts.

Remark 4.19 In view of Milutin's Theorem, it follows that for any compact metric space K and operator $T : C(K) \to X$, T fixes $C(\omega^{\omega\cdot\alpha}+)$ provided $\beta_\varepsilon(T) = \omega^\alpha$. Thus Theorem4.7 follows, letting $\alpha = 1$. Actually, Alspach obtains in [A1] that if $\beta_\varepsilon(T) \geq \omega$ for some $\varepsilon > 0$, K arbitrary, then still T fixes some subspace of $C(K)$ isometric to $C_0(\omega^\omega)$.

We turn next to the basic connection between the Banach-Saks property and $C(\omega^\omega+)$. We first give Schreier's fundamental example showing that $C(\omega^\omega+)$ *fails* the Banach-Saks property; i.e., there exists a weakly null sequence in $C(\omega^\omega+)$, such that no subsequence has its arithmetic averages tending to zero in norm.

Proposition 4.20 *There exists a sequence $U_1, U_2, \ldots$ of compact open subsets of ω^ω such that setting $b_j = X_{U_j}$ for all j, then*

(a) no point of ω^ω belongs to infinitely many of the U_j's

and

(b) For all scalars (c_j) with only finitely many c_j's non-zero

$$\left\|\sum c_j b_j\right\| = \max\left\{\left|\sum_{i=1}^{r} c_{j_i}\right| : j_1 < \cdots < j_r \text{ and } r \leq j_1\right\} . \tag{4.11}$$

Before proving this, we first show that the sequence (b_j) in 4.20 is an "anti-Banach-Saks" sequence. For convenience, we restrict to real scalars.

Proposition 4.21 *Let (b_j) be as in 4.20. Then $b_j \to 0$ weakly. Define a new norm on the span of the b_j's by*

$$|\!|\!|\sum c_j b_j|\!|\!| = \max\left\{\sum_{i=1}^{r} |c_j| : r = j_1 \text{ and } j_1 < \cdots < j_r\right\} . \tag{4.12}$$

Then $\|x\| \leq |\!|\!|x|\!|\!| \leq 2\|x\|$ for all $x \in [b_j]$.

It follows immediately that given $j_1 < j_2 < \cdots$, then $\frac{1}{r}\sum_{i=1}^{r} b_{j_i}$ does not tend to zero in norm as $r \to \infty$. In fact, we have for all k that

$$\left\|\sum_{i=1}^{2k} b_{j_i}\right\| \geq \frac{1}{2}|\!|\!|\sum_{i=1}^{2k} b_{j_i}|\!|\!| \geq \frac{k}{2} . \tag{4.13}$$

We also see the fundamental phenomenon: *any k-terms of the b_j's past the kth are 2-equivalent to the ℓ^1_k-basis.* It also follows, incidentally, that the norm-condition (4.13) alone, insures that $b_j \to 0$ weakly. We prefer however, to give the simpler argument which follows from 4.20. Finally, it follows that the sequence (b_j) in 4.20 is an *unconditional* basic sequence.

Proof of 4.21. Since no k belongs to infintely many of the K_j's, it follows that $X_{K_j} \to 0$ pointwise on $\omega^\omega+$, which immediately yields that $X_{K_j} \to 0$ weakly, by the "baby" version of the Riesz representation theorem. The lower estimate in (4.12) is trivial. But if we fix c_j's, $r \geq 1$ and $r = j_1 < j_2 < \cdots < j_r$, then there is s subset F of $j_1, \ldots, j_r$, with

$$\Big|\sum_{i\in F} c_i\Big| \geq \frac{1}{2}\sum_{i=1}^{r} |c_{j_i}| \ . \tag{4.14}$$

But if we enumerate F as $i_1 < \cdots < i_k$, then trivially $k \leq r \leq j_1 \leq i_1$, hence

$$\Big\|\sum c_j \cdot b_j\Big\| \geq \Big|\sum_{i\in F} c_i\Big| \geq \frac{1}{2}\sum_{i=1}^{r} |c_{j_i}| \geq \frac{1}{2}|\!|\!| c_j b_j |\!|\!| \ . \qquad \square \tag{4.15}$$

Proof of 4.20. We give yet one more (and last!) conceptualization of the compact countable spaces ω^n+ and $\omega^\omega+$. We identify their elements with certain *finite* subsets of $\mathbf{N}$. Let $\mathcal{F}$ be a family of finite subsets of $\mathbf{N}$, so that $\mathcal{F}$ contains no infinite sequences $F_1, F_2, \ldots$ with $F_n \overset{\subset}{\neq} F_{n+1}$ for all n, and such that $\mathcal{F}$ is *closed* under pointwise convergence (where $F_j \to F$ means $X_{F_j} \to X_F$ pointwise on $\mathbf{N}$). It follows easily that $\mathcal{F}$ is then a compact metric space.

Now first let $\mathcal{F}_n$ be the family of all subsets of $\mathbf{N}$ of cardinality as most n. It follows that $\mathcal{F}_n$ is homeomorphic to ω^n+. In fact, we obtain by induction that $\mathcal{F}_n^{(j)} = \mathcal{F}_{n-j}$, so that finally $\mathcal{F}_n^{(0)} = \{\emptyset\}$. Now we "naturally" obtain $\mathcal{F}_\omega$ homeomorphic to $\omega^\omega+$ as follows.

$$\mathcal{F}_\omega \overset{\text{def}}{=} \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \{\alpha \cup n : \ell(\alpha) \leq n \text{ and the least element of } \alpha \geq n\} \ . \tag{4.16}$$

In other words, F_ω consists of all finite sets whose cardinality is at most its least element. It is clear that $\mathcal{F}_\omega$ is indeed compact in the pointwise topology, and moreover it is also clear that $\mathcal{F}_\omega^{(n)} \neq \emptyset$ for all $n = 1, 2, \ldots$ Finally, it is also clear that for each n, the nth term in the above union is homeomorphic to ω^n+, and so we have that $\mathcal{F}_\omega^{(\omega)} = \{\emptyset\}$, whence $\mathcal{F}_\omega$ is homeomorphic to $\omega^\omega+$. Now for each j, define

$$U_j = \{\alpha \in \mathcal{F}_\omega : j \in \alpha\} \ . \tag{4.17}$$

It then easily follows that U_j is a clopen subset of $\mathcal{F}_\omega$, and of course $\phi \notin U_j$ for any j.

It is trivial that no $\alpha \in \mathcal{F}_\omega$ belongs to infinitely many U_j's since α is a finite set. Finally, let (c_j) be a sequence of scalars with only finitely many non-zero elements; then for any $\alpha \in \mathcal{F}_\omega$

$$\Big|\sum c_j b_j(\alpha)\Big| = \Big\{\Big|\sum c_j\Big| : j \in \alpha\Big\} \ . \tag{4.18}$$

But if $\alpha = \{j_1, \ldots, j_r\}$ with $j_1 < j_2 < \cdots < j_r$, then by definition of $\mathcal{F}_\omega$, $r \leq j_1$, and conversely given $j_1 < \cdots j_r$ with $r \leq j_1$, $\{j_1, \ldots, j_r\} \in \mathcal{F}_\omega$. Thus (4.18) yields (4.11), completing the proof. $\square$

Next we discuss Banach-Saks operators on $C(K)$ spaces. Actually, Theorem 4.9 is an immediate consequence of Theorem 4.7 and the following remarkable result ([Bo1], Lemma 17) (see Definition 4.8 for the ε-Szlenk index of an operator)

Theorem 4.22 *Let X and Y be Banach spaces with X separable and $T : X \to Y$ be a given operator. Then if the ε-Szlenk index of T is finite for all $\varepsilon > 0$, T is a Banach-Saks operator. In particular, if the ε-Szlenk index of X is finite for all $\varepsilon > 0$, X has the Banach-Saks property.*

Just to clarify notation, we first give the

Proof of Theorem 4.9. Let $T : C(K) \to Y$ be a non-Banach-Saks operator. It is trivial that then, without loss of generality, we may assume that $C(K)$ is separable, i.e., that K is compact metric. Then by Theorem 4.22, there exists an $\varepsilon > 0$ such that $\beta_\varepsilon(T^*(\mathrm{Ba}(Y^*))) \geq \omega$. But then T fixes $C(\omega^\omega +)$ by Theorem 4.7. $\square$

The initial steps in the proof of Lemma 17 of [Bo1] (given as a lemma there) can be eliminated, using a fundamental dichotomy discovered a few years earlier. Moreover, the details of the proof of Lemma 17 itself do not seem correct. Because of the significance of this result, we give a detailed proof here.

The following is the basic dichotomy discovered By the author in [Ro5]; several proofs have been given since, see e.g. [Mer].

Theorem 4.23 [Ro5] *Let (b_n) be a weakly null sequence in a Banach space. Then (b_n) has a subsequence (y_n) satisfying one of the following mutually exclusive alternatives:*

(a) $\frac{1}{n}\sum_{j=1}^n y'_j$ *tends to zero in norm, for all subsequences (y'_j) of (y_j).*
(b) (y_j) *is a basic sequence so that any k terms past the kth are uniformly equivalent to the ℓ^1_k basis. Precisely, there is a $\delta > 0$ so that for all $k < j_1 < \cdots < j_k$ and scalars $c_1, \ldots, c_k$,*

$$\left\| \sum_{i=1}^k c_i y_{j_i} \right\| \geq \delta \sum_{i=1}^k |c_i| \ . \tag{4.19}$$

Note that it follows immediately that if (y_j) satisfies (b), there is a constant $\eta > 0$ so that for any subsequence (y'_j) of (y_j), $\| \sum_{j=1}^{2k} y'_j \| \geq \eta k$ for all k, hence no subsequence of (y_j) has averages converging to zero in norm. In modern terminology, (y_j) *has a spreading model isomorphic to ℓ^1*. Notice that Schreier's sequence given in Proposition 4.20 is a witness to this general phenomenon.

Now we give the

Proof of Theorem 4.22. Let $T : X \to Y$ be a given operator, and suppose T is non-Banach-Saks. We may assume without loss of generality that $\|T\| = 1$. Choose (x_n) a weakly null sequence in X so that the arithmetic averages of Tx_n do not tend to zero in norm. Assume that $\|x_n\| \leq 1$ for all n. Now choose (b_n) a subsequence of (x_n) so that setting

$y_n = Tb_n$ for all n, then for some $\delta > 0$

$$(y_n) \text{ satisfies } (4.19)\ . \tag{4.20}$$

Let $K = T^*(\mathrm{Ba}(Y^*))$. We shall prove that

$$P_m(\delta, K) \neq \emptyset \quad \text{for all } m = 1, 2, \ldots \tag{4.21}$$

where the sets $P_m(\delta, K)$ are those originally defined by Szlenk in his derivation (as defined above, preceding Proposition 2.17).

We need the following fundamental consequence of (4.19).

For all m and $\alpha = (j_1, \ldots, j_m)$ with $m < j_1 < j_2 < \cdots < j_m$, there exists a $y_\alpha^* \in \mathrm{Ba}(Y^*)$ with

$$|y_\alpha^*(y_j)| \geq \delta \quad \text{for all } j \in \alpha. \tag{4.22}$$

Given $m, \ell \geq 1$, we set

$$\Gamma_{m,\ell} = \{\alpha \subset \{\ell + 1, \ell + 2, \ldots\} : \#\alpha = m\} \tag{4.23}$$

(i.e., $\Gamma_{m,\ell}$ is just all m element subsets of $\mathbf{N}$ past the ℓth term).

We now prove the following claim by induction on m.

Claim: For all m, $\ell \geq m$ and families $\{y_\alpha^* : \alpha \in \Gamma_{m,\ell}\}$ with y_α^* satisfying (4.22) for all $\alpha \in \Gamma_{m,\ell}$, there is a weak*-cluster point of $\{T^*(y_\alpha^*) : \alpha \in \Gamma_{m,\ell}\}$ belonging to $P_m(\delta, K)$. (This is a more delicate version of the apparently incorrect argument in [Bo1]; the author nevertheless greatly admires the ingenuity of Bourgain's discussion there).

The case $m = 1$ is really immediate, just using the ω^*-compactness of K. After all, given any $n > \ell$, then by definition,

$$|(T^* y_{(n)}^*)(b_n)| = |y_{(n)}^*(y_n)| \geq \delta \tag{4.24}$$

hence *any* ω^*-cluster point of $(T^* y_{(n)}^*)$ lies in $P_1(\delta, K)$ since (b_n) is weakly null in $\mathrm{Ba}(X)$.

Now suppose the claim is proved for m, let $\ell \geq m + 1$, and let $\{y_\alpha^* : \alpha \in \Gamma_{m+1,\ell}\}$ be given with y_α^* satisfying (4.22) for all $\alpha \in \Gamma_{m+1,\ell}$. Fix $n > \ell$, and *define* $\tilde{y}_\alpha^*$ for each $\alpha \in \Gamma_{m,n}$ by

$$\tilde{y}_\alpha^* = y_{\alpha \cup \{n\}}^*\ .$$

But then for all $\alpha \in \Gamma_{m,n}$, $\tilde{y}_\alpha^*$ satisfies (4.22), hence by our induction hypothesis, there exists a ω^*-cluster point x_n^* of $\{T^*(\tilde{y}_\alpha^*) : \alpha \in \Gamma_{m,n}\}$ which belongs to $P_m(\delta, K)$. But since $\alpha \cup \{n\} \in \Gamma_{m+1,\ell}$, we have that

$$|\langle T^*(\tilde{y}_\alpha^*), b_n\rangle| = |\langle y_{\alpha \cup \{n\}}^*, y_n\rangle| \geq \delta \tag{4.25}$$

for all $\alpha \in \Gamma_{m,n}$; then also

$$|x_n^*(b_n)| \geq \delta \ . \tag{4.26}$$

But *then* if x^* is a weak*-cluster point of $(x_n^*)_{n=\ell+1}^\infty$, $x^* \in P_{m+1}(\delta, K)$, and of course x^* is indeed a weak*-cluster point of $\{T^*(T^*(y_\alpha^*) : \alpha \in \Gamma_{m+1,\ell}\}$. This completes the induction step of the claim. which then shows (4.21) so the proof of Theorem 4.22 is complete. $\square$

C. Operators fixing $C([0,1])$.

We finally treat Theorem 4.10. We shall sketch the main ideas in the proof given in [Ro2]. We first note, however, that there are two other proofs known, both conceptually different from each other and from that in [Ro2]. L. Weis obtains this result via an integral representation theorem for operators on $C(K)$ spaces [We]. For extensions of this and further complements in the context of Banach lattices see [GJ] and [FGJ]. Finally, the general result 4.18 also yields 4.10, as noted by Bourgain in [Bo1]. Indeed, suppose $T : C(K) \to X$ are given as in the statement of 4.10, where K is totally disconnected. Then 4.18 yields that T fixes $C(L)$ for *every* countable subset L of $[0,1]$. But the family of *all* compact subsets L of $[0,1]$ such that T fixes $C(L)$, forms a Borel subset of the family of all compact subsets L of $[0,1]$ in the Hausdorff metric; the countable ones, however are *not* a Borel set by classical descriptive set theory. Hence there is an uncountable compact $L \subset [0,1]$ so that F fixes L, and then $C(L)$ is isomorphic to $C([0,1])$ by Milutin's theorem. We also note that a refinement of the arguments in [Ro2] yields the following generalization of 4.10, due jointly to H. P. Lotz and H.P. Rosenthal [LR]: *Let E be a separable Banach lattice with E^* weakly sequentially complete, X be a Banach space, and $T : E \to X$ be an operator with $T^*(X^*)$ non-separable. Then T fixes $C([0,1])$.* For extensions of this and further complements in the context of Banach lattices, see [GJ] and [FGJ].

We first introduce some (standard) terminology. Let X be a Banach space, Y be a subspace of X, and W be a subset of $\mathrm{Ba}(X^*)$. We say that W *norms* Y if there exists a constant $\lambda \geq 1$ so that

$$\|y\| \leq \lambda \sup_{w \in W} |w(y)| \quad \text{for all } y \in Y. \tag{4.27}$$

If (4.27) holds, we say that W λ-*norms* Y. Now let K be a compact metric space. Theorem 4.10 then follows immediately from the following stronger statement:

Theorem 4.24 *A non-separable subset of* $\mathrm{Ba}(C(K)^*)$ *norms a subspace of* $C(K)$ *isometric to* $C(\mathbf{D})$.

Indeed, we may obviously assume that $T : C(K) \to X$ has norm one. Then assuming T^*X^* is non-separable, so is $W = T^*(\mathrm{Ba}(X^*))$, and thus 4.24 yields a subspace Y of $C(K)$ with Y isometric to $C(\mathbf{D})$ and $T|Y$ an isomorphism; then T fixes $C(\mathbf{D})$, and so of course $C([0,1])$, which is isometric to a subspace of $C(\mathbf{D})$.

The proof of 4.24 proceeds by reduction to the following almost isometric result.

Lemma 4.25 *Let Z be a subspace of $C(K)^*$ with Z isometric to L^1. Then for every $\varepsilon > 0$,* $\mathrm{Ba}(Z)$ $(1+\varepsilon)$*-norms a subspace of $C(K)$ which is isometric to $C(\mathbf{D})$.*

We shall sketch some of the ideas in the proof of 4.25 later on. We first note that the actual proof of Theorem 4.24 yields the following dividend.

Corollary 4.26 *Let Z be a non-separable subspace of $C(K)^*$. Then for all $\varepsilon > 0$,* $\mathrm{Ba}(Z)$ $(1+\varepsilon)$*-norms a subspace of $C(K)$ which is isometric to $C(\mathbf{D})$.*

Of course 4.26 has the following immediate consequence.

Corollary 4.27 *Let X be a quotient space of $C(K)$ with X^* non-separable. Then X contains a subspace $(1+\varepsilon)$-isomorphic to $C(\mathbf{D})$ for all $\varepsilon > 0$.*

Remark 4.28 The conclusion badly fails for subspaces X of $C(\mathbf{D})$ which are themselves isomorphic to $C(\mathbf{D})$. Indeed, it is proved in [LP] that for every $\lambda > 1$ there exists a Banach space X which is isomorphic to $C(\mathbf{D})$ but contains no subspace λ-isomorphic to $C(\mathbf{D})$; of course X is isometric to a subspace of $C(\mathbf{D})$.

We now take up the route which leads to Lemma 4.25 Say that elements μ and ν of $C(K)^*$ are *pairwise disjoint* if μ and ν are singular, regarding μ, ν as complex Borel measures on K. The next result is proved by a two step transfinite induction.

Lemma 4.29 *Let L be a convex symmetric non-separable subset of* $\mathrm{Ba}(C(K))^*$*. Then there is a $\delta > 0$ so that for all $0 < \varepsilon < \delta$, there exists an uncountable subset $\{\ell_\alpha\}_{\alpha\in\Gamma}$ of L and a family $\{\mu_\alpha\}_{\alpha\in\Gamma}$ of pairwise disjoint elements of* $\mathrm{Ba}(C(K)^*)$ *so that for all α,*

$$\|\mu_\alpha - \ell_\alpha\| \le \varepsilon \quad \text{and} \quad \|\mu_\alpha\| \ge \delta \ . \tag{4.28}$$

Moreover if L is the unit ball of a subspace of $C(K)^*$, one can take $\delta = 1$.

Now of course the family $\{\mu_\alpha/\|\mu_\alpha\| : \alpha \in \Gamma\}$ is isometrically equivalent to the usual $\ell^1(\Gamma)$ basis. But this is also an uncountable subset of a compact metrizable space, $\mathrm{Ba}(C(K)^*)$ in the ω^*-topology. So it follows that we may choose $\alpha_1, \alpha_2, \ldots$ distinct elements of Γ with $(f_n)_{n=1}^\infty$ ω^*-dense in itself where $f_n = \mu_{\alpha_n}/\|\mu_{\alpha_n}\|$ for all n; note that $(f_n)_{n=1}^\infty$ is isometrically equivalent to the usual ℓ^1 basis. A variation of an argument of C. Stegall [St] now yields

Proposition 4.30 *Suppose X is a separable Banach space and (f_n) in X^* is isometrically equivalent to the ℓ^1 basis and ω^*-dense in itself. Then there exists a subspace U of X^*, isometric and ω-isomorphic to $C(\mathbf{D})^*$, such that for all $x \in X$,*

$$\sup_{u\in \mathrm{Ba}(U)} |u(x)| \le \sup_n |f_n(x)| \ . \tag{4.29}$$

Of course U is obtained as $T^*(C(\mathbf{D}))^*$ where T is constructed to be a quotient map of X onto $C(\mathbf{D})$).

For our purposes, we only need that U contains a subspace isometric to L^1. We now complete the proof of Theorem 4.24, using 4.25, 4.29 and 4.30. Let L be as in 4.24. We may assume that L is convex and symmetric, for if $\tilde{L}$ is the closed convex hull of $L \cup -L$, then $\sup_{\ell\in L}|\ell(x)| = \sup_{\ell\in\tilde{L}}|\ell(x)|$ for all $x \in C(K)$. Let δ satisfy the conclusion of Lemma 4.29, and let $0 < \varepsilon$ so that

$$1 - \varepsilon - \frac{\varepsilon}{\delta} > 0 \ . \tag{4.30}$$

Now let $(\ell_\alpha)_{\alpha\in\Gamma}$ and $(\mu_\alpha)_{\alpha\in\Gamma}$ satisfy the conclusion of 4.29. Choose $\alpha_1, \alpha_2, \ldots$ distinct α's so that $(f_n)_{n=1}^\infty$ is ω^*-dense in itself, where $f_n = \mu_{\alpha_n}/\|\mu_{\alpha_n}\|$ for all n. Also let $y_n = \ell_{\alpha_n}$ and $\delta_n = \|\mu_{\alpha_n}\|$ for all n. Next, choose Z a subspace of $C(K)^*$ isometric to L^1 such that for all $x \in C(K)$,

$$\sup_{z\in \mathrm{Ba}\, Z} |z(x)| \le \sup_n |f_n(x)| \tag{4.31}$$

thanks to Proposition 4.30. Finally, choose X a subspace of $C(K)$ with X isometric to $C(\mathbf{D})$) so that

$$(1-\varepsilon)\|x\| \le \sup_{z\in \mathrm{Ba}\, Z} |z(x)| \quad \text{for all } x \in X \ , \tag{4.32}$$

by Lemma 4.25. Now by our definition of the y_n's and δ_n's, we have for all n that (by (4.28))

$$\|f_n - y_n/\delta_n\| \le \frac{\varepsilon}{\delta_n} \le \frac{\varepsilon}{\delta} \ . \tag{4.33}$$

Thus finally fixing $x \in X$ with $\|x\| = 1$, we have

$$1 - \varepsilon \le \sup_n |f_n(x)| \quad \text{by (4.31) and (4.32)} \tag{4.34}$$

$$\le \frac{1}{\delta}\sup_n |y_n(x)| + \frac{\varepsilon}{\delta} \quad \text{by (4.33)} \ . \tag{4.35}$$

Hence

$$(1 - \varepsilon - \frac{\varepsilon}{\delta})\delta \le \sup_n |y_n(x)| \ . \tag{4.36}$$

Thus letting $\lambda = ((1-\varepsilon-\frac{\varepsilon}{\delta})\delta)^{-1}$, we have proved that L λ-norms X. Finally, if L is the unit ball of a non-separable subspace of $C(K)^*$, we may choose $\delta = 1$ by 4.29; but then it follows that since ε may be chose arbitrarily small, λ is arbitrarily close to 1, and this yields Corollary 4.26. $\square$

We finally treat Lemma 4.25. Let then Z be a subspace of $C(K)^*$ which is isometric to L^1. Standard results yield that *there exists a Borel probability measure μ on K, a Borel measurable function θ with $|\theta| = 1$, a compact subset S of K with $\mu(S) = 1$, and a σ-subalgebra $\mathcal{S}$ of the Borel subsets of S such that $(S, \mathcal{S}, \mu|\mathcal{S})$ is a purely non-atomic measure space and $Z = \theta \cdot L^1(\mu|\mathcal{S})$.* (We adopt the notation: $\theta \cdot Y = \{\theta y : y \in Y\}$). The desired isometric copy of $C(\mathbf{D})$ which is $(1+\varepsilon)$-normed by Z is now obtained through the following construction.

Lemma 4.31 *Let μ, E, and $\mathcal{S}$ be as above, and let $\varepsilon > 0$. Then there exist sets $F_i^n \in \mathcal{S}$ and compact subsets K_i^n of S satisfying the following properties for all $1 \le i \le 2^n$ and $n = 0, 1, \ldots$.*

(i) $K_i^n \cap K_{i'}^n = F_i^n \cap F_{i'}^n = \emptyset$ *for any* $i' \neq i$.
(ii) $K_i^n = K_{2i-1}^{n+1} \cup K_{2i}^{n+1}$ *and* $F_i^n = F_{2i-1}^{n+1} \cup F_{2i}^{n+1}$.
(iii) $K_i^n \subset F_i^n$.
(iv) $\frac{(1-\varepsilon)}{2^n} \le \mu(K_i^n)$ *and* $\mu(F_i^n) \le \frac{1}{2^n}$.
(v) $\theta | K_1^0$ *is continuous relative to* K_1^0.

(This is Lemma 1 of [Ro2], with condition (v) added as in the correction to [Ro2].) We conclude our discussion with the

Proof of Lemma 4.25. Let $F = K_1^0$ and let A denote the closure of the linear span of $\{\chi_{K_i^n} : 1 \le i \le 2^n,\ n = 0, 1, 2, \ldots\}$ in $C(F)$. Then A is a subalgebra of $C(F)$ algebraically isometric to $C(\mathbf{D})$. Hence also,

$$Y \stackrel{\text{def}}{=} \bar{\theta} \cdot A \quad \text{is a subspace of } C(F) \text{ isometric to } C(\mathbf{D}). \tag{4.37}$$

($\bar{\theta}$ denotes the complex conjugate of θ, in the case of complex scalars.) Now let $E : C(F) \to C(K)$ be an isometric extension operator, as insured by the Basuk Theorem (Lemma 1.4(b) above). Finally, set $X = E(Y)$. So, evidently X is a subspace of $C(K)$, isometric to $C(\mathbf{D})$.

We claim that

$$\mathrm{Ba}(Z) \quad \frac{1}{1-2\varepsilon} - \text{norms } X \tag{4.38}$$

which yields 4.25. Of course, it suffices to show that for a dense linear subspace X_0 of X

$$\sup_{z \in \mathrm{Ba}\, Z} |z(x)| \ge (1-2\varepsilon)\|x\| \qquad \text{for all } x \in X_0 \ . \tag{4.39}$$

We take X_0 to be the linear span of the functions $E(\bar{\theta} \cdot \chi_{K_i^n})$. So, fix n and let

$$\phi = \bar{\theta} \sum_{i=1}^{2^n} c_i \chi_{K_i^n} \text{ for scalars } c_1, \ldots, c_{2^n} \text{ with } \max_i |c_i| = 1. \tag{4.40}$$

Finally, let $\tilde{\phi} = E(\phi)$. So $\|\tilde{\phi}\| = 1$. Of course we identify the elements of Z with the complex Borel measures in $\theta \cdot L^1(\mu|\mathcal{S})$. So choose i with $|c_i| = 1$ and let $f = \theta \chi_{F_i^n} / \mu(F_i^n)$.

Then $\|f\|_{L^1(\mu)} = 1$, so $f \cdot \mu$ as an element of Z, also has norm 1. We have that

$$
\begin{aligned}
\Big|\int f\tilde{\phi}d\mu\Big| &\geq \Big|\int_{K_i^n} f\tilde{\phi}d\mu\Big| - \int_{F_i^n \sim K_i^n} |f\tilde{\phi}|d\mu \\
&= \frac{\mu(K_i^n)}{\mu(F_i^n)} - \int_{F_i^n \sim K_i^n} |f\tilde{\phi}|d\mu \\
&\geq \frac{\mu(K_i^n)}{\mu(F_i^n)} - \frac{\mu(F_i^n - \mu(K_i^n)}{\mu(F_i^n)} \qquad \text{since } |\tilde{\phi}| \leq 1 \\
&\geq 1 - 2\varepsilon \qquad \text{by Lemma 4.31.}
\end{aligned}
\tag{4.41}
$$

This concludes the proof, and our discussion of Theorem 4.10. $\square$

5 The Complemented Subspace Problem

In its full generality, this problem, (denoted the CSP), is as follows: *Let K be a compact Hausdorff space and X be a complemented subspace of $C(K)$. Is X isomorphic to $C(L)$ for some compact Hausdorff space L?* We first state a few results which hold in general, although most of them are easily reduced to the separable case anyway. (All Banach spaces, subspaces etc., are taken as infinite-dimensional.

Theorem 5.1 [Pe2] *Every complemented subspace of a $C(K)$-space contains a subspace isomorphic to c_0.*

This is an immediate consequence of Corollary 4.2 and Theorem 4.5. The next result refines Milutin's theorem to the non-separable setting.

Proposition 5.2 [D1] *A complemented subspace of a $C(K)$ space is isomorphic to a complemented subspace of $C(L)$ for some totally disconnected L.*

Let us point out, however: it is *unknown* if a (non-separable) $C(K)$ space itself is isomorphic to $C(L)$ for some totally disconnected L.

Later on, we shall give results characterizing c_0 (or rather c) as the smallest of the complemented subspaces of $C(K)$-spaces. Of course Theorem 4.10 characterizes $C([0,1])$ as the largest separable case.

Theorem 5.3 [Ro2] *Let X be a complemented subspace of a separable $C(K)$ space with X^* non-separable. Then X is isomorphic to $C([0,1])$.*

PROOF. Assume then X is complemented in $C(K)$ with $C(K)$ separable, i.e., K is metrizable. Then K is uncountable, since $C(K)^*$ itself must be non-separable. Thus by Milutin's

Theorem, $C(K)$ is isomorphic to $C([0,1])$.

By Theorem 4.10, X contains a subspace isomorphic to $C([0,1])$. By Pełczyński's weak injectivity result (Theorem 3.1), X contains a subspace Y isomorphic to $C([0,1])$ with Y complemented in $C(K)$. Thus by the decomposition method (applying Proposition 1.2 to $C(\mathbf{D})$ instead), X is isomorphic to $C([0,1])$. □

For the next result, recall that a Banach space X is called *primary* if whenever X is isomorphic to $Y \oplus Z$ (for some Banach spaces Y and Z), then X is isomorphic to Y or to Z.

The following result is due to Lindenstrauss and Pełczyński.

Corollary 5.4 [LP] *$C([0,1])$ is primary.*

PROOF. Suppose $C([0,1])$ is isomorphic to $X \oplus Y$. Then X^* or Y^* is non-separable, and hence either X or Y is isomorphic to $C([0,1])$ by the preceding result. □

Remark 5.5 Actually, the stronger result is obtained in [LP]: *Let X be a subspace of $C([0,1])$. Then $C([0,1])$ embeds in either X or $C([0,1])/X$.* Also, it is established in [AB1] and independently, in [Bi], that $C(K)$ is primary for all countable compact K. Thus, all separable $C(K)$-spaces are primary.

We next give characterizations of $C([0,1])$ which follow from Theorem 5.3 and some rather deep general Banach space principles. We assume K is general, although the result easily reduces to the metrizable case.

Theorem 5.6 *Let X be a complemented subspace of $C(K)$. The following are equivalent.*

(1) $C([0,1])$ embeds in X.
(2) ℓ^1 embeds in X.
(3) L^1 embeds in X^*.
(4) X^* has a sequence which converges weakly but not in norm.

PROOF. The implications (2) ⇒ (3) and (1) ⇒ (3) are due to Pełczyński (for general Banach spaces X) [Pe3]. (Actually, (3) ⇒ (2) is also true for general X, by [Pe3] and a refinement due to Hagler [H2].) Of course (1) ⇒ (2) is obvious, and so is (3) ⇒ (4), since ℓ^2 is isometric to a subspace of L^1. We show (4) ⇒ (2) ⇒ (1) to complete the proof. Let then (x_n^*) in X^* tend weakly to zero, yet for some $\delta > 0$,

$$\|x_n^*\| > \delta \text{ for all } n . \tag{5.1}$$

Then for each n, choose x_n in $Ba(X)$ with

$$|x_n^*(x_n)| > \delta \ . \tag{5.2}$$

Now since X is complemented in $C(K)$, X has the Dunford-Pettis property (i.e., X satisfies the conclusion of Theorem 4.1). But then

$$(x_n) \text{ has no weak-Cauchy sequence.} \tag{5.3}$$

Indeed, if a Banach space Y has the Dunford Pettis property, then $y_n^*(y_n) \to 0$ as $n \to \infty$ whenever (y_n^*) is *weakly* null in Y and (y_n) is weak-Cauchy in Y; so (5.3) follows in virtue of (5.2). But then by the ℓ^1-Theorem [Ro4], (x_n) has a subsequence equivalent to the ℓ^1-basis, hence (2) holds.

(2) $\Rightarrow$ (1). Let $P : C(K) \to X$ be a projection and let Y be a subspace of X isomorphic to ℓ^1. Let Z be the conjugation-closed norm-closed unital subalgebra of $C(K)$ generated by Y. Then by the Gelfand-Naimark theorem (which holds in this situation for real scalars also), Z is isometric to $C(L)$ for some compact metric space L. Let $T = P|Z$. Since $T|Y = I|Y$, $T^*(Z^*)$ is non-norm-separable. Hence (1) holds by Theorem 4.10. $\square$

For the remainder of our discussion, we assume the separable situation. Thus, K denotes a compact metric space; a "$C(K)$-space" refers to $C(K)$ for some K, so it is separable.

Now of course Theorem 5.3 reduces the CSP to the case of spaces X complemented in $C(K)$ with X^* separable. If the CSP has an affirmative answer, such an X must be c_0-saturated (see Proposition 3.5). This motivates the following special case of the CSP, raised in the 70's by the author.

Problem 1. Let X be a complemented subspace of $C(K)$ so that X contains a reflexive subspace. Is X isomorphic to $C([0,1])$?

Although this remains open, it was solved in such special cases as: ℓ^2 embeds in X, by J. Bourgain, in a remarkable tour-de-force.

Theorem 5.7 [Bo2] *Let X be a Banach space and let $T : C(K) \to X$ fix a subspace Y of $C(K)$ so that Y does not contain ℓ_n^∞'s uniformly. Then $T^*(X^*)$ is non-separable.*

Of course then T fixes $C([0,1])$ by Theorem 4.10, and so we have the

Corollary 5.8 *Let X be complemented in $C(K)$ and assume X contains a subspace Y which does not contain ℓ_n^∞'s uniformly. Then X is isomorphic to $C([0,1])$.*

For the remainder of our discussion, we focus on spaces X with separable dual. The next result is due to Y. Benyamini, and rests in part on a deep lemma due to M. Zippin ([Z1], [Z2]) which we will also discuss.

Theorem 5.9 [Be3] *Let X be a complemented subspace of $C(K)$. Then either X is isomorphic to c_0 or $C(\omega^\omega+)$ embeds in X.*

The following result is an immediate consequence, in virtue of the decomposition method and weak injectivity of $C(\omega^\omega+)$, i.e., Theorem 3.1.

Corollary 5.10 *A complemented subspace of $C(\omega^\omega+)$ is isomorphic to c_0 or to $C(\omega^\omega+)$.*

Now of course, Theorem 5.9 implies Zippin's remarkable characterization of separably injective spaces, since if $C(\omega^\omega+)$ embeds in X, it also embeds complementably, and hence X cannot be separably injective by Amir's Theorem [A], obtained via Theorem 3.14 above. In reality, Theorem 5.9 rests fundamentally on the main step in [Z1],which may be formulated as follows [Be3]. (Let us call $\beta_\varepsilon(\mathrm{Ba}\, X^*)$ the ε-Szlenk index of X, where β_ε is given in Definition‘1.11.)

Lemma 5.11 [Z1] *Let X be a Banach space with X^* separable, and let $0 < \varepsilon < \frac{1}{2}$. There is a $\delta > 0$ so that if W is a ω^*-compact totally disconnected $(1+\delta)$-norming subspace of $\mathrm{Ba}(X^*)$ and if $\gamma < \omega^{\alpha+1}$ with α the δ-Szlenk index of X, then there exists a subspace Y of $C(W)$ with Y isometric to $C(\gamma+)$ so that for all $x \in X$, there exists a $y \in Y$ with*

$$\|i_W x - y\| \leq (1+\varepsilon)\|i_W x\| \ . \tag{5.4}$$

(Here, $(i_W x)(w) = w(x)$ for all $w \in W$. Also, $i_W = i$ if $W = \mathrm{Ba}\, X^*$.) Zippin also proved in [Z1] the interesting result that for any separable Banach space X and $\delta > 0$, there is a $(1+\delta)$-norming totally disconnected subset of $\mathrm{Ba}(X^*)$. Benyamini establishes the following remarkable extension of this in the main new discovery in [Be3].

Theorem 5.12 *Let X be a separable Banach space and $\varepsilon > 0$. There exists a ω^*-compact $(1+\varepsilon)$-norming subset W of $\mathrm{Ba}(X^*)$ and a norm one operator $E : C(W) \to C(\mathrm{Ba}(X^*))$ so that*

$$\|E i_W x - ix\| \leq \varepsilon\|x\| \quad \text{for all} \quad x \in X \ . \tag{5.5}$$

The preceding two rather deep results hold for general Banach spaces X. In particular, the non-linear approximation resulting from Zippin's Lemma shows that in a sense, the $C(K)$-spaces with K countable play an unexpected role in the structure of general X. The next quite simple result, however, needed for Theorem 5.9, bears solely on the structure of complemented subspaces of $C(K)$-spaces. It yields (for possibly non-separable) X that if X is isomorphic to a complemented subspace of some $C(K)$-space, then X is already complemented in $C(\mathrm{Ba}\, X^*)$ and moreover, the best possible norm of the projection is found there.

Proposition 5.13 [BL] *Let X be given, let $L = \mathrm{Ba}(X^*)$, and suppose $\lambda \geq 1$ is such that for some compact Hausdorff space Ω, there exist operators $U : X \to C(\Omega)$ and $V : C(\Omega) \to X$ with*

$$I_X = V \circ U \quad \text{and} \quad \|U\|\,\|V\| \leq \lambda \ . \tag{5.6}$$

Then $i(X)$ is λ-complemented in $C(L)$.

PROOF. Without loss of generality, $\|U\| = 1$. Let Ω be regarded as canonically embedded in $C(\Omega)^*$. Thus letting $\varphi = U^*|\Omega$, φ maps Ω into L. So of course φ° maps $C(L)$ into $C(\Omega)$. We now simply check that

$$V\varphi^\circ i(x) = x \quad \text{for all} \quad x \in X\ . \tag{5.7}$$

Then it follows that $V\varphi^\circ$ is a projection from $C(L)$ onto iX, and of course

$$\|V\varphi^\circ\| \le \|V\|\,\|\varphi^\circ\| \le \lambda\ . \qquad \square \tag{5.8}$$

Remark 5.14 Theorem 5.12 and the preceding Proposition may be applied to $C(K)$ spaces themselves to obtain that $C(K)$ is $(1+\varepsilon)$-isomorphic to a $(1+\varepsilon)$-complemented subspace of $C(\mathbf{D})$, for all $\varepsilon > 0$. Thus the main result in [Be3] yields another proof of Milutin's Theorem. We prefer the exposition in Section 2, however, for the above result "loses" the isometric fact that $C(K) \overset{cc}{\hookrightarrow} C(\mathbf{D})$ for all K.

The next remarkable result actually yields most of the known positive results in our present context.

Theorem 5.15 [Be3] *Let X^* be separable, with X a Banach space isomorphic to a complemented subspace of some $C(K)$-space. There exists a $\delta > 0$ so that if α is the δ-Szlenk index of X and $\gamma < \omega^{\alpha+1}$, then X is isomorphic to a quotient space of $C(\gamma+)$.*

PROOF. By the preceding result, $i(X)$ is already complemented in $C(L)$ where $L = \mathrm{Ba}(X^*)$ in its ω^*-topology. Let $P : C(L) \to i(X)$ be a projection and let $\lambda = \|P\|$. Now let $0 < \varepsilon$ be such that

$$\varepsilon\lambda < \frac{1}{2}\ . \tag{5.9}$$

Choose $\varepsilon > \delta > 0$ satisfying the conclusion of Zippin's Lemma. Now choose W a $(1+\delta)$-norming totally disconnected ω^*-compact subset of L and a norm one operator E satisfying the conclusion of Theorem 5.12; in particular, (5.5) holds. Finally let $x \in X$, and choose $y \in Y$ satisfying (5.4). Then since $\|E\| = 1$,

$$\|Ey - Ei_W x\| \le \varepsilon \|i_W x\|\ . \tag{5.10}$$

Then by (5.5)

$$\|Ey - ix\| \le \varepsilon(\|i_W x\| + \|x\|) \le 2\varepsilon\|x\|\ . \tag{5.11}$$

Since $Pix = ix$, we have

$$\|PEy - ix\| \le 2\varepsilon\|P\|\,\|x\| \le 2\varepsilon\lambda\|x\|\ . \tag{5.12}$$

Since $2\varepsilon\lambda < 1$ by (5.6) and of course $\|ix\| = \|x\|$, it follows finally by (5.12) that $PE|Y$ maps Y *onto* X, completing the proof. $\square$

We now obtain the

Proof of Theorem 5.9 Suppose first that the ε-Szlenk index of X is finite for all $\varepsilon > 0$. Then by Theorem 5.15, there is a positive integer n so that X is isomorphic to a quotient space of $C(\omega^n+)$. But in turn, $C(\omega^n+)$ is isomorphic to c_0, and so X is thus isomorphic to a quotient space of c_0. Finally, every quotient space of c_0 is isomorphic to a subspace of c_0 by a result of Johnson and Zippin [JZ1]. But X is also a $\mathcal{L}_\infty$-space by a result of J. Lindenstrauss and the author [LR2], and hence also by the results in [JZ1], X is isomorphic to c_0.

If the ε-Szlenk index of X is at least ω for some $\varepsilon > 0$, then X contains a subspace isomorphic to $C(\omega^\omega+)$ by Alspach's result, Theorem 4.7. $\square$

Recall that *a Banach space X is called an $\mathcal{L}_\infty$ space if there is a $\lambda > 1$ so that for all finite dimensional $E \subset X$, there exists a finite-dimensional F with $E \subset F \subset X$ so that*

$$d(F, \ell_n^\infty) \le \lambda \text{ where } n = \dim F \ . \tag{5.13}$$

If λ works, X is called an $\mathcal{L}_{\infty,\lambda}$-space.

Using partitions of unity, it is not hard to see that a $C(K)$-space is an $\mathcal{L}_{\infty,1+}$ space, i.e., it is an $\mathcal{L}_{\infty,1+\varepsilon}$ space for all $\varepsilon > 0$. However, *a Banach space X is an $\mathcal{L}_{\infty,1+}$ space if and only if it is an $L^1(\mu)$-predual*; i.e., X^* *is isometric to $L^1(\mu)$ for some μ*. We shall discuss these briefly later on. The result of [LR2] mentioned above: *A complemented subspace of an $\mathcal{L}_\infty$ space is also an $\mathcal{L}_\infty$-space.* In general, $\mathcal{L}_\infty$-spaces are very far away from $C(K)$-spaces; however the following result due to the author [Ro6], shows that small ones are very nice. (The result extends that of [JZ1] mentioned above.)

Proposition 5.16 [Ro6] *Let X be a $\mathcal{L}_\infty$-space which is isomorphic to a subspace of a space with an unconditional basis. Then X is isomorphic to c_0.*

(This was subsequently extended in [GJ] to $\mathcal{L}_\infty$-spaces which embed in a σ-σ Banach lattice.) Theorem 5.15 actually yields that if X is as in its statement, there exists a countable compact K so that X and $C(K)$ have the same Szlenk index, with X isomorphic to a quotient space of $C(K)$. Remarkably, Alspach and Benyamini prove in [AB2] that for any $\mathcal{L}_\infty$-space X with X^* separable, one has that $C(K)$ is isomorphic to a quotient space of X, K as above. So in particular, using also a result from Section 2, we have

Theorem 5.17 [Be3], [AB2] *Let X be isomorphic to a complemented subspace of a $C(K)$ space with X^* separable. Then the Szlenk index of X is $\omega^{\alpha+1}$ for some countable ordinal α and then X and $C(\omega^{\omega^\alpha}+)$ are each isomorphic to a quotient space of the other.*

Despite the many positive results discussed so far, the eventual answer to the CSP seems far from clear. We conclude this general discussion with two more problems on special cases.

Let then X be isomorphic to a complemented subspace of a $C(K)$-space with X^* separable.

Problem 2. Does X embed in $C(\alpha+)$ for some countable ordinal α? What if $\mathrm{Sz}(X) = \omega^2$?

Finally, what is the structure of complemented subspaces of $C(\omega^{\omega^2}+)$? Specifically,

Problem 3. Let X be a complemented subspace of $C(\omega^{\omega^2}+)$ with $\mathrm{Sz}(X) = \omega^2$. Is X isomorphic to $C(\omega^\omega+)$? If $\mathrm{Sz}(X) = \omega^3$, is X isomorphic to $C(\omega^{\omega^2}+)$ itself?

We next indicate complements to our discussion. Alspach constructs in [A1] a quotient space of $C(\omega^\omega+)$ which does not embed in $C(\alpha+)$ for any ordinal α; thus Problem 2 cannot be positively solved by just going through quotient maps. The remarkable fixing results Theorems 4.7 and 4.9 cannot be extended without paying some price. Alspach proves in [A3] that there is actually a surjective operator on $C(\omega^{\omega^2}+)$ which does not fix $C(\omega^{\omega^2}+)$. This result has recently been extended by Gasparis [Ga] to the spaces $C(\omega^{\omega^{\alpha+1}}+)$ for all ordinals α and an even wider class of counterexamples is given by Alspach in [A3]. Thus an affirmative answer even to Problem 3 must eventually use the assumption that one has a *projection*, not just an operator. We note also results of J. Wolfe [W], which yield rather complicated necessary and sufficient conditions that an operator on a $C(K)$ space fixes $C(\alpha+)$.

Some of the original motivation for the concept of $\mathcal{L}_\infty$-spaces was that these might characterize $C(K)$ spaces by purely local means. However Benyamini and Lindenstrauss discovered this is not the case even for $\mathcal{L}_{\infty,1+}$ spaces. They construct in [BL] a Banach space X with X^* *isometric* to ℓ^1, such that X is *not* isomorphic to a complemented subspace of $C([0,1])$. We note in passing, however, that the CSP is open for separable spaces X (in its statement) which are themselves $L^1(\mu)$ preduals. It is proved in [JZ2] that separable $L^1(\mu)$ preduals X are actually isometric to quotient spaces of $C([0,1])$. Hence if X is such a space and X^* is non-separable, X contains for all $\varepsilon > 0$ a subspace $(1+\varepsilon)$-isomorphic to $C([0,1])$, by the results of [Ro2] discussed above. Also, it thus follows by Theorem 4.5 that separable $L^1(\mu)$ preduals contain isomorphic copies of c_0. J. Bourgain and F. Delbaen prove in [BD] that separable $\mathcal{L}_\infty$ spaces are not even isomorphic to quotients of $C([0,1])$ in general; they exhibit for example an $\mathcal{L}_\infty$ space such that every subspace contains a further reflexive subspace. For further counterexample $\mathcal{L}_\infty$ spaces of a quite general nature, see [Bo-Pe].

Here are some positive results on the structure of separable $\mathcal{L}_\infty$ spaces, which of course yield results on complemented subspaces of $C(K)$ spaces. Results of D.R. Lewis and C. Stegall [LS] and of C. Stegall [St] yield that *if X is a separable $\mathcal{L}_\infty$ space, then X^* is isomorphic to ℓ^1 or to $C([0,1])^*$*. Thus in particular, the duals of complemented subspaces of separable $C(K)$ spaces are classified. It is proved in [JRZ] that *every separable $\mathcal{L}_\infty$ space X has a basis which is moreover shrinking in case X^* is separable.* A later refinement in [NW] yields that the basis (b_α) may be chosen with $d([b_j]_{j=1}^n, \ell_n^\infty) \le \lambda$ for all n (for some λ); of course this *characteristizes* $\mathcal{L}_\infty$ spaces.

We conclude with a brief discussion of the postive solution to the CSP problem in the isometric setting. *Let L be a locally compact 2nd countable metrizable space and let X be a contractively complemented subspace of $C_0(L)$. Then X is isomorphic to a $C(K)$ space.* The reason for this: such spaces X are characterized isometrically as C_σ-spaces. This is proved for real scalars in [LW], and for complex scalars in [FR]. For real scalars, X is a C_σ subspace of $C(L)$ provided there exists an involutive homeomorphism $\sigma : L \to L$ such that $X = \{f \in C(L) : f(\sigma x) = -f(x)$ for all $x \in X\}$. See [FR] for the complex scalar case. It follows by results of Benyamini in [Be1] that such spaces are isomorphic to $C(K)$-spaces; in fact it is proved in [Be1] that separable G-spaces are isomorphic to $C(K)$-spaces. This family of spaces includes closed sublattices of $C(K)$ spaces. It then follows (using the known structure of Banach lattices) that *if a complemented subspace X of a separable $C(K)$-space is isomorphic to a Banach lattice, X is isomorphic to a $C(K)$-space.* On the other hand, it remains an open question, if complemented subspaces of Banach lattices are isomorphic to Banach lattices. We note finally that Benyamini later constructed a counter-example to his result in the non-separable setting, obtaining a non-separable sublattice of a $C(K)$ space which is not isomorphic to a complemented subspace of $C(L)$ for any compact Hausdorff space L [Be2]. For further complements on the CSP in the non-separable setting, see [Z2]; for properties of non-separable $C(K)$ spaces, see [Ne] and [Ziz].

We note finally the following complement to the isometric setting [AB3]. *If a Banach space X is $(1+\varepsilon)$-isomorphic to a $(1+\varepsilon)$-complemented subspace of a $C(K)$-space for all $\varepsilon > 0$, then X is contractively complemented in $C(L)$ where $L = (\mathrm{Ba}(X)^*, \omega^*)$, hence X is a C_σ space.*

References

[A] D. Amir, *Projections onto continuous functions spaces*, Proc. Amer. Math. Soc. **15** (1964), 396–402.

[A1] D.E. Alspach, *Quotients of $C[0,1]$ with separable dual*, Israel J. Math. **29** (1978), 361–384.

[A2] D.E. Alspach, *A quotient of $C(\omega^\omega)$ which is not isomorphic to a subspace of $C(\alpha)$, $\alpha < \omega$*, Israel J. Math. **33** (1980), 49–60.

[Al3] D.E. Alspach, *$C(K)$ norming subsets of $C[0,1]^*$*, Studia Math. **70** (1981), 27–61.

[A3] D.E. Alspach, *Operators on $C(\omega^\alpha)$ which do not preserve $C(\omega^\alpha)$*, Fund. Math. **153** (1997), 81–98.

[AB1] D.E. Alspach and Y. Benyamini, *Primariness of spaces of continuous functions on ordinals*, Israel J. Math. **27** (1977), 64–92.

[AB2] D.E. Alspach and Y. Benyamini, *$C(K)$ Quotients of separable $\mathcal{L}_\infty$ spaces*, Israel J. Math. **32** (1979), 145–160.

[AB3] D.E. Alspach and Y. Benyamini, *A geometrical property of $C(K)$ spaces*, Israel J. Math. **64** (1988), 179–194.

[AA] S.A. Argyros and A.D. Arvanitakis, *A characterization of regular averaging operators and its consequences*, Studia Math. **151** (2002), no.3, 207–226.

[AGR] S.A. Argyros, G. Godefroy, and H.P. Rosenthal, *Descriptive Set Theory and Banach Spaces*, Handbook of the Geometry of Banach spaces, vol.2, W.B. Johnson and J. Lindenstrauss, eds., Elsevier, Amsterdam (to appear).

[Be1] Y. Benyamini, *Separable G spaces are isomorphic to $C(K)$ spaces*, Israel J. Math. **14** (1973), 287–293.

[Be2] Y. Benyamini, *An M-space which is not isomorphic to a $C(K)$ space*, Israel J. Math. **28** (1977), no.1–2, 98–102.

[Be3] Y. Benyamini, *An extension theorem for separable Banach spaces*, Israel J. Math. **29** (1978), 24–30.

[BL] Y. Benyamini and J. Lindenstrauss, *A predual of ℓ_1 which is not isomorphic to a $C(K)$-space*, Israel J. Math. **13** (1972), 246–259.

[BP] C. Bessaga and A. Pełczyński, *Spaces of continuous functions IV (On isomorphic classifications of spaces $C(S)$)*, Studia Math. **19** (1960), 53–62.

[Bi] P. Billard, *Sur la Primarité des espaces $C(\alpha)$*, (French), Studia Math. **62** (1978), no.2, 143–162.

[B] K. Borsuk, *Über Isomorphie der Funktionalräume*, Bull. Int. Acad. Polon. Sci. A 1/3 (1933), 1–10.

[Bo1] J. Bourgain, *The Szlenk index and operators on $C(K)$-spaces*, Bulletin Soc. Math. de Belgique Sér.B **31** (1979), no.1, 87–117.

[Bo2] J. Bourgain, *A result on operators on $C[0,1]$*, J. Operator Theory **3** (1980), 279–289.

[BD] J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}_\infty$ spaces*, Acta Math. **145** (1981), 155–176.

[Bo-Pe] J. Bourgain and G. Pisier, *A construction of $\mathcal{L}_\infty$-spaces and related Banach spaces*, Bol. Soc. Brasil Mat. **14** (1983), no.2, 109–123.

[D1] S. Ditor, *On a lemma of Milutin concerning operators in continuous function spaces*, Trans. Amer. Math. Soc. **149** (1970), 443–452.

[D2] S. Ditor, *Averaging operators in $C(S)$ and lower semicontinuous sections of continuous maps*, Trans. Amer. Math. Soc. **175** (1973), 195–208.

[DJP] J. Diestel, H. Jarchow, and A. Pietsch, *Operator Ideals*, Handbook of the Geometry of Banach spaces, vol.1, pp.437–496, W.B. Johnson and J. Lindenstrauss, eds., Elsevier, Amsterdam, 2001.

[DP] N. Dunford and B.J. Pettis, *Linear operators on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 232–392.

[FR] Y. Friedman and B. Russo, *Contraction Projections on $C_0(K)$*, Trans. Amer. Math. Soc. **273** (1982), 57–73.

[FGJ] T. Figiel, N. Ghoussoub and W.B. Johnson, *On the structure of non-weakly compact operators on Banach lattices*, Math. Ann. **257** (1981), no.3, 317–334.

[Ga] I. Gasparis, *A class of ℓ_1-preduals which are isomorphic to quotients of $C(\omega^\omega)$*, Studia Math. **133** (1999), 131–143.

[GJ] N. Ghoussoub and W.B. Johnson, *Factoring operators through Banach lattices not containing $C(0,1)$*, Math. Z. **194** (1987), no.2, 153–171.

[Gr] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$*, Canadian J. Math. **5** (1953), 129–173.

[H1] J. Hagler, *Embedding of L^1-spaces in conjugate Banach spaces*, Thesis, University of California at Berkeley, 1972.

[H2] J. Hagler, *Some more Banach spaces which contain L^1*, Studia Math. **46** (1973), 35–42.

[JL] W.B. Johnson and J. Lindenstrass, *Basic concepts in the geometry of Banach spaces*, Handbook of the Geometry of Banach spaces, vol.1, pp.1–84, W.B. Johnson and J. Lindenstrauss, eds., Elsevier, Amsterdam, 2001.

[JRZ] W.B. Johnson, H.P. Rosenthal and M. Zippen, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9** (1971), 488–506.

[JZ1] W.B. Johnson and M. Zippin, *On subspaces and quotients of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$*, Israel J. Math. **13** (1972), 311–316.

[JZ2] W.B. Johnson and M. Zippin, *separable L_1 preduals are quotients of $C(\Delta)$*, Israel J. Math. **16** (1973), 198–202.

[Ka] R. Kaufman, *A type of extension of Banach spaces*, Acta Scientiarum Mathematicarum V**26** (1965), 163–166.

[Ki] E. Kirchberg, *On subalgebras of the CAR-algebra*, J. Funct. Anal. **129** (1995), 35–63.

[LS] D.R. Lewis and C. Stegall, *Banach spaces whose duals are isomorphic to $\ell_1(\Gamma)$*, J. Funct. Anal. **12** (1973), 177–187.

[LR1] J. Lindenstrauss and H.P. Rosenthal, *Automorphisms in c_0, ℓ_1 and m*, Israel J. Math. **7** (1969), 227–239.

[LR2] J. Lindenstrauss and H.P. Rosenthal, *the $\mathcal{L}_p$-spaces*, Israel J. Math. **7** (1969), 325–349.

[LP] J. Lindenstrauss and A. Pełczyński, *Contributions to the theory of the classical Banach spaces*, J. Funct. Anal. **8** (1971), 225–249.

[LW] J. Lindenstrauss and Daniel E. Wulbert, *On the classification of the Banach spaces whose duals are L_1 spaces*, J. Funct. Anal. **4** (1969), 332–349.

[LR] H.P. Lotz and H.P. Rosenthal, *Embeddings of $C(\Delta)$ and $L^1[0,1]$ in Banach lattices*, Israel J. Math. **31** (1978), 169–179.

[M] A.A. Milutin, *Isomoprhisms of spaces of continuous functions on compacts of power continuum*, Tieoria Func. (Kharkov), **2** (1966), 150–156 (Russian).

[MS] S. Mazurkiewicz and W. Sierpinski, *Contribution à la topologie des ensembles dénombrables*, Fund. Math. **1** (1920), 17–27.

[Mer] S. Mercourakis, *On Cesàro summable sequences of continuous functions*, Mathematicka **42** (1995), no.1, 87–104.

[NW] N.J. Nielsen and P. Wojtaszczyk, *A remark on bases in $\mathcal{L}_p$-spaces with an application to complementably universal $\mathcal{L}_\infty$-spaces*,

[Ne] S. Negrepontis, *Banach spaces and topology*, Chapter 23, pp.1045–1142, Handbook of Set-Theoretic Topology, K. Kunen and J.E. Vaughan, eds., Elsevier, 1984.

[OR] T. Oikhberg and H.P. Rosenthal, *Extension properties for the space of compact operators*, J. Funct. Anal. **179** (2) (2001), 251–308.

[Pe1] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228.

[Pe2] A. Pełczyński, *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Pol. Sci. ser. Math. Astr. Phys. **10** (1962), 265–270.

[Pe3] A. Pełczyński, *On Banach spaces containing $L_1(M)$*, Studia Math. **30** (1968), 231–246.

[Pe4] A. Pełczyński, *On $C(S)$-subspaces of separable Banach spaces*, Studia Math. **31** (1968), 231–246.

[Ro1] H.P. Rosenthal, *On relatively disjoint families of measures, with some applications to Banach space theory*, Studia Math. **37** (1970), 13–36; Correction, ibid., 31–313.

[Ro2] H.P. Rosenthal, *On factors of $C([0,1])$ with non-separable dual*, Israel J. Math. **13** (1972), 361–378.

[Ro3] H.P. Rosenthal, *On subspaces of L^p*, Ann. Math. **97** (1973), 344–373.

[Ro4] H.P. Rosenthal, *A characterization of Banach spaces containing ℓ^1*, Proc. Nat. Acad. Sci. USA **71** (1974), 2411–2413.

[Ro5] H.P. Rosenthal, *Normalized weakly null sequences with no unconditional subsequences*, in Durham Symposium on the relations between infinite-dimensional and finite-dimensional convexity, Bull. London Math. Soc. **8** (1976), 22–24.

[Ro6] H.P. Rosenthal, *A characterization of c_0 and some remarks concerning the Grothendieck property*, The Longhorn Notes, The University of Texas Functional Analysis Seminar (1982–83), 95–108.

[Ro7] H.P. Rosenthal, *The complete separable extension property*, J. Operator Theory **43** (2000), 324–374.

[Ro8] H.P. Rosenthal, *Banach and operator space structure of C^*-algebras*, to appear.

[Sa] C. Samuel, *Indice de Szlenk des $C(K)$ (K espace topologique compact dé nombrable)*, Seminar on the geometry of Banach spaces, vol.I, II (Paris 1983), 81–91, Publ. Math. Univ. Paris VII.

[S] J. Schreier, *Ein gegenbeispiel sur theorie der schwachen konvergence*, Studia Math. **2** (1930), 58–62.

[Sob] A. Sobczyk, *Projection of the space* (m) *on its subspace* (c_0), Bull. Amer. Math. Soc. **47** (1941), 938–947.

[St] C. Stegall, *Banach spaces whose duals contain* $\ell_1(7)$*, with applications to the study of dual* $L_1(M)$*-spaces*, Trans. Amer. Math. Soc. **176** (1973), 463–477.

[Sz] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable Banach spaces*, Studia Math. **30** (1968), 53–61.

[We] L.W. Weis, *The range of an operator in* $C(X)$ *and its representing stochastic kernel*, Arch. Math. **46** (1986), 171–178.

[W] J. Wolfe, $C(\alpha)$ *preserving operators on* $C(K)$ *spaces*, Trans. Amer. Math. Soc. **273** (1982), 705–719. Elsevier, Amsterdam (to appear).

[Z1] M. Zippin, *The separable extension problem*, Israel J. Math. **26** (1977), no.3-4, 372–387.

[Z2] M. Zippin, *Extensions of bounded linear operators*, Handbook of Geometry of Banach spaces, vol.2, W.B. Johnson and J. Lindenstrauss, eds. Elsevier, Amsterdam (to appear).

[Ziz] V. Zizler, *Nonseparable Banach spaces*, Handbook of Geometry of Banach spaces, vol.2, W.B. Johnson and J. Lindenstrauss, eds.