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Note

Boundedly complete weak-Cauchy basic sequences in Banach spaces with the PCP

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Abstract

It is proved that *every non-trivial weak-Cauchy sequence in a Banach space with the* PCP (*the Point of Continuity Property*) *has a boundedly complete basic subsequence*. The following result, due independently to S. Bellenot and C. Finet, is then deduced as a corollary. *If a Banach space X has separable dual and the PCP, then every non-trivial weak-Cauchy sequence in X has a subsequence spanning an order-one quasi-reflexive space*.

Keywords: Banach space; Point of Continuity Property; Non-trivial weak-Cauchy sequence; Boundedly complete basic sequence

1. Introduction

We use standard Banach space facts and terminology. Let us just recall that a sequence (b_j) in a Banach space is a *basic sequence* provided it is a Schauder-basis for its closed linear span, denoted $[b_j]$. A basic sequence (b_j) is called *boundedly complete* provided whenever scalars (c_j) satisfy $\sup_n \|\sum_{j=1}^n c_j b_j\| < \infty$, then $\sum c_j b_j$ converges. A weak-Cauchy sequence is called *non-trivial* if it does not converge weakly. Our main result goes as follows:

Theorem 1. *Let X be a Banach space with the Point of Continuity Property* (*the PCP*). *Then every non-trivial weak-Cauchy sequence in X has a boundedly complete basic subsequence.*

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Recall that a Banach space is said to have the PCP provided every non-empty closed subset admits a point of continuity from the weak to norm topologies. It is known that separable dual spaces, and more generally, spaces with the Radon–Nikodym property, have the PCP.

It is also known that there are separable Banach spaces with the PCP and failing the RNP [2], and in fact there exists a separable space whose dual is non-separable and has the PCP. (See [6] and [11] for references and further information; a remarkable result of Charles Stegall [14] asserts that a non-separable dual of a separable Banach space fails the RNP.) For an interesting "tree" characterization of Banach spaces having the PCP with separable duals, see [4]. A (quite special) case of our main result is also obtained in [4]. For a recent application of Theorem 1, see [13]. We also obtain the following immediate consequence of Theorem 1 and previously known results: Every semi-normalized basic sequence in a Banach space with the PCP has a boundedly complete subsequence. (In case the Banach space is isomorphic to a subspace of a separable dual space, this follows easily from the l^1-Theorem and Theorem III.2 of [9].)

Of course boundedly complete basic sequences span Banach spaces isomorphic to a dual space; moreover it was previously known that Banach spaces with the PCP have boundedly complete basic sequences (see [6]). (All Banach spaces shall be assumed infinite-dimensional.) This latter result also easily follows from our main result and the "ℓ^1-Theorem" [10]. Indeed, if X is a reflexive Banach space, then any basic sequence is boundedly complete; but if X is non-reflexive with the PCP, then by the ℓ^1-Theorem, either X has a sequence equivalent to the usual ℓ^1-basis, which of course is boundedly complete, or X has a non-trivial weak-Cauchy sequence.

We recall that a basic sequence (b_j) in a Banach space is called (s) (respectively (s.s.)) if (b_j) is a weak-Cauchy sequence so that whenever scalars (c_j) are such that $\sum c_j b_j$ converges (respectively $\sup_n \|\sum_{j=1}^n c_j b_j\| < \infty$), then $\sum c_j$ converges ((s) stands for "summing", (s.s.) stands for "strongly summing"). As shown in [8] (cf. also [12, Proposition 2.2]), every non-trivial weak-Cauchy sequence in a Banach space has an (s)-subsequence. Now it is proved in [12] that a Banach space X contains no isomorph of c_0 if and only if every non-trivial weak-Cauchy sequence in X has an (s.s.) subsequence. Thus if X has the PCP, we obtain a better behaved subsequence, for boundedly complete (s)-sequences are obviously (s.s.). However, the existence of boundedly complete (s)-sequences in a general Banach space appears to be a rare phenomenon. Indeed, W.T. Gowers [7] has constructed a Banach space X containing no (infinite-dimensional) subspace isomorphic either to c_0 or to a dual space. Thus X has no boundedly complete basic sequences, although it is "saturated" with (s.s.) ones, by the above-mentioned result in [12].

Our main result is proved using arguments along the lines of those in S. Bellenot [1] and C. Finet [5], and uses (as do the above authors) the fundamental result of N. Ghoussoub and B. Maurey [6] that every separable Banach space with the PCP has a boundedly complete skipped-blocking decomposition. We prove Theorem 1 by first observing in Proposition 2 that an (s)-sequence is boundedly complete if and only if its difference sequence is skipped-boundedly complete. Then we show that any non-trivial weak-Cauchy sequence in a space with a skipped-boundedly complete decomposition may be refined so that its differences almost lie in the elements of the decomposition in such a way that a skipped-blocking of the differences almost lies in a skipped-blocking of the decomposition, hence is boundedly complete.

Next, we give an argument of Bellenot which yields that any non-trivial weak-Cauchy sequence in a space with separable dual, has a subsequence whose differences form a shrinking basic sequence (Proposition 5). Finally, we observe that if a Banach space B is spanned by a boundedly complete (s)-basis with difference sequence (e_j), and Y denotes the closed linear span of the e_j^*'s in B^*, then the canonical map of B into Y^* has range of codimension one (Proposition 6). These considerations then immediately yield the main result of Bellenot [1] and Finet [5]: *if a Banach space X has separable dual and the PCP, then every non-trivial weak-Cauchy sequence in X has a subsequence spanning an order-one quasi-reflexive space* (Corollary 6 below).

2. The PCP and the notion of the boundedly complete skipped blocking property (the bcsbp, to be defined shortly), were introduced in [2], where it was proved that the bcsbp implies the PCP. Subsequently N. Ghoussoub and B. Maurey proved the remarkable result that the converse is true (for separable spaces) in [6]. (For a later exposition of these results, see [11].)

To define the bcsbp, we first recall that a sequence of non-zero finite-dimensional subspaces (F_j) of a Banach space X is called a *decomposition* of X if $[F_j] = X$ and $F_i \cap [F_j]_{j\neq i} = \{0\}$ for all i. (For (A_j) a sequence of subsets of X, $[A_j]$ denotes the closed linear span of the A_j's.)

Given (F_j) a decomposition and I a finite non-empty interval of integers, we denote the linear span of the F_j's for j in I by F_I.

A sequence (F_j) of non-zero finite-dimensional subspaces of a Banach space is called an FDD provided (F_j) is a Schauder-decomposition for $[F_j]$. That is, for every x in $[F_j]$, there exists a unique sequence (f_j) with $f_j \in F_j$ for all j and $x = \sum f_j$. A classical result of Banach yields that an FDD is a decomposition for its closed linear span.

Definition 1. A decomposition (F_j) for a Banach space X is called a boundedly complete skipped-blocking decomposition if given a sequence (n_j) of non-negative integers with $n_j + 1 < n_{j+1}$ for all j, then $(F_{(n_j,n_{j+1})})$ is a boundedly complete FDD. That is, $(F_{(n_j,n_{j+1})})$ is an FDD so that whenever $f_j \in (F_{(n_j,n_{j+1})})$ for all j and $\sup_n \|\sum_{j=1}^n f_j\| < \infty$, then $\sum f_j$ converges.

Of course we say that X has the bcsbp if X admits a boundedly complete skipped-blocking decomposition.

Definition 2. A sequence (e_j) in a Banach space is called skipped boundedly complete if letting F_j be the span of e_j for all j, then (F_j) is a boundedly complete skipped-blocking decomposition for $[e_j]$.

Remark. The following equivalences are easily established ((x_j) is called a proper subsequence of (b_j) if $(x_j) = (b_{n_j})$ where $N \sim \{n_1, n_2, \ldots\}$ is infinite). *Let (e_j) be a basic sequence in a Banach space. The following are equivalent*:

(i) *(e_j) is skipped boundedly complete.*
(ii) *Every proper subsequence of (e_j) is boundedly complete.*
(iii) *Given a sequence of scalars (c_j) with $c_j = 0$ for infinitely many j and* $\sup_n \|\sum_{j=1}^n c_j e_j\| < \infty$ *, then $\sum c_j e_j$ converges.*

The next result gives some simple equivalences for an (s)-sequence to be boundedly complete. (For a sequence (b_j), (e_j) is called the difference sequence of (b_j) if $e_1 = b_1$ and $e_j = b_j - b_{j-1}$ for all $j > 1$. We recall that if (b_j) is an (s)-sequence, its difference sequence (e_j) is basic; cf. Proposition 2.1 of [12].)

Proposition 2. *Let (b_j) be an* (s)*-sequence with difference sequence (e_j). The following assertions are equivalent*:

(a) *(b_j) is boundedly complete.*
(b) *(e_j) is skipped boundedly complete.*
(c) *(e_j) is a* (c.c.)*-sequence so that whenever $\sup_n \|\sum_{j=1}^n c_j e_j\| < \infty$ and $\lim_{j\to\infty} c_j = 0$, then $\sum c_j e_j$ converges.*

Remark. Note that (e_j) cannot itself be boundedly complete since $(\|\sum_{j=1}^n e_j\|) = (\|b_n\|)$ is a bounded sequence. Also, recall that a basic sequence (e_j) is defined to be (c.c.) provided $(\sum_{j=1}^n e_j)$ is weak-Cauchy and whenever $\sup_n \|\sum_{j=1}^n c_j e_j\| < \infty$, then (c_j) converges. ("(c.c.)" stands for "coefficient converging".)

Proof of Proposition 2. (a) $\Rightarrow$ (b). We use equivalence (iii) in the remark following Definition 2. Suppose c_j's are scalars with $c_j = 0$ for infinitely many j and $\sup_n \|\sum_{j=1}^n c_j e_j\| < \infty$. Since (e_j) is a basic sequence, $\sup_j |c_j| < \infty$. Now for all n,

$$\sum_{j=1}^{n} c_j e_j = (c_1 - c_2)b_1 + \cdots + (c_{n-1} - c_n)b_{n-1} + c_n b_n. \tag{1}$$

It follows that $\sup_n \|\sum_{j=1}^n (c_j - c_{j+1})b_j\| < \infty$, hence $\sum (c_j - c_{j+1})b_j$ converges. Choose $n_1 < n_2 < \cdots$ with $c_{n_j} = 0$ for all j. Then by (1),

$$\sum_{j=1}^{n_i} c_j e_j = \sum_{j=1}^{n_i - 1} (c_j - c_{j+1})b_j \quad \text{for all } i,$$

hence $\lim_{i\to\infty} \sum_{j=1}^{n_i} c_j e_j$ exists, so $\sum c_j e_j$ converges since (e_j) is a basic sequence.

(b) $\Rightarrow$ (c). It follows immediately from Proposition 2.7 of [12] that if (e_j) satisfies (b), (e_j) is (c.c.). Indeed, if the scalars (c_j) satisfy the conditions in (iii) of the remark after Definition 2, then since $\sum c_j e_j$ converges and (e_j) is semi-normalized, $c_j \to 0$, so (e_j) is (c.c.) by 2.7 of [12]. Now let the c_j's satisfy the condition in (c), and choose (n_j) an increasing sequence of indices with $|c_{n_j}| < 1/2^j$ for all j. Since $\sum c_{n_j} e_{n_j}$ converges absolutely, its partial sums are bounded, so defining $c'_j = c_j$ if $j \neq n_i$ for any i and $c'_j = 0$ if $j = n_i$ for some i, then $\sup_k \|\sum_{j=1}^k c'_j e_j\| < \infty$, whence $\sum_{j=1}^k c'_j e_j$ converges by (b), so $\sum c'_j e_j + \sum c_{n_j} e_{n_j}$ converges, i.e., $\sum c_j e_j$ converges.

(c) $\Rightarrow$ (a). Let (α_j) be scalars so that $\sup_n \|\sum_{j=1}^n \alpha_j b_j\| < \infty$. Since (e_j) is (c.c.), (b_j) is (s.s.) by Proposition 2.3 of [12], and hence $\sum \alpha_j$ converges. Now define (c_j) by $c_j = \sum_{i=j}^{\infty} \alpha_i$ for all j. Then of course $c_j \to 0$ and for all n,

$$\sum_{j=1}^{n} c_j e_j = \sum_{j=1}^{n-1} \alpha_j b_j + c_n b_n \quad \text{by (1)}. \tag{2}$$

Thus since $\sup_n \|\sum_{j=1}^n c_j e_j\| < \infty$, $\sum c_j e_j$ converges by (c), so since $c_n \to 0$, $\sum \alpha_j b_j$ converges by (2). $\square$

We next give a simple criterion for a basic sequence to be skipped boundedly complete. For F a non-empty subset of X and $x \in X$, $d(x, F) \stackrel{\mathrm{df}}{=} \inf\{\|x - f\|:\ f \in F\}$.

Lemma 3. *Let (F_j) be a skipped boundedly complete decomposition of a Banach space X, and (e_j) a semi-normalized basic sequence in X. Assume there exist integers $0 = n_0 < n_1 < n_2 < \cdots$ so that*

$$\sum_{j=1}^{\infty} d(e_j, F_{(n_{j-1}, n_{j+1})}) < \infty. \tag{3}$$

Then (e_j) is skipped boundedly complete.

Proof. We may choose (u_j) non-zero vectors so that for all j, $u_j \in F_{(n_{j-1}, n_{j+1})}$ and

$$\|e_j - u_j\| \leqslant 2d(e_j, F_{(n_{j-1}, n_{j+1})}) \quad \text{for all } j. \tag{4}$$

Thus $\sum \|e_j - u_j\| < \infty$ by (3), and it follows by a standard perturbation result that (u_j) is a basic sequence equivalent to (e_j). Thus we need only prove that (u_j) is skipped boundedly complete. Let (m_j) be given with $m_0 = 0$ and $m_{i-1} + 1 < m_i$ for all i; we need only show that $([u_j]_{j \in (m_{i-1}, m_i)})_{i=1}^{\infty}$ is a boundedly complete decomposition. Now this decomposition lies inside the one for the F_j's, which skips $F_{n_{m_1}}, F_{n_{m_2}}, \ldots$. That is, setting $\ell_j = n_{m_j}$ for all j, we have that $[u_i]_{i \in (m_{j-1}, m_j)} \subset F_{(\ell_{j-1}, \ell_j)}$ for all j. Since $(F_{(\ell_{j-1}, \ell_j)})$ is a bounded complete FDD, so is $[u_i]_{i \in (m_{j-1}, m_j)}$. □

We are now prepared for the

Proof of Theorem 1. Let (b_i) be a non-trivial weak-Cauchy sequence in X. We may assume without loss of generality that X is separable, for we could replace X by $[b_i]$. Now by passing to a subsequence of (b_i), we may assume that (b_i) is an (s)-sequence. By the basic result in [6], since X is assumed to have the PCP, there exists a boundedly complete skipped blocking decomposition (F_j) for X. Next, we may assume without loss of generality that

$$b_i \text{ is in the linear span of the } F_j\text{'s for all } i. \tag{5}$$

Indeed, we may choose a sequence (y_i) of non-zero elements of the linear span of the F_j's with $\sum \|b_i - y_i\| < \infty$. It then follows by a standard perturbation argument that (y_i) is a basic sequence equivalent to (b_i); in particular, (y_i) is an (s)-sequence. If then $m_1 < m_2 < \cdots$ are such that (y_{m_i}) is boundedly complete, so is (b_{m_i}).

Now by the definition of a decomposition, for each j there exists a projection Q_j from X onto F_j with kernel $[F_i]_{i \neq j}$. Each Q_j is then a bounded linear projection, although the Q_j's are not in general uniformly bounded. Thus also defining $P_j = \sum_{i=1}^{j} Q_i$ for all j, P_j is again a bounded linear projection for each j.

A simple compactness argument shows that we may choose (b'_j) a subsequence of (b_j) so that

$$\lim_{j \to \infty} P_k(b'_j) \quad \text{exists for all } k = 1, 2, \ldots. \tag{6}$$

Now setting $n_0 = 0$, $n_1 = 1$, we claim we can choose $1 < n_2 < n_3 < \cdots$ and (x_j) a subsequence of (b'_j) so that for all j,

$$x_j \in F_{[1,n_{j+1})} \tag{7}$$

and

$$\left\| P_{n_j}(x_k) - P_{n_j}(x_j) \right\| < \frac{1}{2^j} \quad \text{for all } k > j. \tag{8}$$

Once this is done, we have that (x_j) is the desired boundedly complete subsequence. Indeed, let (e_j) be its difference sequence, fix j, and let $k = n_{j-1}$, $\ell = n_{j+1} - 1$. Then by (7), x_j and x_{j-1} lie in $F_{[1,\ell]}$. It follows that

$$(I - P_k)(x_j - x_{j-1}) \in F_{(n_{j-1},n_{j+1})}. \tag{9}$$

But by (8), $\| P_k(x_j - x_{j-1}) \| < 1/2^{j-1}$. Thus we have

$$d(e_j, F_{(n_{j-1},n_{j+1})}) < \frac{1}{2^{j-1}}. \tag{10}$$

Of course (10) and Lemma 3 yield that (x_j) is boundedly complete.

It remains to construct $n_2 < n_3 < \cdots$ and $m_1 < m_2 < \cdots$ so that $(x_j) \stackrel{\mathrm{df}}{=} (b'_{m_j})$ satisfies (7) and (8).

First, using (6), choose m_1 so that

$$\left\| P_{n_1}\left(b'_{m_1}\right) - P_{n_1}\left(b'_j\right) \right\| < \frac{1}{2} \quad \text{for all } j \geqslant m_1. \tag{11}$$

Next using (5), choose $n_2 > n_1$ so that

$$b'_{m_1} \in F_{[1,n_2]}. \tag{12}$$

Now suppose $j > 1$ and m_{j-1} and n_j have been chosen. Then using (6), choose $m_j > m_{j-1}$ so that

$$\left\| P_{n_j}\left(b'_{m_j}\right) - P_{n_j}\left(b'_k\right) \right\| < \frac{1}{2^j} \quad \text{for all } k > m_j. \tag{13}$$

Finally, choose $n_{j+1} > n_j$ so that

$$b'_{m_j} \in F_{[1,n_{j+1}]}. \tag{14}$$

This completes the inductive construction of the m_j's and n_j's. Now (14) and (13) yield that (7) and (8) hold for all j, completing the proof. □

Remark. The following consequence of the main result of [12, Theorem 1.1], complementary to Theorem 1, is motivated by a question of F. Chaatit.

Suppose (b_j) is a semi-normalized non-weakly null basic sequence in a Banach space, so that whenever (c_j) is a sequence of scalars with $\sup_n \|\sum_{j=1}^n c_j b_j\| < \infty$ and $\sum c_j$ convergent, then $\sum c_j b_j$ converges. Then either (b_j) has a convex block basis equivalent to the summing basis, or (b_j) has a boundedly complete subsequence.

To see this, since (b_j) is non-weakly null, and (b_j) is basic, either (b_j) has a non-trivial weak-Cauchy subsequence or a subsequence equivalent to the ℓ^1-basis, by the ℓ^1-Theorem. Of course in the latter case, the subsequence is boundedly complete. In the former case, by the c_0-Theorem (Theorem 1.1 of [12]), either (b_j) has a convex block basis equivalent to the summing basis, or an (s.s.)-subsequence (b'_j). But then of course (b'_j) satisfies the same hypotheses as (b_j), whence (b'_j) is boundedly complete.

Corollary 4. *Let X be a Banach space with the PCP. Then every semi-normalized basic sequence in X has a boundedly complete subsequence.*

Proof. Let (x_j) be a semi-normalized basic sequence in X. If (x_j) has a weakly convergent subsequence (x'_j), then (x'_j) must converge weakly to zero, for no basic sequence can converge weakly to something non-zero. Thus by a result in [6], (x'_j) has a boundedly complete subsequence.

If (x_j) has a subsequence (x'_j) equivalent to the ℓ^1-basis, then of course (x'_j) is boundedly complete. If (x_j) has no weakly convergent subsequence and no subsequence equivalent to the ℓ^1-bases, (x_j) has a weak-Cauchy subsequence (x'_j) by the ℓ^1-Theorem [10]. Of course then (x'_j) is a non-trivial weak-Cauchy sequence, so (x'_j) has a boundedly complete subsequence by Theorem 1. □

Remarks. 1. It follows from the results of [6] that if X is a Banach space with the PCP, then every normalized weakly null tree in X has a boundedly complete branch. (See [4] for the relevant definitions.) We can of course assume that X is separable. Thus X has a boundedly complete skipped blocking decomposition, by [6], and it is not hard to see that the claimed result follows from this.

2. We do not know the answer to the following question. Is the converse to Corollary 4 valid if it is assumed that X has no subspace isomorphic to ℓ^1? The converse is false without this assumption, for by a result in [3], there exists a subspace X of L^1 with the Strong Schur Property, failing the PCP.

We conclude with a discussion of the above mentioned result of S. Bellenot and C. Finet. Recall that a basic sequence (x_j) in a Banach space X is *shrinking* if $[x_j^*] = [x_j]^*$, where $[x_j^*]$ are the functionals biorthogonal to the x_j's (in $[x_j]^*$). It is a standard result that a basic sequence (e_j) is shrinking if and only if every f in X^* satisfies the condition

$$\left\| f|[e_i]_{i=n}^{\infty} \right\| \to 0 \quad \text{as } n \to \infty. \tag{15}$$

The proof of the next result is as in [1], and is given here for the sake of completeness.

Proposition 5. *Let X be a Banach space with X^* separable and (x_j) be a non-trivial weak-Cauchy sequence in X. Then (x_j) has an* (s)*-subsequence (b_j) whose difference sequence (e_j) is shrinking.*

Proof. Let $\{d_1, d_2, \ldots\}$ be a countable dense subset of X^*. By Proposition 2.2 of [12] and a simple compactness argument, we may choose (b_j) an (s)-subsequence of (x_j) so that (setting $b_0 = 0$),

$$\sum_{i=1}^{\infty} \left| b_j(d_i) - b_{j-1}(d_i) \right| < \infty \quad \text{for all } i. \tag{16}$$

Letting (e_j) be the difference sequence of (b_j), (16) yields that every f in X^* satisfies (15). Indeed, first if $f = d_j$ for some j, letting $\tau = \max_i \|e_i^*\|$, we have that

$$f\left(\sum_{i=k}^{\ell} c_i e_i\right) \leqslant \tau \left(\sum_{i=k}^{\ell} \left| f(e_i) \right|\right) \left\| \sum_{i=k}^{\ell} c_i e_i \right\|$$

for all scalars $c_1, \ldots, c_\ell$. Hence

$$\left\| f | [e_i]_{i=k}^{\infty} \right\| \leqslant \tau \sum_{k}^{\infty} \left| f(e_i) \right| \to 0 \quad \text{as } k \to \infty, \text{ by (16).}$$

Finally, if f is arbitrary, let $\varepsilon > 0$ and choose j so that $\|f - d_j\| < \varepsilon$. Then

$$\lim_{n \to \infty} \left\| f | [e_i]_{i=n}^{\infty} \right\| \leqslant \lim_{n \to \infty} \left\| d_j | [e_i]_{i=n}^{\infty} \right\| + \varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (15) holds. □

Remark. It is evident that if (b_j) is an (s)-sequence with difference sequence (e_j), then (e_j) is shrinking if and only if (b_j) spans a codimension-one subspace of $[b_j]^*$. Indeed, suppose $X = [b_j]$ and let G be the weak*-limit of the b_j's in X^{**}; let $\mathbf{s}$ be the summing functional. Then $G(\mathbf{s}) = 1$ and $G(b_j^*) = 0$ for all j. Hence $[b_j^*] \neq X^*$. Of course $\mathbf{s} = e_1^*$ and $b_j^* = e_j^* - e_{j+1}^*$ for all j. Hence if (e_j) is shrinking, $[b_j^*] \oplus [\mathbf{s}] = X^*$. But conversely if $Y \overset{\text{df}}{=} [e_j^* - e_{j+1}^*]_{j=1}^{\infty}$ is codimension one in X^*, then since e_1^* does not belong to Y, $X^* = [e_j^*]$.

The next result shows that the span of a boundedly complete (s)-sequence naturally embeds as a codimension-one subspace of a certain dual space. For Y a linear subspace of X^*, the dual of X, we define the canonical map $T : X \to Y^*$ by $(Tx)(y) = y(x)$ for all $x \in X$, $y \in Y$.

Proposition 6. *Let (b_j) be a boundedly complete* (s)*-basis for a Banach space B, (e_j) its difference sequence, and $T : B \to [e_j^*]^*$ the canonical map. Then TB is a codimension-one subspace of $[e_j^*]^*$.*

Proof. $[e_j^*]^*$ may be canonically identified with $B((e_j))$, the set of all sequences (c_j) so that $\sup_n \|\sum_{j=1}^{n} c_j e_j\| < \infty$. In fact, if $F \in [e_j^*]^*$, then $F = \sum_{j=1}^{\infty} F(e_j^*) T e_j$, the convergence being weak*, and $F \to (F(e_j^*))_{j=1}^{\infty}$ is the desired isomorphism. Since (e_j) is an (s)-sequence, $(\sum_{j=1}^{n} e_j)$ is a weak-Cauchy sequence, and it follows that $G \overset{\text{df}}{=} \sum_{j=1}^{\infty} T e_j$ is an element of $[e_j^*]^*$ which does not belong to TB, hence TB is of codimension at least one. Now conversely, suppose

$F \in [e_j^*]^*$, so $F = \sum_{j=1}^\infty F(e_j^*) T e_j$, the convergence being weak*. Of course since T is an (into) isomorphism,

$$\sup_n \left\| \sum_{j=1}^n F(e_j^*) e_j \right\| < \infty. \tag{17}$$

Since (b_j) is boundedly complete, (e_j) is a (c.c.)-sequence (by Proposition 2(c)), hence

$$\lim_{n\to\infty} F(e_j^*) \stackrel{\text{df}}{=} c \quad \text{exists.} \tag{18}$$

But then we have that

$$\sup_n \left\| \sum_{j=1}^n \bigl(F(e_j^*) - c\bigr) e_j \right\| < \infty. \tag{19}$$

Thus by Proposition 2(c), $\sum_{j=1}^\infty (F(e_j^*) - c) e_j$ converges to an element b of B. But then $F = Tb + cG$. This proves $[e_j^*]^* = TB \oplus [G]$. □

The above mentioned result of S. Bellenot and C. Finet now follows directly.

Corollary 7. *(See [1] and [5]). Let X have the* PCP *and suppose X^* is separable. Then every non-trivial weak-Cauchy sequence in X has a boundedly complete subsequence spanning an order-one quasi-reflexive space.*

Proof. Let (x_j) be a non-trivial weak-Cauchy sequence in X. By Theorem 1, (x_j) has a boundedly complete (s)-subsequence (x'_j). By Proposition 4, (x'_j) has a further subsequence (b_j) whose difference sequence (e_j) is shrinking; thus $[e_j^*] = B^*$, where $B = [b_j]$. Then the map T of Proposition 6 is simply the canonical embedding of B in B^{**}, whence since B^{**}/B is one-dimensional by Proposition 6, B is order-one quasi-reflexive. □

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