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A METRIC CHARACTERIZATION OF NORMED LINEAR SPACES

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ABSTRACT. Let X be a linear space over a field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, equipped with a metric ρ. It is proved that ρ is induced by a norm provided it is translation invariant, real scalar "separately" continuous, such that every 1-dimensional subspace of X is isometric to $\mathbf{K}$ in its natural metric, and (in the complex case) $\rho(x, y) = \rho(ix, iy)$ for any $x, y \in X$.

1. Introduction and main results. Recall that a *linear space*, also called a *vector space*, X over a field of scalars $\mathbf{K}$ (either $\mathbf{R}$ or $\mathbf{C}$) is a set, endowed with compatible operations of addition and multiplication by a scalar. A *topological linear space* is, in addition, equipped with a topology compatible with these operations. The reader is referred to [**3**, Chapter 1] for a lucid introduction into this topic. We use the term *real*, respectively *complex*, *linear space* to indicate whether we are working with real or complex scalars.

A function $\rho : X \times X \to [0, \infty)$ is called a *metric* if

1. $\rho(x, y) = \rho(y, x)$ for any $x, y \in X$.

2. $\rho(x, y) = 0$ if and only if $x = y$.

3. $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ for any $x, y, z \in X$ (the triangle inequality).

A metric ρ is called *translation invariant* if, in addition, $\rho(x, y) = \rho(x + z, y + z)$ for any $x, y, z \in X$.

A *norm* is a function $\| \cdot \| : X \to [0, \infty)$ with the following properties:

1. Positive homogeneity: $\|\lambda x\| = |\lambda| \, \|x\|$ for any $x \in X$ and $\lambda \in \mathbf{K}$.

2. Positive definiteness: $\|x\| = 0$ if and only if $x = 0$.

3. Triangle inequality: $\|x + y\| \le \|x\| + \|y\|$ for any $x, y \in X$.

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A linear space equipped with a norm (or a metric) is called a *normed*, respectively *metric*, *space*. A norm or a metric defines the topology of a linear space as follows: $E \subset X$ is open if and only if for any $x \in E$ there exists $\varepsilon > 0$ such that any $y \in X$ satisfying $\|x - y\| < \varepsilon$, respectively, $\rho(x, y) < \varepsilon$, belongs to E.

Clearly, any norm induces a metric on X (just set $\rho(x, y) := \|x - y\|$). In this case, $\|x\| = \rho(x, 0)$. In this paper we give a geometric description of metrics on linear spaces which are induced by norms. In other words, we establish that metric spaces with certain properties are, in fact, normed spaces.

The exposition is fairly self-contained. The only major result we need is a description of isometries between normed linear spaces, proved by Mazur and Ulam in the 1930s, see the original paper [**2**] or [**1**, Chapter XI]: if ϕ is a bijective isometry between real normed linear spaces X and Y, then ϕ is an affine map, that is, there exists a linear isometry $T : X \to Y$ such that $\phi(x) = \phi(0) + Tx$ for any $x \in X$.

Below we formulate the main results of our paper. They will be proved in Section 2. Finally, Section 3 contains a counterexample showing that the assumptions of our theorems cannot be significantly weakened.

Theorem 1.1. *Let X be a linear space over a field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, equipped with a metric ρ. Assume*:

1. *ρ is translation invariant, that is, $\rho(x, y) = \rho(x + z, y + z)$ for any $x, y, z \in X$.*

2. *Multiplication by real scalars is continuous*: *for any $x \in X$, the map $[0, 1] \to X : t \mapsto tx$ is continuous.*

3. *Every one-dimensional affine subspace of X is isometric to $\mathbf{K}$. In addition, if $\mathbf{K} = \mathbf{C}$, then $\rho(x, y) = \rho(ix, iy)$ for any $x, y \in X$.*

Then ρ induces a norm on X. In other words, $\|x\| = \rho(x, 0)$ is a norm.

The conclusion of this theorem fails if condition (2) is omitted, see Example 3.1. However, we can do away with (2) by strengthening (3):

Theorem 1.2. *Let X be a linear space over a field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, equipped with a translation invariant metric ρ. Assume that one of the three statements holds for all $x \in X$:*

1. $\mathbf{K} = \mathbf{R}$, *and* $\{\lambda x \mid -1 \leq \lambda \leq 1\}$ *is isometric to* $[-\rho(x, 0), \rho(x, 0)]$.

1′. $\mathbf{K} = \mathbf{R}$, *and* $\{\lambda x \mid 0 \leq \lambda \leq 1\}$ *is isometric to* $[0, \rho(x, 0)]$.

2. $\mathbf{K} = \mathbf{C}$, *and* $\{\lambda x \mid \lambda \in \mathbf{C}, |\lambda| \leq 1\}$ *is isometric to* $\{\lambda \in \mathbf{C} \mid |\lambda| \leq \rho(x, 0)\}$.

Then ρ induces a norm on X.

Our main tool is

Lemma 1.3. *Suppose $\phi : \mathbf{C} \to \mathbf{C}$ is a bijection satisfying, for all $x, y \in \mathbf{C}$,*

1. $|\phi(x + y)| = |\phi(x) + \phi(y)|$;
2. $|\phi(ix)| = |\phi(x)|$;
3. *ϕ is separately real continuous, that is, $\phi(t_n x) \to \phi(tx)$ whenever $t_n \to t$ (t_n, $t \in \mathbf{R}$).*

Then ϕ is either linear or conjugate linear: either $\phi(z) = \alpha z$ for all $z \in \mathbf{C}$, or $\phi(z) = \alpha \overline{z}$ for all $z \in \mathbf{C}$.

Remark. To prove Theorem 1.1, we only need the equality $|\phi(z)| = a|z|$ for some $a > 0$. However, this fails if (3) is not assumed.

2. Proof of the main theorem. We start by stating a well known (and easily proved) result concerning translation invariant metrics.

Proposition 2.1. *Suppose ρ is a translation invariant metric on a linear space X, and define, for $x \in X$, $\|x\| := \rho(x, 0)$. Then, for any $x, y \in X$,*

1. $\|x\| = \|-x\|$.
2. $\|x\| = 0$ *if and only if* $x = 0$.
3. $\|x + y\| \leq \|x\| + \|y\|$.

We state the next two results in greater generality than is required for the proof of Theorem 1.1.

Lemma 2.2. *Suppose X and Y are real linear spaces, $\|\cdot\|$ is a norm on Y, and let $\phi : X \to Y$ be a map for which $\|\phi(x_1 + x_2)\| = \|\phi(x_1) + \phi(x_2)\|$ for any $x_1, x_2 \in X$. Then, for any $x \in X$, we have:*

1. $\phi(0) = 0$.
2. $\phi(-x) = -\phi(x)$.
3. $\|\phi(rx)\| = |r|\|\phi(x)\|$ *for any* $r \in \mathbf{Q}$.

Proof. (1) $2\|\phi(0)\| = \|\phi(0) + \phi(0)\| = \|\phi(2 \cdot 0)\| = \|\phi(0)\|$, hence $\|\phi(0)\| = 0$, which implies $\phi(0) = 0$.

(2) $0 = \|\phi(x + (-x))\| = \|\phi(x) + \phi(-x)\|$, hence $\phi(x) + \phi(-x) = 0$.

(3) It suffices to show that

$$\|\phi(mx)\| = m\|\phi(x)\| \tag{2.1}$$

whenever $m \in \mathbf{N}$ and $x \in X$. Indeed, $\|\phi(x)\| = \|\phi(n(x/n))\| = n\|\phi(x/n)\|$, $n \in \mathbf{N}$, and therefore, $\|\phi(mx/n)\| = m\|\phi(x/n)\| = (m/n)\|\phi(x)\|$ when $m, n \in \mathbf{N}$. Moreover, $\|\phi(-mx/n)\| = (m/n)\|\phi(x)\|$.

We use induction to prove (2.1). This equality clearly holds for $m = 1$. For $m = 2$, we have

$$\|\phi(2x)\| = \|\phi(x + x)\| = \|2\phi(x)\| = 2\|\phi(x)\|.$$

Now suppose (2.1) has been established for $m = 1, 2, \ldots, 2k$. Then

$$\|\phi((2k+2)x)\| = \|\phi(2((k+1)x))\| = 2\|\phi((k+1)x)\| = 2(k+1)\|\phi(x)\|.$$

Therefore,

$$\begin{aligned} 2(k+1)\|\phi(x)\| &= \|\phi((2k+2)x)\| \\ &= \|\phi((2k+1)x) + \phi(x)\| \leq \|\phi((2k+1)x)\| + \|\phi(x)\|, \end{aligned}$$

which implies $\|\phi((2k+1)x)\| \geq (2k+1)\|\phi(x)\|$. On the other hand,

$$\|\phi((2k+1)x)\| = \|\phi(2kx) + \phi(x)\| \leq (2k+1)\|\phi(x)\|.$$

Thus, $\|\phi((2k+1)x)\| = (2k+1)\|\phi(x)\|$. We have established (2.1) for $m = 2k+1, 2k+2$. $\square$

Remark. ϕ as above is not necessarily a linear isometry: consider, for instance, $\phi : \mathbf{R} \to \mathbf{R}^2 : t \mapsto (t, \sin t)$, where $\mathbf{R}^2$ is equipped with the norm $\|(x,y)\| = \max\{|x|, |y|\}$.

Lemma 2.3. *Suppose X is a topological linear space, Y is a normed linear space, $\phi : X \to Y$ is a bijection satisfying*:

1. *$\phi|_Z$ is separately real continuous for all one-dimensional subspaces Z of X, that is, $\phi(tx) = \lim_n \phi(t_n x)$ whenever $x \in X$ and $\lim_n t_n = t$ for real (t_n), t.*

2. *$\|\phi(x_1 + x_2)\| = \|\phi(x_1) + \phi(x_2)\|$ for any $x_1, x_2 \in X$.*

3. *In the complex case, $i\phi(Z) = \phi(Z)$ for all one-dimensional subspaces Z of X, and $\|\phi(ix)\| = \|\phi(x)\|$ for any $x \in X$.*

Then ϕ is either linear or conjugate linear.

Proof. For any $x \in X$ define $|||x||| := \|\phi(x)\|$. Show first that $|||\lambda x||| = |\lambda| \cdot |||x|||$ for any $x \in X$ and $\lambda \in \mathbf{R}$. To this end, find a sequence of rational numbers (r_n) converging to λ. By the previous lemma, $|||r_n x||| = |r_n| \cdot |||x|||$. By continuity of ϕ on $\mathbf{R}x$, $\phi(\lambda x) = \lim_n \phi(r_n x)$, hence

$$|||x||| = \lim_n \|\phi(r_n x)\| = \lim_n |r_n| \cdot |||x||| = |\lambda| \cdot |||x|||.$$

In the real case, we conclude that $|||\cdot|||$ is a norm (all the conditions on $|||\cdot|||$ can now be easily verified). Therefore, ϕ is a bijective isometry between normed spaces $(X, |||\cdot|||)$ and $(Y, \|\cdot\|)$. By Mazur-Ulam's theorem, ϕ is a linear map.

Now consider the complex case. The reasoning above shows that ϕ is "real linear," that is, $\phi(\lambda x) = \lambda\phi(x)$ for any $x \in X$ and $\lambda \in \mathbf{R}$. It remains to show that either $\phi(ix) = i\phi(x)$ for every $x \in X$ (in this case, ϕ is linear), or $\phi(ix) = -i\phi(x)$ for every $x \in X$ (ϕ is conjugate linear).

Fix $x \in X$. Let $Z = \mathbf{C}x$, $y_1 = \phi(x)$, $y_2 = \phi(ix)$. By (3), $y_2 = \lambda y_1$ for some $\lambda \in \mathbf{C}$. But $\|\phi(ix)\| = \|\phi(x)\|$, hence $|\lambda| = 1$, and $\|y_1\| = \|y_2\|$.

Then

$$\|\phi(x+ix)\| = \|(1+\lambda)y_1\| = \sqrt{2+2\mathrm{Re}\,\lambda}\,\|y_1\|,$$

and

$$\|\phi(i(x+ix))\| = \|\phi(ix)-\phi(x)\| = \|(\lambda-1)y_1\| = \sqrt{2-2\mathrm{Re}\,\lambda}\,\|y_1\|.$$

However, $\|\phi(x+ix)\| = \|\phi(i(x+ix))\|$, hence $\mathrm{Re}\,\lambda = 0$, and $\lambda = \pm i$. Thus, $\phi((a+ib)x)$ equals to either $(a+ib)\phi(x)$, or $(a-ib)\phi(x)$. That is, $\phi|_Z$ is either linear, or conjugate linear.

Now suppose, for the sake of contradiction, that ϕ is linear on $Z_1 = \mathbf{C}x_1$, and conjugate linear on $Z_2 = \mathbf{C}x_2$. ϕ is real linear, hence

$$\phi(i(x_1+x_2)) = \phi(ix_1)+\phi(ix_2) = i\phi(x_1) - i\phi(x_2). \tag{2.2}$$

On the other hand, ϕ must be either linear or conjugate linear on $\mathbf{C}(x_1+x_2)$. Therefore, either

$$\phi(i(x_1+x_2)) = i\phi(x_1+x_2) = i\phi(x_1)+i\phi(x_2),$$

or

$$\phi(i(x_1+x_2)) = -\,i\phi(x_1+x_2) = -\,i\phi(x_1) - i\phi(x_2).$$

In either case, x_1 or x_2 must equal 0 for the equality (2.2) to hold. Therefore, ϕ is either linear on the whole space X, or conjugate linear on the whole space X. $\square$

Lemma 1.3 follows easily from Lemma 2.3 (just take $X = Y = \mathbf{C}$). In the real case ($X = Y = \mathbf{R}$) we obtain:

Corollary 2.4. *Suppose $\phi : \mathbf{R}\to\mathbf{R}$ is a continuous bijection, and $|\phi(x+y)| = |\phi(x)+\phi(y)|$ for any $x, y \in \mathbf{R}$. Then ϕ is linear, that is, there exists $\alpha \in \mathbf{R}$ such that $\phi(x) = \alpha x$ for any $x \in \mathbf{R}$.*

Proof of Theorem 1.1. To show that $\|\cdot\| = \rho(\cdot, 0)$ is a norm, it suffices to prove that

$$\|\lambda x\| = |\lambda|\,\|x\| \tag{2.3}$$

for any $\lambda \in \mathbf{K}$ and $x \in X$. We begin by considering the real case. Fix a nonzero $x \in X$, and consider $Z = \{\lambda x \mid \lambda \in \mathbf{R}\}$. There exists a bijective isometry $T : Z \to \mathbf{R}$. Assume without loss of generality that $T(0) = 0$. Then $|T(z)| = \|z\|$ for any $z \in Z$. Define $\phi : \mathbf{R} \to \mathbf{R}$ by setting $\phi(\lambda) = T(\lambda x)$. For any $\lambda_1, \lambda_2 \in \mathbf{R}$, we have

$$\begin{aligned}|\phi(\lambda_1 - \lambda_2)| &= |T(\lambda_1 x - \lambda_2 x)| = \rho((\lambda_1 x - \lambda_2 x), 0) = \rho(\lambda_1 x, \lambda_2 x) \\ &= |T(\lambda_1 x) - T(\lambda_2 x)| = |\phi(\lambda_1) - \phi(\lambda_2)|.\end{aligned}$$

By our assumption, $\phi(0) = 0$, hence ϕ is an odd function ($\phi(-\lambda) = -\phi(\lambda)$ for any $\lambda \in \mathbf{R}$). Moreover, the map $\lambda \mapsto \rho(\lambda x, 0)$ is continuous, hence ϕ is continuous. By Corollary 2.4, ϕ is linear, that is, $\phi(\lambda) = \alpha\lambda$. Here, $|\alpha| = |\phi(1)| = \rho(x, 0)$. Thus,

$$\|\lambda x\| = \rho(\lambda x, 0) = |\phi(\lambda)| = |\lambda|\,\|x\|.$$

This establishes (2.3) in the real case.

In the complex case, fix $x \in X \backslash \{0\}$, and consider a subspace $Z = \mathbf{C}x \subset X$ and a bijective isometry $T : Z \to \mathbf{C}$ (that is, $\rho(\lambda_1 x, \lambda_2 x) = |T(\lambda_1) - T(\lambda_2)|$ whenever $\lambda_1, \lambda_2 \in \mathbf{C}$). As in the real case, we set $\phi(\lambda) = T(\lambda x)$, and show that $|\phi(\lambda_1 - \lambda_2)| = |\phi(\lambda_1) - \phi(\lambda_2)|$. In particular, $\|\lambda x\| = \rho(\lambda x, 0) = |\phi(\lambda)|$, and therefore,

$$|\phi(\lambda)| = \rho(\lambda x, 0) = \rho(i\lambda x, 0) = |\phi(i\lambda)|$$

for any $\lambda \in \mathbf{C}$. By Lemma 1.3, either $\phi(\lambda) = \alpha\lambda$, or $\phi(\lambda) = \alpha\overline{\lambda}$, with $|\alpha| = |\phi(1)| = \|x\|$. Thus, $\|\lambda x\| = |\phi(\lambda)| = |\lambda|\|x\|$ for any $\lambda \in \mathbf{C}$. $\square$

Proof of Theorem 1.2. As before, we only have to show that $\|\lambda x\| = |\lambda|\|x\|$. Consider the cases (1) and (2) first. Observe first that $\|\lambda x\| \leq \|x\|$ whenever $|\lambda| \leq 1$. Indeed, let $D(c) = \{\lambda \in \mathbf{K} \mid |\lambda| \leq c\}$. We know that $D(1)x$ is isometric to $D(\rho(0, x))$. Therefore, $D(1)\lambda x$ is isometric to $D(\rho(0, \lambda x))$. If $|\lambda| \leq 1$, then $D(1)\lambda x \subset D(1)x$, and therefore, $D(\rho(0, \lambda x))$ is isometric to a subset of $D(\rho(0, x))$. This is possible only if $\rho(0, x) \geq \rho(0, \lambda x)$.

By the above, $\rho(0, x) = \rho(0, \omega x)$ whenever $|\omega| = 1$.

Fix $x \in X$ for which $\rho(0,x) = 1$, and consider the function $\phi : [0,\infty)\to[0,\infty) : \lambda \mapsto \rho(0,\lambda x)$. It suffices to show that ϕ is the identity function. We already know that ϕ is nondecreasing. The triangle inequality implies the subadditivity of ϕ, that is, $\phi(\lambda_1+\lambda_2) \leq \phi(\lambda_1)+\phi(\lambda_2)$.

Moreover, $\phi(0) = 0$, and $\lim_{\lambda\to 0^+}\phi(\lambda) = 0$ (otherwise, there can be no isometry between $D(1)x$ and $D(\rho(0,x)))$. Therefore, ϕ is uniformly continuous on $[0,\infty)$. Indeed, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\phi(\varepsilon) < \delta$. If $x < y < x+\delta$, we have, by the above,

$$\phi(x) \leq \phi(y) \leq \phi(x) + \phi(\delta) < \phi(x) + \varepsilon.$$

We next show that $\phi(n\lambda) = n\phi(\lambda)$ for any $n \in \mathbf{N}$ and $\lambda \in [0,\infty)$. Clearly, the equality holds for $n = 0,1$. Suppose $\phi((n-1)\lambda) = (n-1)\phi(\lambda)$, and show that $\phi(n\lambda) = n\phi(\lambda)$. Consider an isometry $\psi : D(n\lambda)x\to D(\phi(n\lambda))$. $\rho(0,\omega n\lambda) = \phi(n\lambda)$ whenever $|\omega| = 1$, hence $\psi(0) = 0$. Assume without loss of generality that $\psi(n\lambda x) = \phi(n\lambda)$. Then there exists ω such that $|\omega| = 1$ and $\psi((n-1)\omega\lambda x) = (n-1)\phi(\lambda)$. Then

$$\begin{aligned}\phi(n\lambda) - (n-1)\phi(\lambda) &= \big|\psi(n\lambda x) - \psi((n-1)\omega\lambda x)\big| \\ &= \rho(n\lambda x, (n-1)\omega\lambda x) = \rho((n-(n-1)\omega)\lambda x, 0) \\ &= \phi(|n-(n-1)\omega|\lambda) \geq \phi(\lambda).\end{aligned}$$

On the other hand, the triangle inequality implies $\phi(n\lambda) \leq n\phi(\lambda)$. Thus, $\phi(n\lambda) = n\phi(\lambda)$.

Suppose, for the sake of contradiction, that $\phi(\lambda) < \lambda$ for some λ. Then, for sufficiently large n,

$$\phi(n\lambda) = n\phi(\lambda) < n\lambda - 1 < \lfloor n\lambda \rfloor = \phi(\lfloor n\lambda \rfloor) \leq \phi(n\lambda).$$

The possibility of $\phi(\lambda) > \lambda$ is ruled out the same way.

The case $(1')$ is dealt with in a similar way. As above, we show that $\rho(0,\lambda x) = \lambda\rho(0,x)$ for any $x \in X$ and $\lambda \geq 0$. Moreover, $\rho(0,x) = \rho(0,-x)$ due to the translation invariance of ρ. Thus, $\rho(0,\lambda x) = |\lambda|\rho(0,x)$ for any $x \in X$ and $\lambda \in \mathbf{R}$. $\square$

3. A counterexample. We shall show that Theorem 1.1 fails if condition (2) (real continuity) is dropped. For the sake of convenience, we define the "real multiplication map" $M_x : \mathbf{R}\to\mathbf{R}^N$ (or $\mathbf{R}\to\mathbf{K}$): for x in $\mathbf{R}^N$ (or $\mathbf{K}$, respectively), we set $M_x(t) = tx$. The "inverse" M_x^{-1} maps $\mathbf{R}x$ to $\mathbf{R}$.

Example 3.1. There exists a translation invariant metric ρ on $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, satisfying assumptions (1) and (3) of Theorem 1.1, such that ρ is not induced by a norm. Moreover, for any $x \in \mathbf{K}\backslash\{0\}$, the real multiplication map M_x, as well as its inverse, are discontinuous everywhere.

The construction is based on the following lemma.

Lemma 3.2. (a) *There exists a bijection $\psi : \mathbf{R}\to\mathbf{R}$ such that $\psi(0) = 0$, $\psi(1) = 1$, and $\psi(\sum_{k=1}^n r_k x_k) = \sum_{k=1}^n r_k \psi(x_k)$ whenever $n \in \mathbf{N}$, $r_k \in \mathbf{Q}$, and $x_k \in \mathbf{R}$. Moreover, for every interval (a, b),*

$$\sup_{x,y\in(a,b)} |\psi(x) - \psi(y)| = \sup_{x,y\in(a,b)} |\psi^{-1}(x) - \psi^{-1}(y)| = \infty.$$

Consequently, the map $N_x : t \mapsto \psi(tx)$ and its inverse are discontinuous everywhere whenever $x \neq 0$.

(b) *Suppose G is a countable semi-group of unitary operators on $\mathbf{R}^N$ ($N \in \mathbf{N}$). Then there exists a bijection $\phi : \mathbf{R}^N\to\mathbf{R}^N$, such that $\phi(\sum_{k=1}^n r_k U_k x_k) = \sum_{k=1}^n r_k U_k \phi(x_k)$ whenever $n \in \mathbf{N}$, $r_k \in \mathbf{Q}$, $U_k \in G$, and $x_k \in \mathbf{R}^N$. Moreover,*

$$\sup_{x,y\in E\cap L} |\phi(x) - \phi(y)| = \sup_{x,y\in E\cap L} |\phi^{-1}(x) - \phi^{-1}(y)| = \infty$$

for every open set $E \subset \mathbf{R}^N$ and any affine subspace $L \subset \mathbf{R}^N$ of dimension at least 1. Consequently, the map $N_x : t \mapsto \phi(tx)$ and its inverse are discontinuous everywhere whenever $x \neq 0$.

Proof. Consider the countable group F, containing all rational numbers, entries of unitaries $U \in G$, as well as products, inverses, and sums of its members. In the case (a), the group G consists of just one operator – the identity, and $F = \mathbf{Q}$. An application of Zorn's lemma produces a *Hamel basis* for $\mathbf{R}$ over F, that is, an uncountable family $(b_\alpha)_{\alpha\in\mathcal{I}} \subset (0,\infty)$ such that any $x \in \mathbf{R}\backslash\{0\}$ can be uniquely represented as $x = \sum_{k=1}^n f_k b_{\alpha_k}$, with $n \in \mathbf{N}$ and $f_k \in F\backslash\{0\}$ (here $b_{\alpha_1}, \dots, b_{\alpha_n}$ are distinct). Moreover, we can assume that $1 = b_\alpha$ for some $\alpha \in \mathcal{I}$.

Pick distinct $\beta_1, \beta_2 \in \mathcal{I}$ for which $b_{\beta_1}, b_{\beta_2} \neq 1$. Let $a_{\beta_1} = b_{\beta_2}$, $a_{\beta_2} = b_{\beta_1}$, and $a_\alpha = b_\alpha$ if $\alpha \in \mathcal{I}\backslash\{\beta_1,\beta_2\}$. Define ψ by setting

$$\psi\Big(\sum_{k=1}^n f_k b_{\alpha_k}\Big) = \sum_{k=1}^n f_k a_{\alpha_k}.$$

It is easy to see that ψ is rational-linear, $\psi(0) = 0$, and $\psi(1) = 1$. ψ is a bijection, since any non-zero $x \in \mathbf{R}$ has a unique decomposition as a linear combination of b_α's with coefficients from F.

To show that ψ is discontinuous everywhere, observe that

$$S = \Big\{ \sum_{k=1}^n f_k b_{\alpha_k} \Big| n \in \mathbf{N},\ f_k \in F,\ \alpha_k \in \mathcal{I}\backslash\{\beta_1,\beta_2\} \Big\}$$

is dense in $\mathbf{R}$ (indeed, Fb_α is dense in $\mathbf{R}$ for any α). Thus, $S + Nb_{\beta_1}$ is dense in $\mathbf{R}$ for any $N \in \mathbf{N}$. Therefore, for every $a, b \in \mathbf{R}$ and $N \in \mathbf{N}$ there exist $s, t \in (a,b)$ such that $s \in S$ (hence $\psi(s) = s$) and $t \in S + Nb_{\beta_1}$ (hence $\psi(t) = t + N(b_{\beta_2} - b_{\beta_1})$). In particular, $|\psi(s) - \psi(t)| \geq N|b_{\beta_2} - b_{\beta_1}| - (b-a)$. N can be selected to be arbitrarily large, hence $\sup_{s,t\in(a,b)} |\psi(s) - \psi(t)| = \infty$. The equality $\sup_{s,t\in(a,b)} |\psi^{-1}(s) - \psi^{-1}(t)| = \infty$ is proved in the same manner.

To establish (b), define

$$\phi(x_1, \dots, x_N) = (\psi(x_1), \dots, \psi(x_N))$$

for $(x_1, \dots, x_N) \in \mathbf{R}^N$. Then, for any $r_k \in \mathbf{Q}$, $U_k = (U_{kij})_{i,j=1}^N \in G$, and $x_k = (x_{kl})_{l=1}^N$, $1 \leq k \leq n$, we have

$$\begin{aligned}
\phi\Big(\sum_{k=1}^n r_k U_k x_k\Big) &= \Big(\psi\Big(\sum_{k=1}^n r_k \sum_{j=1}^N U_{kij} x_{kj}\Big)\Big)_{i=1}^N \\
&= \sum_{k=1}^n r_k \Big(\sum_{j=1}^N U_{kij}\psi(x_{kj})\Big)_{i=1}^N \\
&= \sum_{k=1}^n r_k U_k \Big(\psi(x_{kj})\Big)_{j=1}^N = \sum_{k=1}^n r_k U_k \phi(x_k).
\end{aligned}$$

Suppose E and L are an open subset of $\mathbf{R}^N$ and an affine subspace of $\mathbf{R}^N$, respectively, with $L \neq \{0\}$. For $1 \leq k \leq N$, denote by P_k the kth coordinate projection, that is, $P_k(x_1, x_2, \dots, x_N) = (0, \dots, 0, x_k, 0, \dots, 0)$. Clearly, there exists k for which $P_k(E \cap L)$ contains an open interval (a, b). Then

$$\sup_{x,y \in E \cap L} |\phi(x) - \phi(y)| \geq \sup_{s,t \in (a,b)} |\psi(s) - \psi(t)| = \infty.$$

Similarly, one can show that

$$\sup_{x,y \in E \cap L} |\phi^{-1}(x) - \phi^{-1}(y)| = \infty.$$

Thus, ϕ has all the desired properties. $\square$

Construction of Example 3.1. In the real case, define $\rho(x, y) = |\psi(x-y)|$ for $x, y \in \mathbf{R}$ (here ψ is the function from Lemma 3.2(a)). Then $(\mathbf{R}, \rho)$ is isometric to $(\mathbf{R}, |\cdot|)$ via ψ, and ρ is a translation invariant. Moreover, by Lemma 3.2, for any interval $(a, b) \subset \mathbf{R}$ and $x \in \mathbf{R} \backslash \{0\}$,

$$\sup_{s,t \in (a,b)} \rho(M_x(t), M_x(s)) = \sup_{s,t \in (a,b)} |\psi(tx) - \psi(sx)| = \infty.$$

Therefore, multiplication by scalars is not continuous in $(\mathbf{R}, \rho)$ (condition (2) of Theorem 1.1 fails). Thus, $\|x\| = \rho(x, 0)$ is not a norm. The discontinuity of M_x^{-1} is proved in the same way.

In the complex case, consider the group G of operators on $\mathbf{R}^2$, generated by the matrix $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (that is, G consists of the identity I, U, $U^2 = -I$, and $U^3 = -U$). Note that $\mathbf{R}^2$ can be canonically identified with $\mathbf{C}$ (the vector (x, y) corresponds to $x + iy$). Then U can be viewed as the operator of multiplication by i. For $z_1, z_2 \in \mathbf{C}$, let $\rho(z_1, z_2) = |\phi(\operatorname{Re} z_1, \operatorname{Im} z_1) - \phi(\operatorname{Re} z_2, \operatorname{Im} z_2)|$. By construction, ρ is invariant under translation and multiplication by i, and $(\mathbf{C}, \rho)$ is isometric to $(\mathbf{C}, |\cdot|)$. The discontinuity of M_x and M_x^{-1} is proven as in the real case. This allows us to conclude that $\|x\| = \rho(x, 0)$ is not a norm. $\square$

Remark. In fact, we have shown that Lemma 1.3 fails if condition (3) is not satisfied.

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