SEVERAL NEW CHARACTERIZATIONS OF BANACH SPACES CONTAINING ℓ^1

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Abstract. Several new characterizations of Banach spaces containing a subspace isomorphic to ℓ^1, are obtained. These are applied to the question of when ℓ^1 embeds in the injective tensor product of two Banach spaces.

Notations and terminology. All Banach spaces are taken as infinite dimensional, "subspace" means "closed linear subspace," "operator" means "bounded linear operator." If W is a subset of a Banach space, $[W]$ denotes its closed linear span. $\mathfrak{c}$ denotes the cardinal of the continuum, i.e., $\mathfrak{c} = 2^{\aleph_0}$; this is also identified with the least ordinal of cardinality $\mathfrak{c}$. For $1 \le p < \infty$, $\ell^p_{\mathfrak{c}}$ denotes the family f of all scalar valued functions defined on $\mathfrak{c}$ with $\|f\|_p = (\sum_{\alpha<\mathfrak{c}} |f(\alpha)|^p)^{1/p} < \infty$. Finally, we recall that a scalar-valued function defined on a compact metric space K is called universally measurable if it is measurable with respect to the completion of every Borel measure on K.

Throughout this paper, the symbols X, Y, Z, B, E shall denote Banach spaces. $\operatorname{Ba} X$ denotes the closed unit ball of X. Recall that an operator $T : X \to Y$ is called Dunford-Pettis if T maps weakly compact sets in X to norm compact sets in Y. Also, $\mathcal{L}(X,Y)$ (resp. $K(X,Y)$) denotes the space of operators (resp. of compact operators) from X to Y, $\mathcal{L}(X) = \mathcal{L}(X,X)$, $K(X) = K(X,X)$. A bounded subset W of X^* is said to *isomorphically norm* X if there exists a $C < \infty$ such that

$$\|x\| \le C \sup_{w\in W} |w(x)| \quad \text{for all} \quad x \in X \ .$$

In case $C = 1$ and $W \subset \operatorname{Ba} X^*$, we say that W *isometrically norms* X. $X \overset{\vee}{\otimes} Y$, $X \overset{\wedge}{\otimes} Y$ denote the injective, respectively projective, tensor products of X and Y. See [DU], [Gr2] for terminology and theorems in this area.

Main results. Our first main result gives several equivalences for a Banach space to contain an isomorph of ℓ^1. We have included many previously known ones, to round out the list; also, we use some of them later on. As far as I know, the equivalences of 1. with the following are new: 2, 3, 4, 6, 7, 8, 11, 12, 13, 14, 19. (Of course some of the implications were previously

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known or are obvious. Also, the same construction proves that 2, 3, 4 and 6 imply 1, but I thought it useful nevertheless to list these explicitly.) For other equivalences, cf. [H1], [Hay], [G1], Theorem II.3 of [G2], [Ro4], and several of the remarks following the proof of Theorem 1.

Theorem 1. *Let X be given. Then the following are equivalent.*

1. *ℓ^1 is not isomorphic to a subspace of X.*
2. *Every integral operator from Y to X^* is compact, for any Y.*
3. *Every integral operator from ℓ^1 to X^* is compact.*
4. *Every integral operator on X^* is compact.*
5. *Every integral operator from X to Y is compact, for any Y.*
6. *Every integral operator from X to X^* is compact.*
7. *Every operator from L^1 to X^* is Dunford-Pettis.*
8. *Every Dunford Pettis operator from X to Y is compact, for arbitrary Y.*
9. *Every w^*-compact subset of X^* is the norm-closed convex hull of its extreme points.*
10. *If K is a weak*-compact subset of X^* which isomorphically norms X, then $[K] = X^*$.*

The remaining equivalences assume that X is separable.

11. *Every unconditional family in X^* is countable.*
12. *Every unconditional family in X^* has cardinality less than $\mathfrak{c}$.*
13. *$\mathcal{L}(X^*, \ell^\infty)$ has cardinality $\mathfrak{c}$.*
14. *$\mathcal{L}(X^*)$ has cardinality $\mathfrak{c}$.*
15. *X^{**} has cardinality $\mathfrak{c}$.*
16. *X^{**} has cardinality less than $2^{\mathfrak{c}}$.*
17. *If K is a weak*-compact subset of X^* and (x_n) is a bounded sequence in X, then setting $\hat{x}_n(k) = k(x_n)$ for all k and n, any point-wise cluster point of $(\hat{x}_n)$ belongs to the first Baire class on K.*
18. *There exists an isomorphically norming w^*-compact subset K of X^* so that if (x_n) is as in 17, then $(\hat{x}_n)$ has a point-wise cluster point which is universally measurable on K.*
19. *There exists a K as in 18 so that if (x_n) is as in 17, the cardinality of the set of point-wise cluster points of $(\hat{x}_n)$ on K, is less than $2^{\mathfrak{c}}$.*

The implications $7 \Rightarrow 1$, $12 \Rightarrow 1$ follow quickly from a classical theorem of Pełczyński [P], and the second of these does not require the separability of X. We prove a generalization of Pełczyński's result in the Appendix.

After proving Theorem 1 and discussing some complements, we apply it in some detail to the question of when ℓ^1 embeds in the injective tensor product of two Banach spaces.

Proof. 1 $\Rightarrow$ 2. Let $T : Y \to X^*$ be an integral operator. Thus there exists a probability space $(\Omega, \mathcal{S}, \mu)$ and operators $U : Y \to L^\infty(\mu)$ and $V : L^1(\mu) \to X^*$ such that the following commutative diagram holds

$$\begin{tikzcd} L^\infty(\mu) \arrow[r, "i"] & L^1(\mu) \arrow[d, "V"] \\ Y \arrow[u, "U"] \arrow[r, "T"] & X^* \end{tikzcd} \tag{1}$$

Here, $i : L^\infty(\mu) \to L^1(\mu)$ denotes the canonical injection. Suppose T is not compact. Then there exists a bounded sequence (y_n) such that (Ty_n) has no convergent subsequence. Then (as is standard), after passing to a subsequence if necessary, we may assume that there is a $\delta > 0$ so that

$$\|Ty_n - Ty_m\| \ge \delta \quad \text{for all} \quad n \ne m \ . \tag{2}$$

Since i is weakly compact, so is T; so again after passing to a subsequence if necessary, we may assume that (Ty_n) converges weakly. But now if we consider the sequence (z_n) defined by

$$z_n = y_{2n} - y_{2n-1} \quad \text{for all} \quad n \ , \tag{3}$$

then (z_n) is also bounded of course, and

$$Tz_n \to 0 \quad \text{weakly as} \quad n \to \infty \ , \quad \text{and} \quad \|Tz_n\| \ge \delta \quad \text{for all} \quad n \ . \tag{4}$$

Thus it follows by the Hahn-Banach theorem that we may choose a sequence (x_n) in the unit ball of X such that

$$|(Tz_n)(x_n)| \ge \frac{\delta}{2} \quad \text{for all} \quad n \ . \tag{5}$$

Now by the ℓ^1-Theorem [Ro1], (x_n) has a weak-Cauchy subsequence, so let us just assume that (x_n) is itself weak-Cauchy (note that obviously (5) holds for subsequences (z'_n, x'_n) of the pair (z_n, x_n)).

But now it follows that we may choose $n_1 < n_2 < \cdots$ such that

$$|Tz_{n_{k+1}}(x_{n_k})| < \frac{\delta}{4} \quad \text{for all} \quad k \ . \tag{6}$$

Indeed, let $n_1 = 1$; since $Tz_n \to 0$ w^*, we may choose $n_2 > n_1$ such that $|Tz_{n_2}(x_{n_1})| < \frac{\delta}{4}$. Having chosen n_k, choose $n_{k+1} > n_k$ such that (6) holds. But now it follows that

$$|Tz_{n_{k+1}}(x_{n_{k+1}} - x_{n_k})| > \frac{\delta}{4} \quad \text{for all} \quad k \ , \tag{7}$$

and

$$x_{n_{k+1}} - x_{n_k} \to 0 \text{ weakly.} \tag{8}$$

Thus finally, after just re-lettering everything we have that

$$\begin{aligned} &(z_n) \text{ is a bounded sequence in } Y,\ (x_n) \text{ and } (Tz_n) \text{ both} \\ &\text{converge to zero weakly, and (5) holds for some } \delta > 0. \end{aligned} \tag{9}$$

Now regarding $X \subset X^{**}$, we may thus write that

$$|\langle Tz_n, x_n\rangle| = |\langle T^* x_n, z_n\rangle| \geq \frac{\delta}{2} \text{ for all } n\ . \tag{10}$$

But T^* is also integral, and in fact admits the factorization

$$\begin{array}{ccc} L^\infty(\mu) & \xrightarrow{\ i\ } & L^1(\mu) \\ \big\uparrow{\scriptstyle V^*} & & \big\downarrow{\scriptstyle \tilde U} \\ X^{**} & \xrightarrow{\ T^*\ } & Y^* \end{array} \tag{11}$$

where $\tilde U = U^* \mid L^1(\mu)$, $L^1(\mu)$ regarded as contained in $L^1(\mu)^{**} = (L^\infty(\mu))^*$.

But finally, since $L^\infty(\mu)$ has the Dunford-Pettis property and $\tilde U i$ is weakly compact and $V^* x_n \to 0$ weakly,

$$\|T^* x_n\| = \|\tilde u i V^*(x_n)\| \to 0 \text{ as } n \to \infty\ . \tag{12}$$

Of course since (z_n) is bounded, this contradicts (10).

$2 \Rightarrow 3$, $2 \Rightarrow 4$, and $2 \Rightarrow 6$ are trivial.

Next, we show that not $1 \Rightarrow$ not 3, not 4, and not 6, establishing the equivalence of 1, 2, 3, 4, and 6. We shall need the following basic fact concerning integral operators: *If $Y \subset Z$, B is complemented in B^{**}, and T is an integral operator from Y to B, then T extends to an integral operator from Z to B.* To see this, choose a probability measure space $(\Omega, \mathcal{S}, \mu)$ and operators $U : Y \to L^\infty(\mu)$ and $V : L^1(\mu) \to B$ so that the following diagram commutes:

$$\begin{array}{ccc} L^\infty(\mu) & \xrightarrow[\ i\ ]{} & L^1(\mu) \\ \big\uparrow{\scriptstyle U} & & \big\uparrow{\scriptstyle V} \\ Y & \xrightarrow[\ T\ ]{} & B \end{array}$$

(where i is the canonical "identity" map). Then since $L^\infty(\mu)$ has the Hahn Banach extension property, we may choose an operator $\tilde U : Z \to L^\infty(\mu)$ extending U, (with $\|\tilde U\| = \|U\|$). Thus $Vi\tilde U$ is an integral operator extending T, proving our assertion.

Now suppose ℓ^1 embeds in X. Then by a theorem of Pełczyński [P]

$$(C([0,1]))^* \text{ embeds in } X^*\ . \tag{13}$$

(See the Appendix for a more general result.)

Thus in particular,

$$L^1 \text{ embeds in } X^* . \tag{14}$$

Let $Q : \ell^1 \to C[0,1]$ be a quotient map, and let $i : C[0,1]) \to L^1$ be the canonical "identity" map and also let $U : L^1 \to X^*$ be an isomorphic embedding.

Then $T = UiQ$ is a non-compact integral operator from ℓ^1 to X^*, proving not 3. But if we let Y be a subspace of X and $V : Y \to \ell^1$ a surjective isomorphism, then the map $S = TV$ is a non-compact integral operator from Y to X^*, and so has an integral operator extension from X to X^*, proving not 6. Finally, let Z be a subspace of L^1 isometric to ℓ^1, and $A : Z \to \ell^1$ a surjective isometry. Then TA is a non-compact integral operator from Z to X^*, and so has an integral operator extension from X^* to X^* by the basic fact above, proving not 4.

$2 \Rightarrow 5$. Let $T : X \to Y$ be an integral operator. Then $T^* : Y^* \to X^*$ is also integral. Hence T^* is compact, so T is compact. (The result that $1 \Rightarrow 5$ follows from a result due to Pisier; see remark 4 below.)

$5 \Rightarrow 6$ is trivial.

$1 \Rightarrow 7$. Suppose to the contrary that $T : L^1 \to X^*$ is a non Dunford-Pettis operator. It follows that we may choose a sequence (f_n) in L^1 so that $f_n \to 0$ weakly but $\|Tf_n\| \not\to 0$. Therefore we may choose a subsequence (f'_n) of (f_n) so that for some $\delta > 0$,

$$\|Tf'_n\| > \delta \text{ for all } n . \tag{15}$$

For each n, choose $x_n \in X$ with $\|x_n\| = 1$ and

$$|\langle Tf'_n, x_n\rangle| > \delta . \tag{16}$$

By the ℓ^1-Theorem, by passing to a further subsequence if necessary, we may assume that (x_n) is a weak-Cauchy sequence. But then it follows that

$$(T^* x_n) \text{ is a weak Cauchy sequence in } L^\infty \tag{17}$$

(where we regard X as canonically embedded in X^*).

Since L^1 has the Dunford-Pettis property, it follows that

$$|\langle f'_n, T^* x_n\rangle| \to 0 \text{ as } n \to \infty , \tag{18}$$

which contradicts (16). (A Banach space B has the Dunford-Pettis property if every weakly compact operator $T : B \to Y$ is Dunford-Pettis, for all Y. By fundamental results of Grothendieck [G1], this is equivalent to: (b_n) weakly null in B, (f_n) weakly null in B^* implies $f_n(b_n) \to 0$ as $n \to \infty$. It is then a standard exercise to show that in fact if (b_n) is weakly null and (f_n) is weak-Cauchy, still $f_n(b_n) \to 0$ as $n \to \infty$.)

Not 1$\Rightarrow$ not 7. Since L^1 is isomorphic to a subspace of X^* when 1 fails, this is immediate: An (into) isomorphism $T : L^1 \to X^*$ is obviously not Dunford-Pettis.

$1 \Rightarrow 8$. Let (x_n) be a bounded sequence in X. Choose (by the ℓ^1-Theorem) (x'_n) a weak Cauchy subsequence of (x_n); then given (n_i) and (m_i) strictly increasing sequences of positive integers, $(x'_{n_i} - x'_{m_i})$ is weakly null, and hence by hypothesis

$$\|T(x'_{n_i} - x'_{m_i})\| = \|Tx'_{n_i} - T'x_{m_i}\| \to 0 \quad \text{as} \quad i \to \infty \ .$$

This implies (Tx'_n) is a Cauchy sequence in Y, so it converges since Y is complete; thus T is compact.

$8 \Rightarrow 5$. Integral operators are Dunford-Pettis operators because $L^\infty(\mu)$-spaces have the Dunford-Pettis property and integral operators factor through the "identity" map $i : L^\infty(\mu) \to L^1(\mu)$, for some probability measure μ; of course i is weakly compact.

$1 \Rightarrow 9$. Suppose to the contrary that $(u_\alpha)_{\alpha<w_1}$ is an uncountable unconditional family in X^*; let U be the norm closure of its linear span. Assume $\|u_\alpha\| = 1$ for all α; letting (u^*_α) be the functions in U^* biorthogonal to (u_α), choose K so that $\|u^K_\alpha\| \le K$ for all α. Now we have the following fundamental claim:

$$\text{Given } f \in U^*, \text{ then } W_f = \{\alpha : f(u_\alpha) \ne 0\} \text{ is countable.} \tag{19}$$

If this were false, say then $f = U^*$, $\|f\| = 1$, and W_f is countable. Then we may pass to an uncountable subset Γ of W_f such that there exists a $\delta > 0$ so that

$$|f(u_\alpha)| \ge \delta \quad \text{for all} \quad \alpha \in \Gamma \ . \tag{20}$$

But now a standard argument shows that

$$(u_\alpha)_{\alpha\in\Gamma} \quad \text{is equivalent to the natural basis of} \quad \ell^1(\Gamma) \ . \tag{21}$$

Thus $\ell^1(\Gamma)$ embeds in X^*, which implies ℓ^1 embeds in X, by a result of Pełczyński [P], Hagler [H1], a contradiction.

$1 \Rightarrow 9$. This is due to R. Haydon [Hay]. For X separable, this had previously been proved by E. Odell and myself [OR].

$9 \Rightarrow 10$. If K satisfies the hypothesis of 10, so does $\tilde K = \{\alpha k : \alpha$ is a scalar, $|\alpha| = 1$, $k \in K\}$. But then it follows that the w^*-closed convex hull W of $\tilde K$ has non-empty interior, and hence since then also Ext $W \subset \tilde K$ and 9 implies W is the norm closed convex hull of Ext W, $[K] = [\tilde K] = X^*$.

$10 \Rightarrow 1$. This follows from a result of G. Godefroy [G1]. Indeed, assume that ℓ^1 embeds in X. Then it is proved in [Gr1] that there exists an equivalent norm $\|\!|\cdot|\!\|$ on X such that if K denotes the w^* closure of the extreme points of the ball of X^* in the dual norm induced by

$|||\cdot|||$, then $[K] \neq X^*$. But of course then K is an isomorphically norming w^*-compact subset of X^* in its original norm, proving that 10 does not hold. There is a minor point here that requires explanation, however. The cited result of Godefroy's requires the fundamental case where $X = \ell^1$ itself, for the proof. But the argument given in [G1] is only valid for the case of complex scalars. The result for real scalars may be deduced from the work in [G1] as follows: First, to avoid ambiguity, let $\ell^p_{\mathbb{R}}$, resp. $\ell^p_{\mathbb{C}}$, denote ℓ^p for real scalars, resp. for complex scalars, $p = 1$ or ∞. It is proved in [G1] that for complex scalars, there exists an equivalent norm $|||\cdot|||$ on $\ell^1_{\mathbb{C}}$ so that if K is as above, and Y is the closed linear span of K over the complex scalars, then there exists an infinite subset M of $\mathbb{N}$ with infinite complement so that

$$\inf\{|y(n) - y(n')| : n \in M,\ n' \in N \sim M\} = 0 \quad \text{for all} \quad y \in Y \ . \tag{22}$$

Now let $|||\cdot|||^*$ be the dual norm induced on $\ell^\infty_{\mathbb{C}}$, and just regard $(\ell^\infty_{\mathbb{C}}, |||\cdot|||^*)$ as a real Banach space. Now if we take the standard norm on $\ell^\infty_{\mathbb{C}}$ and regard this as a real Banach space, we obtain $\ell^\infty_{\mathbb{R}} \oplus \ell^\infty_{\mathbb{R}}$ under the norm

$$\|(a_j) \oplus (b_j)\| = \sup_j \sqrt{a_j^2 + b_j^2} \quad \text{for} \quad (a_j), (b_j) \in \ell^\infty_{\mathbb{R}} \tag{23}$$

which is obviously equivalent to the standard norm on $\ell^\infty_{\mathbb{R}} \oplus \ell^\infty_{\mathbb{R}}$, which of course is isometric to $\ell^\infty_{\mathbb{R}}$. Then it follows also that the norm $|||\cdot|||^*$ must be equivalent to the standard norm $\|\cdot\|_\infty$ on $\ell^\infty_{\mathbb{R}} \oplus \ell^\infty_{\mathbb{R}}$ (where obviously we take the isomorphism $(a_j + ib_j) \to ((a_j), (b_j))$ for $(a_j + ib_j) \in \ell^\infty_{\mathbb{C}}$).

Now let c_{00} denote the space of all sequences of reals which are ultimately zero. Define a norm $|||\cdot|||$ on $c_{00} \oplus c_{00}$ by

$$(a_j) \oplus (b_j) = \sup\Big\{ \big|\sum(\alpha_j a_j + \beta_j b_j)\big| : |||(\alpha_j) \oplus (b_j)|||^* = 1 \Big\} \ . \tag{24}$$

It follows easily that $|||\cdot|||$ is equivalent to the ℓ^1-norm on $\ell^1_{\mathbb{R}} \oplus \ell^1_{\mathbb{R}}$. Moreover, we have that a bounded sequence (f_n) in $\ell^\infty_{\mathbb{C}}$ converges in the w^* topology on $\ell^\infty_{\mathbb{C}}$ induced by $\ell^1_{\mathbb{C}}$ iff

$$\lim_{bn\to\infty} f_n(j) \quad \text{exists for all} \quad j \tag{25}$$

iff

$$\lim_{n\to\infty} \operatorname{Re} f_n(j) \quad \text{and} \quad \lim_{n\to\infty} \operatorname{Im} f_n(j) \quad \text{exist for all} \quad j \tag{26}$$

iff

$$\{(\operatorname{Re} f_n \oplus \operatorname{Im} f_n)_{n=1}^\infty \quad \text{converges in the } w^*\text{-topology on} \quad \ell^\infty_{\mathbb{R}} \oplus \ell^\infty_{\mathbb{R}}\} \ . \tag{27}$$

It follows that the w^*-topology on $\ell^\infty_{\mathbb{C}}$ is the same as that on $\ell^\infty_{\mathbb{R}} \oplus \ell^\infty_{\mathbb{R}}$, and hence the unit ball of $(\ell^\infty_{\mathbb{R}} \oplus \ell^\infty_{\mathbb{R}}, |||\cdot|||^*)$ is compact in the w^*-topology; thus by the bipolar theorem, that ball is precisely the dual ball of $(\ell^1_{\mathbb{R}} \oplus \ell^1_{\mathbb{R}}, |||\cdot|||)$; hence the set K defined above is exactly the same as that defined for the real scalars case above. Finally, it follows that if $Y_{\mathbb{R}}$ denotes the norm closed linear span of K over real scalars, then we have that if $(a_j)_{j=1}^\infty \oplus (b_j)_{j=1}^\infty \in Y_{\mathbb{R}}$, $(a_j+ib_j)_{j=1}^\infty \in Y$, and hence by (22), we obtain that

$$\inf\{\max\{|a_n - a'_n|, |b_n - b'_n|\} : n \in M,\ n' \in \mathbb{N} \sim M\} = 0 \tag{28}$$

which obviously implies that $Y_{\mathbb{R}} \neq \ell^\infty_{\mathbb{R}} \oplus \ell^\infty_{\mathbb{R}}$. Thus it follows $X = (\ell^1_{\mathbb{R}} \oplus \ell^1_{\mathbb{R}}, \|\!|\cdot|\!\|)$ is isomorphic to $\ell^1_{\mathbb{R}}$ but for K as defined above, the w^*-closed convex ball of K equals $\mathrm{Ba}\, X^*$ but $[K] \neq X^*$, completing the proof for real scalars.

$11 \Rightarrow 12$. This is trivial.

It follows from the main result in [OR] that every element of X^{**} is the limit of a weak Cauchy sequence in X, which yields 15. ($1 \Rightarrow 15$ is also given in [OR].)

$1 \Rightarrow 13$. It is easily seen that

$$\operatorname{card} \mathcal{L}(X^*, \ell^\infty) \geq \mathfrak{c}\ . \tag{29}$$

Indeed, just fix $z \in \ell^\infty$, $z \neq 0$, $x \in X$, $x \neq 0$, and note that the operator T on X^* defined by $T(x^*) = x^*(x)z$ is thus non-zero; hence

$$r \to rT \ \text{ is a 1–1 map of } \ \mathbb{R} \ \text{ into } \ \mathcal{L}(X^*, \ell^\infty)\ . \tag{30}$$

Thus it remains to prove

$$\operatorname{card} \mathcal{L}(X^*, x) \leq \mathfrak{c}\ . \tag{31}$$

Now since $\mathrm{Ba}\, \ell^1$ is w^*-dense in $\mathrm{Ba}(\ell^\infty)^*$ by Goldstein's Theorem. then

(32) There exists a countable subset D of $\mathrm{Ba}\, \ell^\infty$ which is weak*-dense in it.

But then it follows that if $T \in \mathcal{L}^1(X^*, \ell^\infty)$, T^* is w^*-continuous and is thus determined by its values on D. Thus we deduce that

$$\operatorname{card} \mathcal{L}(X^*, \ell^\infty) \leq \operatorname{card}(X^{**})^D = \mathfrak{c}^{\aleph_0} = \mathfrak{c}\ . \tag{33}$$

$13 \Rightarrow 14$ is obvious since the dual of any separable Banach space is isometric to a subspace of ℓ^∞.

$14 \Rightarrow 15$ is trivial since X^{**} is isometric to a subspace of $\mathcal{L}(X^*)$.

$15 \Rightarrow 16$ is trivial.

$1 \Rightarrow 17$. It follows by the ℓ^1-theorem that given $(\hat{x}_{n_i})$ a subsequence of $(\hat{x}_n)$, then (x_{n_i}) has a weak-Cauchy subsequence, which implies that $(\hat{x}_{n_i})$ has a subsequence pointwise convergent on K. Now the results of [Ro2] prove the conclusion of 17 (and also imply, by the way, that any pointwise cluster point of $(\hat{x}_n)$ is the limit of a pointwise convergent subsequence of (x_n)).

$17 \Rightarrow 18$. Let K be the unit ball of X^*. Then of course $(\hat{x}_n)$ has a pointwise cluster point on K, which is just an element of X^{**} restricted to K. This is a Baire-one functions on K by [OR], and so of course is Borel measurable and hence universally measurable.

$17 \Rightarrow 19$. The cardinality of the class of Baire-one functions on K equals $\mathfrak{c} < 2^{\mathfrak{c}}$.

$18 \Rightarrow 1$. It suffices to prove that any bounded sequence (x_n) in X has a subsequence (x_{n_i}) so that $(\hat{x}_{n_i})$ converges pointwise on K. For then, it follows by the Hahn-Banach, Riesz-representation, and bounded convergence theorems that (x_{n_i}) is a weak-Cauchy sequence, and so 1 holds. But if this is not the case, then we find a bounded sequence (x_n) in X such that $(\hat{x}_n)$ has no pointwise convergent subsequence on K. It now follows by a theorem of Bourgain-Fremlin-Talagrand [BFT] that $(\hat{x}_n)$ has a subsequence, none of whose pointwise cluster points are universally measurable on K, thus contradicting 18. (For an alternate proof of the cited result, see Theorem 3.18 in [Ro3].)

It remains to prove that 12, 16, and 19 imply 1. We shall prove the contrapositive implications instead. So we assume for the rest of the proof that ℓ^1 embeds in X and X is separable. (We don't need the separability assumption for the first two implications.)

$\ell^1_{\mathfrak{c}}$ is isometric to the space of atomic Borel measures on $[0,1]$, and so isometric to a subspace of $C[0,1)^*$. Thus by (8),

$$\ell^1_{\mathfrak{c}} \text{ is isomorphic to a subspace of } X^*, \tag{34}$$

so 12 does not hold.

It also follows that 16 does not hold, for by (34), $\ell^\infty_{\mathfrak{c}}$ is isomorphic to a quotient space of X^{**}, and hence

$$\operatorname{card} X^{**} \geq \mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}} \ . \tag{35}$$

(These implications are of course known.)

Not $1 \Rightarrow$ Not 19. Let K be a w^*-compact isomorphically norming subset of X^*. Then it follows again by the theorems cited in the proof of $18 \Rightarrow 1$, that if (x_n) is a bounded sequence in X such that $(\hat{x}_n)$ converges pointwise on K, then (x_n) is a weak Cauchy sequence in X. Thus there must exist some bounded sequence (x_n) in X such that $(\hat{x}_n)$ has no pointwise convergent subsequence on K. Hence this implication follows (since obviously the family of pointwise cluster points on K has cardinality at most $2^{\mathfrak{c}}$) from

Lemma 2. *Let K be a compact Hausdorff space and (f_n) be a bounded sequence of continuous scalar-valued functions on K which has no pointwise convergent subsequence. Then the family F of pointwise cluster points of (f_n) on K has cardinality at least $2^{\mathfrak{c}}$.*

Proof. We may obviously assume the f_n's are real valued since either the real or imaginary parts of the f_n's have no pointwise convergent subsequence, and the real or imaginary part of a pointwise cluster point of (f_n) is also a pointwise cluster point of the real or imaginary parts of the f_n's. By the proof of the ℓ^1-Theorem ([Ro1]; see also [Ro3]), there exists a subsequence (f'_n) of (f_n) with the following property:

> $(*)$ There exist real numbers r and δ with $\delta > 0$ such that setting $A_n = \{k : f'_n(k) \le r - \delta\}$ and $B_n = \{k : f'_n(k) \ge r + \delta\}$, then these are non-empty sets for all n, such that $((A_n, B_n))$ is a Boolean independent sequence; that is, if one sets $+A_n = A_n$ and $-A_n = B_n$, then for any infinite sequence (ε_j) with $\varepsilon_j = \pm 1$ for all j,

$$\bigcap_{j=1}^{n} \varepsilon_j A_j \quad \text{is non-empty for all} \quad n\ . \tag{36}$$

This implies

$$\bigcap_{j=1}^{\infty} \varepsilon_j A_j \quad \text{is non-empty,} \tag{37}$$

since for all n, f'_n is continuous, and thus A_n, B_n are closed and non-empty, and so (37) follows by the compactness of K. Now let $\mathcal{U}$ be the family of all non-principal ultrafilters on $\mathbb{N}$ (cf. [CN] for the definition and standard properties of ultrafilters). Then as is classical, U may be identified the $\beta\mathbb{N} \sim \mathbb{N}$, where $\beta\mathbb{N}$ denotes the Stone-Cěach compactification of $\mathbb{N}$, and so by a classical theorem in topology (cf. Theorem 2, page 132 of [E]),

$$\operatorname{card} \mathcal{U} = 2^{\mathfrak{c}}\ . \tag{38}$$

(This is also explicitly given in 7.4 Corollary, page 146 of [CN].)

For each $U \in \mathcal{U}$, define a function f_U on K by

$$f_U(k) = \lim_{n \in U} f'_n(k) \quad \text{for all} \quad k \in K\ . \tag{39}$$

Then as is standard, f_U is a pointwise cluster point of (f'_n) and hence of (f_n). Thus to complete the proof of the Lemma, it suffices to show that

$$f_U \ne f_V \quad \text{if} \quad u \ne v\ , \quad U, V \in \mathcal{U}\ . \tag{40}$$

Given $U \ne V$ in u, choose M an infinite subset of $\mathbb{N}$ such that $M \in U$, $M \notin V$. Then as cofinite sets belong to any non-principal ultrafilter,

$$\mathbb{N} \sim M \stackrel{\text{def}}{=} L \quad \text{is infinite and thus belongs to} \quad V\ . \tag{41}$$

(The latter statement holds since M is infinite and $M \notin V$.) Now define (ε_j) by

$$\varepsilon_j = 1 \ \text{ if } \ j \in M \ \text{ and } \ \varepsilon_j = -1 \ \text{ if } j \in L \ . \tag{42}$$

Let k be a point in $\bigcap_{j=1}^{\infty} \varepsilon_j A_j$ (such exists by (37)). Thus by definition of $((A_n, B_n))$

$$f'_n(k) \le r - \delta \ \text{ for all } \ n \in M \ . \tag{43}$$

Now it follows that if $U_M = \{M \cap A : A \in U\}$, then U_M is a non-principal ultrafilter on M, and

$$f_U(k) = \lim_{k \in u_\mu} f'_n(k) \ . \tag{44}$$

Thus it follows by (43) that

$$f_U(k) \le r - \delta \ . \tag{45}$$

By exactly the same reasoning, we obtain that

$$f_V(k) \ge r + \delta \ . \tag{46}$$

Thus (45) and (46) show $f_U \ne f_V$, completing the proof of the Lemma, and thus the proof of Theorem 1. $\square$

Remarks.

1a. The equivalence we have used in the proof of $10 \Rightarrow 1$ is actually quantitative. That is, we have the following fact.

Proposition A. *Given X and $\lambda \ge 1$, the following are equivalent:*

(i) *There exists a weak*-compact λ-norming subset K of $\mathrm{Ba}\, X^*$ so that $[K] \ne X^*$.*

(ii) *There exists a norm $|||\cdot|||$ on X so that*

$$\|\cdot\| \le |||\cdot||| \le \lambda\|\cdot\| \qquad (\|\cdot\| \ \textit{the original norm})$$

with $[K] \ne X^*$, where K is the w^*-closure of the extreme points of $\mathrm{Ba}(X^*, |||\cdot|||^*)$.

Proof. (i) $\Rightarrow$ (ii): Let $W = \{\alpha k : |\alpha| = 1,\ k \in K\}$. Then W is also w^*-compact. Then it follows by the geometrical form of the Hahn-Banach Theorem that if $\tilde{K}$ denotes the w^*-closed convex hull of W, then

$$\tilde{K} \subset \mathrm{Ba}\, X \subset \lambda\, \mathrm{Ba}\, \tilde{K} \ ;$$

in turn, we then easily obtain a norm $|||\cdot|||$ on X satisfying the inequality in (ii) such that $\tilde{K} = \mathrm{Ba}(X^*, |||\cdot|||^*)$. But then it follows that $\mathrm{Ext}\, \tilde{K} \subset W$ which implies (ii). (ii) $\Rightarrow$ (i) is immediate for K is then λ-norming. $\square$

Now the arguments in [G1] yield

Proposition B. *There exists an absolute constant C so that if ℓ^1 embeds in X, there exists a C-norming w^*-compact subset K of* $\mathrm{Ba}\, X^*$ *with* $[K] \neq X^*$.

The question then arises:

Problem 1. *If ℓ^1 embeds in X, is it so that given $\varepsilon > 0$, there exists a $\lambda \leq 1 + \varepsilon$ satisfying (i) of Proposition A?*

The delicate nature of the proof of $10 \Rightarrow 1$ leads me to conjecture that the answer is negative for $X = \ell^1$ itself. Of course if so, it is natural to ask: What is the optimal value of C in Proposition B? Now of course if $X = C(K)$ for some uncountable compact metric space, $\lambda = 1$ works. However $\lambda = 1$ does not work for $X = \ell^1$. For then by the arguments sketched above, we would have that the norm-closed linear span of the w^*-closure of $\mathrm{Ba}\, \ell^\infty$ would be unequal to ℓ^∞. But standard arguments show that the norm-closed convex hull of $\mathrm{Ext}\, \mathrm{Ba}\, \ell^\infty = \mathrm{Ba}\, \ell^\infty$.

1b. Some interesting equivalences are also obtained in [G1], complementary to the above discussion. The following notion is introduced there:

Definition. Let K be a non-empty closed bounded convex subset of X^*, and $W \subset K$. W is called a *boundary* of K if for all $x \in X$, $\sup[\mathrm{Re}(\chi x) \mid W]$ is attained on W.

The following result is obtained in [G1], generalizing the theorem of [OR] that $9 \Rightarrow 1$ for separable X, and yielding an analogy of James' famous characterization of weakly compact convex sets.

Theorem. *Assume X is separable. Then the following are equivalent.*

(i) *ℓ^1 does not embed in X.*

(ii) *for all w^*-compact subsets K of X^*, if W is a boundary of K, then K is the norm-closed convex hull of W.*

(iii) *if K is a closed bounded convex subset of X^* which is a boundary for itself, then K is w^*-compact.*

2. I am indebted to Welfeng Chen for a stimulating conversation concerning the implication $1 \Rightarrow 12$.

3. Suppose X is separable and X^* is non-separable. A remarkable result of Stegall yields that then X^* has a biorthogonal family of cardinality the continuum [S]. Since there are now known many separable spaces X not containing ℓ^1 with X^* nonseparable (cf. [AMP], [H2], [J3], [K], [LS], [Ro5]), we cannot significantly weaken the unconditionality assertion in 11 and

12. This same result incidentally shows that $\operatorname{dens} X^{**} = \operatorname{dens} X^* = \mathfrak{c}$; thus assuming ℓ^1 does not embed in X, we obtain that all cardinal measures of the size of X^*, X^{**}, $\mathcal{L}(X^*)$ yield the same result. (For a metric space M, $\operatorname{dens} M$ denotes the least cardinality of a dense subset.)

4. The following essentially known result gives equivalences for the embedability of ℓ^1 in X and the structure of p-absolutely summing operators on X, analogous to the equivalences 1. through 4. of Theorem 1.

Theorem. *Let X be given. The following are equivalent.*

1. *ℓ^1 does not embed in X.*
2. *For all Y and $1 \le p < \infty$, every p-absolutely summing operator from X to Y is compact.*
3. *Every 2-summing operator from X to ℓ^2 is compact.*
4. *Every 2-summing operator from X to X^* is compact.*

Proof. $1 \Rightarrow 2$. This is due to G. Pisier (Corollary 1.7, part (iii) of [Pi]). For the sake of completeness, we give the argument. Let $T : X \to Y$ be a p-absolutely summing operator. By a fundamental theorem of Pietsch (cf., [LP] or Theorem 1.3 of [Pi]), there exists a regular Borel probability measure μ on $K = \operatorname{Ba} X^*$ endowed with the w^*-topology and a $C < \infty$ so that

(a) $\|Tx\| \le C(\int_K |w(x)|^p \ell\mu(w))^{1/p}$ for all $x \in X$.

Now let (x_n) be a bounded sequence in X, and choose a weak-Cauchy subsequence (x'_n) by the ℓ^1-Theorem. We claim that

(b) (Tx'_n) converges in the norm topology of Y.

To prove this, since Y is a Banach space, we employ the following elegant characterization of Cauchy sequences, due to Pełczyński: It suffices to show that

(c) $\|Tx'_{n_i} - Tx'_{m_i}\| \to 0$ as $i \to \infty$ for all strictly increasing sequences $(n_i), (m_i)$ of $\mathbb{N}$.

But given such sequences, $(x_{n_i} - x_{m_i})$ is a weakly null sequence in X. Now letting $i : C(K) \to L^p(\mu)$ be the natural injection and $U : X \to C(K)$ the canonical map given by: $(Ux)(k) = k(x)$ for all $x \in X$, $k \in K$, then since i is weakly compact and $C(K)$ has the Dunford-Pettis property,

(d) $\|iU(x'_{n_i} - x'_{m_i})\| \to 0$ as $i \to \infty$

But (a) yields that

(f) $\|T(x'_{n_i} - x'_{m_i})\| \le C\|iU(x'_{n_i} - x'_{m_i}\|_{L^p(\mu)}$ for all i.

$2 \Rightarrow 3$, $2 \Rightarrow 4$ are trivial. Now suppose ℓ^1 embeds in X. Thus by Pełczyński's theorem, (8) holds, and since ℓ^2 embeds in L^1, which thus embeds in X^*, it is obviously enough to show that condition 3 of the Theorem fails to hold. Now simply let $T : \ell^1 \to \ell^2$ be defined by: $Te_j = b_j$ for all j, where e_j is the ℓ^1-basis, (b_j) the ℓ^2-basis. It follows that T is absolutely summing, by Grothendieck's fundamental theorem [G2] (although the direct proof of this elementary fact is much simpler). Let Z be a subspace of X isomorphic to ℓ^1 and let $S : Z \to \ell^1$ be a surjective isomorphism. It follows that TS is absolutely summing. Hence TS is 2-absolutely summing, which implies TS is 2-integral and hence TS entends to a 2-integral operator $V : X \to \ell^2$. V is thus 2-absolutely summing, but since T is not compact, neither is V. $\square$

Comment. It thus follows that $1 \Rightarrow 5$ of Theorem 1 can be deduced from the above Theorem, since integral operators are asbolutely summing.

5. The proof of $6 \Rightarrow 1$ yields an integral non-compact operator from ℓ^1 to L^1 (and also $6 \Rightarrow 1$ follows from this and Pełczyński's theorem cited there). The following is a more natural example of such an operator. Letting (e_n) be the ℓ^1 basis, define T by: $Te_n = \sin 2\pi nx$ for all n. T is obviously not compact. To see that T is an integral operator, define $S : C[0,1] \to c_0$ by: $(Sf)_n = \int_0^1 f(x) \sin 2\pi nx\, dx$ for all $n \in N$. Then S is an integral operator, because if we define $V : L^1 \to c_0$ by : $(Vf)_n = \in_0^1 f(x) \sin 2\pi nx$ for all $n \in N$, then $S = Vi$, where $i : C[0,1] \to L^1$ is the canonical map. Therefore S^* is integral; it is easily verified that $S^* = T$.

If K satisfies the hypothesis of 9, then if we renorm X by $|||x||| = \sup\{|k(x)| : k \in K\}$, K now isometrically norms $(X, |||\cdot|||)$, and so also $\Omega = \{\alpha k : \alpha$ is a scalar, $|\alpha| = 1$, $k \in K\}$ isometrically norms $(X, |||\cdot|||)$ and is w^*-compact. Hence the w^*-closed convex hull of Ω equals D, the unit ball of X^* in $|||\cdot|||^*$, and moreover the extreme points of D are contained in Ω. Thus by 9, the norm-closed convex hull of $\Omega = D$, so of course $[\Omega] = [K] = X^*$.

7. Results of Bourgain in [Bo1], [Bo2] yield the following remarkable result.

Theorem. *Assume X is separable. Then the following are equivalent.*

1. *ℓ^1 does not embed in X.*
2. *Any closed bounded convex subset of X^* with the Radon-Nikodym property (RNP) is separable.*
3. *Any subspace of X^* with the RNP is separable.*
4. *There does not exist a compact subset K of $E\,\mathrm{Ba}\,X^*$, the extreme points of $\mathrm{Ba}\,X^*$, such that K is homeomorphic to the Cantor set and the canonical map $X \to C(K)$ is surjective.*
5. *There does not exist a subset of the extreme points of $\mathrm{Ba}\,X^*$ equivalent to the basis of $\ell^1_{\mathfrak{c}}$.*

(Note that $3 \Rightarrow 1$ follows from Pełczyński's theorem and does not require the separability of X, for $\ell^1_{\mathfrak{c}}$ has the RNP.) He then deduces in [Bo2] that if X is non-separable, $E \operatorname{Ba} X^*$ is non-separable in norm. (EW denotes the extreme points of W.) For if ℓ^1 is not isomorphic to a subspace of X, $X^* = [E \operatorname{Ba} X^*]$ by the result of Haydon [Hay]. If ℓ^1 embeds in X, then there is a separable subspace Y of X so that $\operatorname{card} E \operatorname{Ba} Y^* = \mathfrak{c}$ by $4 \Rightarrow 1$ of the above Theorem, but as is standard, every extreme pont of $\operatorname{Ba} Y^*$ lifts to an extreme point of $\operatorname{Ba} X^*$, so $\operatorname{card} E \operatorname{Ba} X^* \geq \mathfrak{c}$. The Theorem established by Bourgain in [Bo1] which gives $4 \Rightarrow 1$, also yields the following striking improvement of Pełczyński's theorem cited in the proof of $12 \Rightarrow 1$ of our Theorem 1. *If ℓ^1 embeds in X, X separable, then for all $1 > \varepsilon > 0$, there exists a subset K of $E \operatorname{Ba} X^*$ homeomorphic to the Cantor set, such that for all $f \in (1-\varepsilon) \operatorname{Ba} C(K)$, there exists an $x \in X$ with $\|x\| < 1$ and $f(k) = k(x)$ for all $k \in K$.* It follows that *for all $\varepsilon > 0$, $E \operatorname{Ba} X^*$ has a subset $(1+\varepsilon)$-equivalent to the basis for $\ell^1_{\mathfrak{c}}$.*

8. Theorem 3.18 of [Ro3] yields the following result, strengthening the result in [BFT] used in proving $18 \Rightarrow 1$ of Theorem 1.

Theorem. *Let K be a compact metric space and (f_n) a bounded sequence in $C(K)$ with no pointwise convergent subsequence. Then there exists a subsequence (f'_n) of (f_n) and a Borel probability measure μ on K such that no point-wise cluster point of (f'_n) is μ-measurable.*

It follows that we may replace the implications $18, 19 \Rightarrow 1$ by the following stronger statement: *If ℓ^1 embeds in X (assumed separable) and K is a weak*-compact norming subset of X^*, there exists a bounded sequence (x_n) in X such that $(\hat{x}_n)$* (as defined in 17) *has $2^{\mathfrak{c}}$ pointwise cluster points on K, none of which are universally measurable.*

Applying Lemma 2, we obtain from the Theorem

Corollary A. *If (f_n) satisfies the hypotheses of the Theorem, there exists a Borel probability measure μ on K such that there is a set W of cardinality $2^{\mathfrak{c}}$, consisting of point-wise cluster points of (f_n), none of which are μ-measurable.*

Here is another applcation, due to Stegall.

Corollary B. *Let W be a weakly pre-compact subset of a Banach space. Then the closed convex hull of W is weakly pre-compact.*

(W is called weakly pre-compact of every sequence in W has a weak-Cauchy subsequence.)

Proof. It obviously suffices to prove that $\operatorname{co} W$, the convex hull of W, is weakly pre-compact. Any sequence in $\operatorname{co} W$ is contained in the convex hull of a countable subset of W, so we may assume w.l.g. W is separable. Note incidentally that any weakly pre-compact set is bounded, since weak-Cauchy sequences are bounded by the uniform boundedness principle. Thus, suppose that $C < \infty$, $\|w\| \leq C$ for all $w \in W$, and to the contrary, there exists a sequence (f_n) in $\operatorname{co} W$ with no weak-Cauchy subsequence. Now letting $K = (\operatorname{Ba} X^*, w^*)$ and $\hat{x}(k) = k(x)$ for all $x \in W$ and $k \in K$, $(\hat{f}_n) \subset C(K)$ thus has no pointwise convergent subsequence on K, and so by the above Theorem, choose μ a Borel probability measure on K and (f'_n) a subsequence of (f_n) such that no point-wise cluster point of $(\hat{f}'_n)$ is μ-measurable. Now letting $i : C(K) \to L^1(\mu)$ be the canonical map, it follows by the bounded convergence theorem that $i(W)$ is a relatively compact subset of $L^1(\mu)$. But then $coi(W) = icoW$ is also a relatively compact set. So then, choose (f''_n) a subsequence of (f'_n) such that $(\widehat{if''_n})$ converges in $L^1(\mu)$, and finally choose $(\tilde{f}_n)$ a subsequence of (f''_n) such that (f_n) converges μ-almost everywhere, to a function g in $L^1(\mu)$. But then any point-wise cluseter point of $(\widehat{\tilde{f}_n})$ equals g almost everywhere, and is hence μ-measurable; of course all such are point-wise cluster points of $(\hat{f}'_n)$, a contradiction. $\square$

Corollary C. *ℓ^1 does not embed in X iff there exists a weakly precompact subset of X isomorphically norming X^*.*

Proof. If ℓ^1 does not embed in X, $W = \operatorname{Ba} X$ is weakly precompact, by the ℓ^1-Theorem. Suppose conversely W is a weakly precompact subset of X, isomorphically norming X^*. It is easily seen that $\tilde{W} \stackrel{\text{def}}{=} \{\alpha w : |\alpha| = 1\}$ is also weakly pre-compact and hence $\overline{\operatorname{co}}\tilde{W}$, the closed convex hull of $\tilde{W}$, is weakly pre-compact by Corollary B. The Hahn-Banach Theorem (geometrical form) implies $\overline{\operatorname{co}}\tilde{W}$ contains $E \operatorname{Ba} X$ for some $\varepsilon > 0$, showing that every bounded sequence in X has a weak-Cauchy subsequence. $\square$

It is interesting to compare the equivalences 1–3 of Theorem 1 with the following equivalences, given in [DU] (see Theorem 8, page 175 of [DU]; $3 \Rightarrow 1$ is due to Uhl and $1 \Rightarrow 3$ is due to Stegall).

Theorem. *Let X be given. Then the following are equivalent.*

1. *X^* has the Radon-Nikodym property (RNP).*
2. *Every integral operator from Y to X^* is nuclear, for all Y.*
3. *Y^* is separable for all separable $Y \subset X^*$.*

However this does not yield the sharp equivalence analogous to the one we obtain in $1 \Leftrightarrow 3$ in Theorem 1.

Problem 2. *Let X be a separable Banach space. When does there exist an integral non-nuclear operator on X^*?*

Now if X^* has the Radon-Nikodym property, then by the above Theorem, every operator on X^* is nuclear. But also, if X^{**} has the RNP, then: $T : X^* \to X^*$ integral implies $T^* : X^{**} \to X^{**}$ integral implies T^* is nuclear implies T is nuclear, since X^* is contractively complemented in X^{***}. Now if X^* has the approximation property (ap) in addition, then (again by Grothendieck's results), $(K(X))^* = (X^* \overset{\vee}{\oplus} X)^* = X^{**} \overset{\wedge}{\oplus} X^*$ isometrically. This is so because Grothendieck's results yield that $(K(X))^* = I(X^*)$, the space of integral operators on X^* (because X has the ap). But then in the case X^{**} has the RNP, we also have that the integral norm of T on X^* equals the integral norm of T^{**}, which equals the nuclear norm of T^{**}, which equals the nuclear norm of T^*, which is then the same as its projective tensor norm. Also, $(X^{**} \overset{\vee}{\oplus} X^*)^* = \mathcal{L}(X^{**})$ isometrically. We may thus summarize this discussion as follows.

Proposition 3. *Let X be a Banach space such that X^* has the ap and X^* or X^{**} have the RNP. Then every integral operator on X^* is nuclear, $K(X)^* = X^{**} \overset{\wedge}{\oplus} X^*$ isometrically and $K(X)^{**} = \mathcal{L}(X^{**})$ isometrically.*

We show later on, however, that there exist separable Banach space X so that X^{**} has the metric approximation property, X and X^* fail the RNP, yet still every integral operator on X^* is nuclear.

Remark. The above discussion also shows that *given X and Y, then if X^* or Y^* has the ap and either X^* or Y^* has the RNP, then $(X \overset{\vee}{\oplus} Y)^* = X^* \overset{\wedge}{\oplus} Y^*$ isometrically.* A slightly weaker result then this is due to Grothendieck ([Gr2]; see also [DFS]).

We now pass to a detailed discussion of the following problem:

Problem 3. *Under what conditions on Banach spaces X and Y is it so that ℓ^1 embeds in $X \overset{\vee}{\otimes} Y$, the injective tensor product of X and Y?*

We solve this problem, for separable Banach spaces X and Y such that X^* or Y^* have the bounded approximation property (bap), in Theorem 6 below. This also yields a partial answer to Problem 2. (Of course under the ap assumption, the problem reduces to the study of separable spaces anyway, because if e.g. X^* has the bap, then we show in Lemma 9 that given X_0 a separable subspace of X, there exists a separable subspace X_1, of X with $X_1 \supset X_0$ such that X_1^* has the bounded approximation property. Thus if ℓ^1 embeds in $X \overset{\vee}{\otimes} Y$ and X^*

or Y^* have the bap, there must exist X_1, Y_1 separable subspaces of X and Y so that X_1^* or Y_1^* has the bap and ℓ^1 embeds in $X_1 \overset{\vee}{\otimes} Y_1$).

We shall freely use the standard (but rather non-trivial!) results concerning tensor products of Banach spaces, due to Grothendieck [Gr2], as also exposed in [DU]. For Banach spaces X and Y, $I(X,Y)$ denotes the space of integral operators from X to Y and $N(X,Y)$ the space of nuclear operators from X to Y. For the definitions, including the norms, of these spaces, see the above references.

Our next result provides as with sufficient conditions for ℓ^1 to *not* embed in $X \overset{\vee}{\otimes} Y$.

Corollary 4. *Let X and Y be given Banach spaces, neither containing an isomorph of ℓ^1, such that X^* or Y^* has the RNP. Then ℓ^1 does not embed into $X \overset{\vee}{\otimes} Y$.*

Proof. It obviously suffices to prove this for the case where X and Y are both separable. By a result of Grothendieck, we also have that

$$(X \overset{\vee}{\otimes} Y)^* = I(X,Y^*) = N(X,Y^*) \tag{47}$$

(via trace duality). Furthermore,

$$N(X,Y^*) \ \text{ is isometric to a quotient space of } \ X^* \overset{\wedge}{\otimes} Y^* \tag{48}$$

which implies that

$$[N(X,Y^*)]^* \ \text{ is isometric to a subspace of } \ X^* \overset{\wedge}{\otimes} Y^* = \mathcal{L}(X^*,Y^{**}) \ . \tag{49}$$

Now assuming Y^* has the RNP, then Y^* is separable, (by [S]). But since Y^* is a separable Banach space, Y^{**} is isometric to a subspace of ℓ^∞, and hence by Theorem 1, part 11, $\mathcal{L}(X^*,Y^{**})$ has cardinality $\mathfrak{c}$, and so we have proved that $(X \overset{\vee}{\otimes} Y)^{**}$ has cardinality $\mathfrak{c}$, using (47)–(49). Thus ℓ^1 does not embed in $X \overset{\vee}{\otimes} Y$ by Theorem 1, part 15 (i.e., by [OR]). $\square$

We shall show later on that Corollary 4 does not solve problem 2. In fact, the following consequence of the result of R. Haydon mentioned above is crucial for the solution.

Proposition 5. *Let X and Y be Banach spaces such that ℓ^1 does not embed in $X \overset{\vee}{\otimes} Y$. Then $I(X,Y^*)$ equals the closure of the finite rank operators from X to Y^* (endowed with the integral norm on $I(X,Y^*)$).*

Proof. Let $K = \{x^* \otimes y^* : \|x^*\|,\ \|y^*\| \le 1,\ x^* \in X^*,\ y^* \in Y^*\}$. Then K is a w^*-compact subset of $\mathrm{Ba}(X \overset{\vee}{\otimes} Y)^*$ which isometrically norms $X \overset{\vee}{\otimes} Y$. By the argument given in the proof of $1 \Rightarrow 9 \Rightarrow 10$ of Theorem 1, it follows from [Hay] that $[K] = (X \overset{\vee}{\otimes} Y)^* = I(X,Y^*)$. This proves Proposition 5. $\square$

Remarks. 1. For X and Y separable, Proposition 5 follows by Theorem 1, part 10.

2. Of course this result holds if we interchange X and Y in its statement, which is more naturally given in the language of tensor products: *For separable Banach spaces X and Y, if ℓ^1 does not embed in $X\overset{\vee}{\otimes}Y$, then $(X\overset{\vee}{\otimes}Y)^*$ equals the closure of $X\otimes Y$ in the space of integral bilinear forms on $X^*\times Y^*$.*

3. It is apparently an open problem if the nuclear and integral norms coincide or are equivalent on $F(X,Y^*)$, X, Y given ($F(Z,W)$ denotes the space of finite rank operators from Z to W). Of course if this should be the case, for X,Y satisfying the hypotheses of the Proposition, then its conclusion can be strengthened to state: *Then every integral operator from X to Y^* is nuclear.*

We may now give a definitive solution to Problem 2, under an approximation property assumption.

Theorem 6. *Let X and Y be separable Banach spaces such that X^* or Y^* has the bounded approximation property. Then the following are equivalent*

1. *ℓ^1 does not embed in $X\overset{\vee}{\otimes}Y$.*
2. $\operatorname{card}\mathcal{L}(X^*,Y^{**})=\mathfrak{c}$.
3. $\operatorname{card}\mathcal{L}(X^*,Y^{**})<2^{\mathfrak{c}}$.

Moreover when this occurs, every integral operator from X to Y^* is nuclear, and consequently

$$(X\overset{\vee}{\otimes}Y)^*=X^*\overset{\wedge}{\otimes}Y^*\ ,\quad \textit{hence}\quad (X\overset{\vee}{\otimes}Y)^{**}=\mathcal{L}(X^*,Y^{**})\ .$$

Remark. As in the preceding result, we may (obviously) interchange X and Y in the statement of Theorem 6.

Proof. Suppose first that 1 holds. The approximation property assumption insures that the integral and nuclear norms are equivalent on $F(X,Y^*)$, and these in turn are equivalent to the projective tensor product norm. The final statement now follows from Proposition 5 and the fact (due to Grothendieck) that $(X\overset{\vee}{\otimes}Y)^*=I(X,Y^*)$; of course then 2 follows by Theorem 1 (i.e., by [OR]).

$2\Rightarrow 3$ is trivial.

Now suppose that 3 holds, and assume that

$$X^* \ \text{ has the approximation property}\ . \tag{50}$$

Let $K=\{x^*\otimes y^*:(x^*,y^*)\in \operatorname{Ba}X^*\times\operatorname{Ba}Y^*)\}$, endowed with its w^*-topology as a subset of $I(X,Y^*)=(X\overset{\vee}{\otimes}Y)^*$. Thus K is a w^*-compact isometrically norming subset of $(X\overset{\vee}{\otimes}Y)^*$.

To prove that 1 holds it suffices to prove that given (A_n) a bounded sequence in $X\overset{\vee}{\otimes}Y$, then defining $\hat{A}_n(x^*\otimes y^*) = \langle x^*\otimes y^*, A_n\rangle$ for all n and $x^*\otimes y^*\in K$, then

$$\hat{A}_n \text{ has a pointwise convergent subsequence.} \tag{51}$$

If (51) is false, then by 19 of Theorem 1,

$$\text{The family } \mathcal{F} \text{ of pointwise cluster points of } (\hat{A}_n) \text{ has cardinality } 2^{\mathfrak{c}}. \tag{52}$$

(Of course we can directly apply Lemma 2 to also see that (52) holds.) Now suppose that $f : K \to \mathbb{K}$ is a pointwise cluster point of $(\hat{A}_n)$ (where $\mathbb{K}$ denotes the scalar field), and suppose that $\|A_n\| \le C$ for all n, $C < \infty$. Now we identify $X\overset{\vee}{\otimes}Y$ with $K_*(X^*,Y)$, the Banach space of compact operators in $\mathcal{L}(X^*,Y)$ which are weak*-norm continuous on bounded subsets of X^*. (This is legitimate by results in [Gr2].) Of course trivially $K_*(X^*,Y)\subset\mathcal{L}(X^*,Y^{**})$. Then we claim that

(53) there exists a unique $T_f\in\mathcal{L}(X^*,Y^{**})$ such that

(i) $\langle T_f(x^*),y^*\rangle = f(x^*\otimes y^*)$ for all $(x^*,y^*)\in \operatorname{Ba}X^*\times\operatorname{Ba}Y^*$

and moreover

(ii) $T_f\ne T_g$ if $f,g\in\mathcal{F}$, $\quad f\ne g$.

Indeed, we may choose a net $(n_\alpha)_{\alpha\in D}$ so that

$$\lim_\alpha A_{n_\alpha}(x^*\otimes y^*) = f(x^*\otimes y^*) \text{ for all } x^*\otimes y^*\in K\ . \tag{54}$$

But then it follows easily that

$$\lim_\alpha A_{n_\alpha}(x^*\otimes y^*) \overset{\text{def}}{=} G_f(x^*,y^*) \text{ exists for all } (x^*,y^*)\in X^*\times Y^*\ , \tag{55}$$

and moreover

$$G_f \text{ is a bilinear form on } X^*\times Y^* \text{ with norm bounded by } C. \tag{56}$$

Of course then there is a unique $T_f\in\mathcal{L}(X^*,Y^{**})$ such that

$$\langle T_f(x^*),y^*\rangle = G_f(x^*,y^*) \text{ for all } (x^*,y^*)\in X^*\times Y^* \tag{57}$$

and moreover it is obvious that $f\ne g\in\mathcal{F}\Rightarrow G_f\ne G_g\Rightarrow T_f\ne T_g$. Thus $\operatorname{card}\mathcal{L}(X^*,Y^{**}) = 2^{\mathfrak{c}}$, contradicting 3.

The final statement of Theorem 6 follows by Proposition 5, Remark 3 following its proof, and the theorem of Grothendieck that $X^*\overset{\wedge}{\otimes}X^* = N(X,Y^*)$ since X^* has the bap. $\square$

Corollary 7. *Let X and Y satisfy the hypotheses of Theorem 6 and suppose ℓ^1 does not embed in either space. Then if every operator from X^* to Y^{**} has separable range, ℓ^1 does not embed in $X\overset{\vee}{\otimes}Y$. In particular, this holds if every operator from X^* to Y^{**} is compact.*

Proof. Since ℓ^1 does not embed in Y^{**}, $\operatorname{card} Y^{**} = \mathfrak{c}$, and hence the cardinality of the family of countable subsets of Y^{**} is $\mathfrak{c}$. If Z is a separable subspace of Y^{**}, we may choose a countable dense subset Z_0 of Z, and so we have established that

(58) the cardinality of the family of separable subspaces of Y^{**} equals $\mathfrak{c}$.

Now for each separable non-zero subspace Z of Y^{**},

$$\mathcal{L}(X^*, Z) \text{ has cardinality } \mathfrak{c} \tag{59}$$

by condition 13 of Theorem 1. Since the union of a family of sets of cardinality $\mathfrak{c}$ has cardinality $\mathfrak{c}$ provided each of the sets has cardinality $\mathfrak{c}$, (58) and (59) imply that 3 of Theorem 6 holds, and hence ℓ^1 does not embed in $X \overset{\vee}{\otimes} Y$ by Theorem 6. The final statement of the Corollary now trivially follows. $\square$

Problem 4. *Let X and Y satisfy the hypotheses of Theorem 6, and suppose every integral operator from X to Y^* is nuclear. Is it so that ℓ^1 does not embed in $X \overset{\vee}{\otimes} Y$ unless ℓ^1 embeds in X or Y?*

Problem 4, in turn, suggests

Problem 5. *If X and Y are given such that L^1 embeds in $X^* \overset{\wedge}{\otimes} Y^*$; does L^1 embed in X^* or Y^*?*

By the results in [H1] and [P], an affirmative answer holds for particular X, Y iff ℓ^1 embeds in X or Y. An affirmative answer to Problem 5 implies an affirmative answer to Problem 4. For suppose to the contrary, that X, Y provide a counterexample to Problem 4. Since ℓ^1 embeds in $X \overset{\vee}{\otimes} Y$, L^1 embeds in $(X \overset{\vee}{\otimes} Y)^* = I(X, Y^*)$. But since ℓ^1 does not embed in X or Y, L^1 does not embed in X^* or Y^*, and so if the answer to the second part of 5 is affirmative, L^1 does not embed in $X^* \overset{\wedge}{\otimes} Y^*$, which thus cannot equal $I(X, Y^*)$ contradicting the final statement in Theorem 6. (See remark 4 following proof of Theorem 11 for further comments on Problem 5.)

Remark. If we do not deal with dual spaces in Problem 5, then the answer is negative. A remarkable result of Talagrand [T] asserts the existence of separable spaces X and Y so that L^1 embeds in $X \oplus Y$ but L^1 does not embed in X or Y. It follows, since L^1 is isomorphic to a finite codimensional subspace of itself, that also L^1 embeds in $X \oplus Y_0$ if Y_0 is of codimension 1 in Y. Let x_0, y_0 be norm one elements of X^*, Y^* respectively; choose x_0^*, y_0^* norm one elements of X^*, Y^* with $x_0^*(x_0) = y_0^*(y_0) = 1$, and let $\tilde{Y} = y_0^\perp$. Then $X \otimes [y_0]$ and $[x_0] \otimes \tilde{Y}$ are isometric to X and $\tilde{Y}$ in $X \overset{\wedge}{\otimes} Y$, and $I \otimes (y_0^* \otimes y_0)$ is a contractive projection from $X \overset{\wedge}{\otimes} Y$ onto $X \otimes [y_0]$

with $[x_0] \otimes \tilde{Y}$ contained in its kernel. It follows that $(X \oplus [y_0]) + ([x_0] \otimes \tilde{Y})$ is isomorphic to $X \oplus \tilde{Y}$ in $X \hat{\otimes} Y$ and thus L^1 embeds in $X \hat{\otimes} Y$.

Of course this counterexample cannot simply lead to a counterexample for Problem 4, because if L^1 embeds in $X^* \oplus Y^*$, ℓ^1 embeds in $X \oplus Y$ by [H1] and [P], and then obviously ℓ^1 embeds in X or Y. Nevertheless, I am inclined to believe that the answer to problem 3 and hence to problem 4 is negative. Leter on, we give examples where in fact ℓ^1 does not embed in X^* or Y^*, both X^* and Y^* have the metric approximation property and fail the RNP, and indeed $(X \check{\otimes} Y)^* \neq X^* \hat{\otimes} Y^*$. However the proof is rather delicate, using specific properties of these spaces, and I'm inclined to believe there is no technique general enough to give an affirmative answer to problem 4.

The next result summarizes consequences of the previous results in the context of Problem 2.

Theorem 8. *Let X be a given Banach space. Consider the following two properties*

P_1. Every integral operator on X^* is nuclear.

P_2. ℓ^1 does not embed in $K(X)$.

1. *If P_1 holds, ℓ^1 does not embed in X.*
2. *If X^* has the approximation property and X^* or X^{**} has the RNP, then P_1 holds.*
3. *If X^* has the bounded approximation property and P_2 holds, P_1 holds.*
4. *If X^* has the bounded approximation property, then P_2 holds if and only if for all separable subspaces Z and Y of X^* and X respectively such that Z^* has the bounded approximation property, the cardinality of $\mathcal{L}(Z^*, Y^{**})$ is less than $2^{\mathfrak{c}}$ (iff the cardinality equals $\mathfrak{c}$).*

1. This follows from Theorem 1, part 4, for if 1 is false, this implies there exists an integral operator on X^* which is not compact, hence not nuclear.

2. This is Proposition 3.

3. If X^* has the bap, then $K(X) = X^* \check{\otimes} X$ (because X has the ap), and hence

$$K(X)^* = I(X^*) = X^{**} \hat{\otimes} X^*$$

by Proposition 5, since the hypotheses imply that the integral, nuclear, and projective tensor norms are equivalent on $X^{**} \otimes X^*$. Of course the final equality implies that P_1 holds.

To show 4, we need

Lemma 9. *If X^* has the bap, then very separable subspace E of X is contained in a separable subspace Z of X such that Z^* is isomorphic to a complemented subspace of X^*.*

We first complete the proof of Theorem 8, then prove Lemma 9.

1. As in the proof of 3, $K(X) = X^* \overset{\vee}{\otimes} X$ and so the "only if" statement follows immediately from Theorem 6. To see "if", suppose to the contrary that ℓ^1 embeds in $K(X)$. Then we can choose separable subspaces E and Y of X and X^* respectively such that ℓ^1 embeds in $E \overset{\vee}{\otimes} Y$. But if we choose Z satisfying the conclusion of Lemma 9, then Z^* has the bap, and then ℓ^1 does not embed in $Z \overset{\vee}{\otimes} Y$, by Theorem 6, a contradiction. $\square$

Proof of Lemma 9. Since X^* has the bap, so does X, and so we may choose $1 < \lambda < \infty$ so that for all finite-dimensional subspaces F of X, there exists a finite rank operator T on X such that

$$\|T\| \le \lambda \ \text{ and } \ T \mid F = I \mid F \ . \tag{60}$$

Let $e_0, e_1, e_2, \ldots$ be an enumeration of a countable dense subset of E. Choose T_1 a finite rank operator on X satisfying (60), where $F = [e_0, e_1]$ and "T" $= T_1$. Suppose $n \ge 1$ and finite rank operators T_1, T_n have been chosen. Now let $F = [\text{Range } T_n, [e_{n+1}]]$. Choose T_{n+1} satisfying (60) with "T" $= T_{n+1}$.

This completes the inductive construction of the T_n's; Let $F_n = T_n(X)$ and $Z = \overline{\cup F_j}$. Note that $F_n \subset F_{n+1}$ and $e_n \in F_n$ for all n; hence $E \subset Z$. It follows that

$$T_n(X) \subset Z \text{ for all } n \text{ and } T_n(z) \to z \text{ in norm for all } z \in Z \ . \tag{61}$$

Now the compactness of the unit ball of X^* in the weak* topology implies that there exists a net $(T_{n_\alpha})_{\alpha \in \infty}$ (i.e., a subnet of the sequence (T_j) and an operator P on X^* such that

$$T^*_{n_\alpha}(x^*) \to P(x^*) \ w^* \ \text{ for all } \ x^* \in X^* \ . \tag{62}$$

(62) shows that $\|P\| \le \lambda$ (so P is indeed bounded). Now if $x^* \in Z^*$, then (61) implies that $T^*_n(x^*) = 0$ for all n, which implies

$$P(x^*) = 0 \ \text{ for all } \ x^* \in Z^+ \ . \tag{63}$$

Now by our construction, $T_{n+1} \mid \text{Range } T_n = I \mid \text{Range } T_n$ for all n, which implies that for all k and $n > k$, $T_n T_k = T_k$, and so taking adjoints, $T^*_k T^*_n = T^*_k$ for all k and $n > k$. But then for $x^* \in X^*$, we deduce, thanks to the weak*-continuity of T^*_k, that

$$\lim_\alpha T^*_k T^*_{n_\alpha}(x^*) = T_k P(x^*) = T_k(x^*) \ . \tag{64}$$

In turn, (64) implies, after taking another limit, that $P^2 = P$; hence P is a projection. Finally,

$$\text{if } P(x^*) = 0, \text{ then } x^* \in Z^\perp. \tag{65}$$

For if not, we would find $z \in Z$, $\langle x^*, z\rangle \neq 0$. But then

$$\langle Px^*, z\rangle = \lim_\alpha \langle T^*_{n_\alpha}, x^*, z\rangle = \lim_\alpha \langle x^*, T_{n_\alpha} z\rangle = (x^*, z) \neq 0\ . \tag{66}$$

(63) and (65) show that $Z^\perp$ is the kernel of P^*. Thus it follows that PX^* is isomorphic to $X^*/Z^\perp$, which of course is isometric to Z^*. □

Remark. It is known that the conclusion of the Lemma holds without any approximation property assumptions, but we included a proof of our needed result for completeness. We only used the hypothesis of the Lemma to obtain that X has the bounded approximation property. With a little more care in the proof, we may thus obtain the following result: *If X has the λ-bap and E is a separable subspace of X, there is a separable subspace Z of X with the λ-bap, containing E, such that Z^* is λ-isomorphic to a λ-complemented subspace of X^*.*

Of course Theorem 8 suggests a possible solution to Problem 2, namely that P_1 and P_2 are equivalent properties, if X^* has the bap.

Problem 6. *If X^* has the bap, does P_1 imply P_2?*

Of course Problems 4 and 6 are linked, for an affirmative answer to Problem 4 implies an affirmative answer to Problem 6.

We now give examples illustrating Theorem 6. The examples show that the conditions of Corollary 4 are not necessary, to insure that ℓ^1 does not embed in $X \overset{\vee}{\otimes} Y$. The examples also show that there exist separable Banach spaces X and Y not containing ℓ^1 isomorphically, so that there exists an integral non-nuclear operator from X to Y^*. The examples are natural generalizations of the James tree space JT. We first define these spaces and summarize some of their properties.

Let $\mathcal{D}$ be the dyadic tree; that is, $\mathcal{D}$ is the set of all finite sequences of 0's and 1's, ordered by extension. Let $\tau : \mathcal{D} \to N$ be the standard bijection, given as

$$\emptyset,\ (0),\ (1),\ (00),\ (01),\ (10),\ (11),\ \cdots \tag{67}$$

and let (e_j) be the unit vectors bases of $c_{0,0}$ the space of all sequences of scalars which are ultimately zero. A non-empty subset s of $\mathcal{D}$ is called a *segment* provided

(68) s is totally ordered, and whenever $\alpha < \gamma < \beta$ in $\mathcal{D}$ with $\alpha, \beta \in s$, then $\gamma \in s$.

A maximal totally ordered subset of $\mathcal{D}$ is called a *branch* of $\mathcal{D}$; equivalently, this is a segment containing $\emptyset$, unbounded above; branches can also be identified with infinite sequences of 0's and 1's, where if (ε_n) is such a sequence, the corresponding branch is $\{\emptyset, (\varepsilon_1), (\varepsilon_1, \varepsilon_2), (\varepsilon_1, \varepsilon_2, \varepsilon_3), \ldots\}$.

Given a segment s of $\mathcal{D}$, define a functional s^* on c_{00} by

$$s^*(f) = \sum_{j\in s} f_{\tau(j)} \ . \tag{69}$$

Now fix p, $1 < p < \infty$, and define a norm $\|\cdot\|_{JT_p}$ on c_{00} by

$$\|f\|_{JT_p} = \max\left\{ \left(\sum_{i=1}^k |s_i^*(f)|^p \right)^? : k \geq 1 \text{ and } s_1,\dots,s_k \text{ are disjoint segments of } \mathcal{D} \right\} . \tag{70}$$

We define JT_p to be the completion of (c_{00}, JT_p). JT_2 is the space defined as JT in [LS], discovered in [J3], and the proofs of the properties of JT_p that we require are the same as those for JT. (For an alternate exposition, see [B].)

Let us first note that

$$s^* \text{ is a norm one functional on } JT_p \text{ for any segment } s. \tag{71}$$

We refer to the functionals β^*, for β a branch of $\mathcal{D}$, as *branch functionals.* For W a non-empty subset of $\mathcal{D}$, set

$$\widetilde{W} = [e_j : \tau(j) \in W] \ . \tag{72}$$

Then it is obvious that for any segment s,

$$\tilde{s} \text{ is contractively complemented in } JT_p, \text{ with complement } \widetilde{\mathcal{D} \sim s}. \tag{73}$$

Let us also note that given a branch β of $\mathcal{D}$, $\beta \sim (\varepsilon_j)_{j=1}^\infty$, then if we let $\beta_\alpha = e_{\tau^{-1}(\varepsilon_1,\dots,\varepsilon_{n-1})}$, $j = 1,\dots,$ then

$$\begin{gathered}(\beta_j)_{j=1}^\infty \text{ is isometrically equivalent to the boundedly complete basis for } J \\ \text{where } J \text{ denotes the quasi-reflexive order one space of James.}\end{gathered} \tag{74}$$

In particular, $(\beta_j)_{j=1}^\infty$ dominates the summing basis, and the branch functional β^* restricted to $\tilde{\beta}$ is the summing functional.

We denote by $\mathcal{D}_n$ the family of elements of $\mathcal{D}$ of length n; then letting $\alpha_1^n,\dots,\alpha_{2^n}^n$ be an enumeration of $\mathcal{D}_n$, and setting $e_j^n = e_{\tau^{-1}(\alpha_j^n)}$ for all j, we have that

$$\begin{gathered}\tilde{\mathcal{D}}_n \text{ is contractively complemented in } JT_p, \text{ by } \widetilde{\sim \mathcal{D}_n} \ , \\ \text{and } (e_j^n)_{j=1}^{2^n} \text{ is isometrically equivalent to the } \ell_{2^n}^p \text{ basis.}\end{gathered} \tag{75}$$

We now summarize the rest of the properties of JT_p that we shall need (some of which are decidedly non-trivial). (Throughout, r^* denotes the conjugate index to r, $\frac{1}{r} + \frac{1}{r^*} = 1$.)

Theorem 10.

1. *(e_j) is a boundedly complete monotone basis for JT_p.*

2. *$(JT_p)^*$ is the norm-closed linear space of the functionals (e_j^*) biorthogonal to (e_j), and the branch functionals.*
3. *The set of branch functionals $\mathcal{B}$ is homeomorphic to the Cantor set, in the weak*-topology. If for $\alpha \in \mathcal{D}$, $\mathcal{B}_\alpha = \{\beta^* : \alpha \in \beta\}$, then $(\mathcal{B}_a)_{\alpha\in\infty}$ is a family of w^*-clopen subsets of $\mathcal{B}$, forming a base for its topology.*
4. *Letting $(JT_p)_*$ be the closed linear span of the biorthogonal functionals (e_α^*), then*

$$(JT_p)^*/(JT_p)_* \ \text{ is isometric to } \ \ell_{\mathfrak{c}}^{p^*} \ . \tag{76}$$

5. *Every normalized weakly null sequence in JT_p has a subsequence equivalent to the ℓ^p-basis.*

(5. was established for JT in [AI]; see [B] for an alternate treatment of the proof of Theorem 10 for JT, which immediately generalizes to the case of JT_p.)

We may draw some immediate consequences. First, since $(JT_p)_*$ is thus canonically embedded in its double dual, JT_p is canonically complemented in $(JT_p)^*$, by $((JT_p)_*)^\perp$. Hence it follows from Theorem 10 part 4 that

$$JT_p^{**} \ \text{ is isometric to } \ JT_p \oplus \ell_{\mathfrak{c}}^p \ . \tag{77}$$

Of course JT_p has the Radon-Nikodym property, being a separable dual. Then obviously by (77), JT_p^{**} also has the Radon-Nikodym property and the metric approximation property. So we deduce that

$$(JT_p)_*,\ JT_p,\ (JT_p)^*, \ \text{ and } \ JT_p^{**} \ \text{ all have the metric approximation property.} \tag{78}$$

Of course it also follows immediately that ℓ^1 does not embed in JT_p^{**}.

We may now formulate our final main result; which shows in particular that for $1 < p < \infty$,

$$\ell^1 \ \text{ embeds in } \ JT_p \overset{\vee}{\otimes} JT_p \ \text{ iff } \ 2 \le p < \infty \ . \tag{79}$$

(I am indebted to J. Diestel for showing me the important special case: ℓ^1 embeds in $JT \overset{\vee}{\otimes} JT$.)

Theorem 11. *Let $1 < p, q < \infty$. Then the following are equivalent.*

1. *ℓ^1 embeds in $JT_p \overset{\vee}{\otimes} JT_q$.*
2. *$p^* \le q$.*
3. *There exists an integral non-nuclear operator from JT_p to $(JT_q)^*$.*

Remark. Of course since JT_p, JT_q satisfy the hypotheses of Theorem 6, we may add as a 4th equivalence

4. card $\mathcal{L}_{p,q} = 2^{\mathfrak{c}}$, *where $\mathcal{L}_{p,q} = \mathcal{L}((JT_p)^*, (JT_p)^{**})$.*

Proof. Suppose first that $p^* \le q$. We show that 3 holds, which implies that 1 holds by Theorem 6. Let μ be the "Cantor probability measure" on the Borel sets of $\mathcal{B}^q$, associated to the family of clopen sets $(\mathcal{B}^q_\alpha)_{\alpha\in\mathcal{D}}$. That is, μ is the unique Borel measure so that

$$\mu(\mathcal{B}^q_\alpha) = \frac{1}{2^n} \text{ if } \alpha \in \mathcal{D}_n \text{ for } n = 0,1,2,\dots \ . \tag{80}$$

(Since we are dealing with different spaces, we are denoting by $\mathcal{B}^q$ the set of Banach functionals on JT_q, $\mathcal{B}^q_\alpha$ its associated clopen-set basis.) Now we define a map $V : L^1(\mu) \to (JT_q)^*$ by

$$Vf = \int_{\mathcal{B}} f(w)w^*\,d\mu(w) \text{ for } f \in L^1(\mu) \ . \tag{81}$$

The integral denotes the weak*-integral; thus, we easily verify that given $x \in JT_q$, then since $w^* \to w^*(x)$ is a continuous function of norm at most $\|x\|$ on $\mathcal{B}$, then for $f \in L^1(\mu)$

$$x \to \int_{\mathcal{B}} f(w)w^*(x)\,d\mu(w) \tag{82}$$

is in $(JT_q)^*$ and has norm at most $\|f\|_{L^1(\mu)}$ thus (81) is well defined, V is indeed a linear operator, and in fact $\|V\| \le 1$. Then we have that for all $\alpha \in \mathcal{D}$,

$$\begin{cases} (V\chi_{\mathcal{B}_\alpha})(e_{\tau(\beta)}) = \frac{1}{2^{|\alpha|}} & \text{if } \beta \in \mathcal{B}_\alpha \\ V\chi_{\mathcal{B}_\alpha}(e_{\tau(\beta)}) = 0 & \text{if } \beta \notin \mathcal{B}_\alpha \end{cases} \tag{83}$$

where we set $|\alpha| = n$ if $\alpha \in \mathcal{D}_n$. Next, let $\varphi : \mathcal{B}^q \to \mathcal{B}^p$ be the canonical homeomorphism such that $\varphi(\mathcal{B}^q_\alpha) = \mathcal{B}^p_\alpha$ for all α. Now define $U : JT_p \to C(\mathcal{B}^q)$ by

$$(Ux)(\beta^*) = \varphi(\beta^*)(x) \text{ for all } x \in JT_p \ , \quad \beta \in \mathcal{B}^q \ . \tag{84}$$

$\square$

Then it is obvious that U is a linear contraction and thus

$$T = VU \text{ is an integral operator from } JT_p \text{ to } (JT_q)^* \ . \tag{85}$$

Thus by trace duality, we have (by Grothendieck's fundamental theory) given $x = \sum_{i=1}^n x_i \otimes y_i \in JT_p \otimes JT_q$, then

$$\Big|\sum_{i=1}^n Tx_i(y_i)\Big| \le \Big\|\sum_{i=1}^n x_i \otimes y_i\Big\| \tag{86}$$

(and $\sum_{i=1}^n x_i \otimes y_i \to \sum_{i=1}^n Tx_i(y_i) = \operatorname{tr} T \sum_{i=1}^n y_i \otimes x_i$ is a well defined linear functional; we used in (86) that the integral norm of T is at most 1). To prove that T is integral but not nuclear, we shall show that there exists a $G \in (JT_p \overset{\vee}{\otimes} JT_q)^{**}$ such that

$$G(T) = 1 \text{ and } G(f) = 0 \text{ for all } f \in JT_p^* \overset{\wedge}{\otimes} JT_q^* \ . \tag{87}$$

Of course to prove the last claim in (87), it suffices to prove that

$$G(x^* \otimes y^*) = 0 \text{ for all } x^* \in JT_p^*,\ y^* \in JT_q^* \ . \tag{88}$$

Let us call $x^* \otimes y^*$ a *basic tensor* if x^*, y^* are each either a biorthogonal functional or a summing functional in JT_p^*, respectively JT_q^*. Thanks to part 2 of Theorem 10 and the continuity of G, it actually suffices to prove that (88) holds for all basic tensors $x^* \otimes y^*$.

Now fix n, and define $A_n \in JT_p \otimes JT_q$ by

$$A_n = \sum_{i=1}^{2^n} e_{\alpha_i^n} \otimes e_{\alpha_i^n} \ . \tag{89}$$

Now let P_n be the canonical projection from JT_p onto $[e_i^n]_{i=1}^{2^n}$; thus P_n^* is the canonical projection from $(JT_p)^*$ onto $[e_i^{n*}]_{i=1}^{2^n}$. Let $Q_n : [e_i^{n*}]_{i=1}^{2^n} \to [e_i^n]_{i=1}^{2^n}$ be the linear map such that $Q_n e_i^{n*} = e_i^n$ for $1 \le i \le 2^n$, where e_i^{n*} denotes an element of $(JT_p)^*$ and e_i^n denotes an element of JT_q. Then

$$A_n = Q_n P_n \ . \tag{90}$$

Thanks to (89), $(e_i^{n*})_{i=1}^{2^n}$ is isometrically equivalent to the $\ell_{2^n}^{p^*}$ bases, $(e_i^n)_{i=1}^{2^n}$ is isometrically equivalent to the $\ell_{2^n}^q$ basis, and hence since 2 holds, $\|Q_n\| = 1$, $\|P_n\| = 1$, and so by (90)

$$\|A_n\| \le 1 \ . \tag{91}$$

We shall define G satisfying (88) and (89) in two steps. First, let $G_1 \in (JT_p \overset{\vee}{\otimes} JT_q)^{**}$ be a w^*-cluster point of $(A_n)_{n=1}^\infty$. Of course (91) shows that $\|G_1\| \le 1$. We have that letting $\langle , \rangle$ be the pairing between $I(JT_p, (JT_q)^*)$ and $JT_p \otimes JT_q$ given after (70) then for all n,

$$\begin{aligned} \langle T, A_n\rangle &= \sum_{i=1}^{2^n} T(e_i^n)(e_i^n) \\ &= \sum_{i=1}^{2^n} \frac{1}{2^n} \ \text{ using (98)} \\ &= 1 \ . \end{aligned} \tag{92}$$

Thus $G_1(T) = 1$.

Next, we claim that for all basic tensors x^* and y^*,

$$G_1(x^* \otimes y^*) = \lim_{n\to\infty} \sum_{j=1}^{2^n} x^*(e_i^n) y^*(e_i^n) \tag{93}$$

(part of this assertion is the claim that this limits exists) and so that setting $H = G_1$, then

$$\begin{gathered} H(x^* \otimes y^*) = 0 \text{ unless there is a branch } \gamma \text{ so that } x^* = \gamma^* \text{ (in } (JT_p)^*), \\ y^* = \gamma^* (\text{in } (TJ_p)^*), \text{ and then } H(\gamma^* \otimes \gamma^*) = 1 \ . \end{gathered} \tag{94}$$

Now it is obvious that the limit in (93) is zero if one of x^* or y^* is a biorthogonal functional. Suppose that there are branches $\gamma \ne \beta$ such that $x^* = \gamma^*$ and $y^* = \beta^*$, and choose k so that the branches split at level k; i.e., if $\gamma = (\gamma_j)_{j=0}^\infty$ and $\beta = (\beta_j)_{j=0}^\infty$, then $\gamma_k \neq \beta_k$, which implies that $\gamma_n \neq \beta_n$ for all $n > k$. But then if $n > k$, there are unique $i \neq j$ such that $\alpha_i^n \in \gamma$, $\alpha_j^n \in \beta$, and hence

$$\gamma^*(e_\ell^n)\beta^*(e_\ell^n) = 0 \quad \text{for all} \quad 1 \le \ell \le 2^n \ . \tag{95}$$

Finally, if $x^* = \gamma^* = y^*$, then for each n, there is exactly one i with $\alpha_i^n \in \gamma$, and then $\gamma^*(e_i^n) = 1$, $\gamma^*(e_j^n) = 0$, $j \neq i$, and hence $\sum_{\ell=1}^{2^n} \gamma^*(e_\ell^n)\gamma^*(e_\ell^n) = 1$ for all n. (Actually, (A_n) converges in the w^* operator topology on $\mathcal{L}_{p,q}$ to the operator S such that $S(e_j^*) = 0$ for all j, and $S(\gamma^*) = \gamma^{**}$ for all branches γ, where $\gamma^{**}(\beta^*) = \delta_{\gamma\beta}$ all β and $\gamma^{**}(e_j^*) = 0$ all j; $(\gamma^{**})_{\gamma\in\mathcal{B}}$ is in fact the basis for $\ell_{\mathfrak{c}}^q$ in $(JT_q)^{**}$).

Now let $\mathcal{F}$ be the family of all finite non-empty subsets of $\mathcal{B}$, directed by inclusion. For each n and $F = \{\gamma_1, \dots, \gamma_n\} \in \mathcal{F}$ of cardinality n, choose

$$m_n \ge n \quad \text{such that the branches } \gamma_1, \dots, \gamma_n \text{ have split at level } m_n \tag{96}$$

That is, we may choose n different integers $i_1, \dots, i_n$ in $\{1, 2, \dots, 2^{m_n}\}$ such that

$$\alpha_{i_j}^{m_n} \in \gamma_j \quad \text{for} \quad 1 \le j \le n \ . \tag{97}$$

Define $B_F \in J_p \otimes J_q$ by

$$B_F = \sum_{j=1}^n e_{i_j}^{m_n} \otimes e_{i_j}^{m_n} \ . \tag{98}$$

We have, just as in the definition of the A_n's, that

$$\|B_F\| \le 1 \quad \text{for all} \quad F \in \mathcal{F} \ . \tag{99}$$

Now let G_2 be a w^*-cluster point of $(B_F)_{F\in\mathcal{F}}$ in $(JT_p \otimes JT_q)^{**}$. Then

$$G_2(T) = 0 \ . \tag{100}$$

Indeed, we have that

$$\langle T, \beta_n \rangle = \sum_{j=1}^n \frac{1}{2^{m_n}} \le \frac{n}{2^n} \to 0 \quad \text{as} \quad n \to \infty \ . \tag{101}$$

Moreover, we claim that setting $H = G_2$, then (94) holds for all basic tensors $x^* \otimes y^*$. Indeed, if x^* or y^* is a biorthogonal functional, then obviously $\langle B_n, x^* \otimes y^* \rangle = \sum_{j=1}^n x^*(e_{i_j}^{m_n})y^*(e_{i_j}^{m_n}) = 0$ for n sufficiently large. But also if γ, β are difference branches of $\mathcal{B}$, then for n sufficiently large, the branches will have split at level m_n, which implies that $\gamma^*(e_{i_j}^{m_n})\beta^*(e_{i_j}^{m_n}) = 0$ for all $1 \le j \le n$. Finally, suppose γ is a given branch. Then for all $F \in \mathcal{F}$ with $\gamma \in \mathcal{F}$, $F = \{\gamma_1, \dots, \gamma_n\}$ with say $\gamma = \gamma_j$,

$$\langle B_F, \gamma^* \otimes \gamma^* \rangle = \gamma_j^*(e_{i_j}^{m_n})\gamma_j^*(e_{i_j}^{m_n}) = 1 \ . \tag{102}$$

It follows that our cluster point G_n must satisfy

$$G_2(\gamma^* \otimes \gamma^*) = 1 \ . \tag{103}$$

Now finally, let $G = G_1 - G_2$. Then we have by (92), (100) and the fact that (94) holds for $H = G_i$, $i = 1, 2$, that G satisfies (87) and (88), and so $G \perp JT_p \hat{\otimes} JT_q$, $G(T) = 1$, whence T is not nuclear.

Now suppose that $q < p^*$. To show that 1 holds, we only need to show that JT_p and JT_q satisfy the assumptions of Corollary 7; but of course these spaces satisfy all the assumptions preceding the final one, so we only need to prove that

(104) Every operator from $(JT_p)^*$ to $(JT_q)^{**}$ has separable range.

Let $P : (JT_q)^{**}$ be the projection from $(JT_q)^{**}$ onto JT_q, with kernel$((JT_q)_*)^\perp$, which we know by Theorem 10 is isometric to $\ell_{\mathfrak{c}}^q$; for the sake of notational simplicity, let us just set $((JT_q)_*)^\perp = \ell_{\mathfrak{c}}^q$. Now let $T : (JT_p)^* \to (JT_q)^{**}$ be a given operator, and set

$$A = PT \quad \text{and} \quad B = (I - P)T \ . \tag{105}$$

Obviously A has separable range, and B is an operator from $(JT_p)^*$ to $\ell_{\mathfrak{c}}^q$. We claim that

$$B \ \text{ is compact.} \tag{106}$$

It suffices to prove that $B^* : \ell_{\mathfrak{c}}^{q^*} \to (JT_p)^{**}$ is compact. To do that, we shall show that

(107) Every normalized weakly null sequence in $(JT_p)^{**}$
has a subsequence equivalent to the ℓ^p basis.

This will complete the proof. Indeed, to show that B^* is compact, it suffices (by reflexivity of $\ell_{\mathfrak{c}}^{q^*}$) to show that

(108) given (x_n) a normalized weakly null sequence in $\ell_{\mathfrak{c}}^{q^*}$, then $\|B^*(x_n)\| \to 0$.

Suppose this is not the case. But if (x_n) fails to satisfy (108), there exists a subsequence (x_n') such that (x_n') is equivalent to the ℓ^{q^*} basis and by (107), $B^*(x_n')$ is equivalent to the ℓ^p basis. But our assumption $q < p^*$ is the same as $p < q^*$, and we have thus deduced that the ℓ^{q^*} basis dominates the ℓ^p basis, a contradiction. Of course once we know that B^* and hence B, is compact, then B has separable range, and thus T has separable range, finishing the proof.

Since $(JT_p)^{**} = JT_p \oplus \ell_{\mathfrak{c}}^p$ and so both factors have the desired property, by Theorem 8 part 5, (107) is easily seen, but for completeness, here's the argument. Let (x_n) be a normalized weakly null sequence in $(JT_p)^{**}$.

We may assume (by passing to a subsequence) that (x_n) is a basic sequence. Now if there is a subsequence (x_n') of (x_n) such that $\|Px_n'\| \to 0$, we easily obtain a further subsequence (x_n'')

such that $((I-P)x''_n)$ is equivalent to (x''_n), and hence by passing to a further subsequence (x''_n) if necessary, that (x'''_n) is equivalent to the ℓ^p basis. Thus we may now assume that

$$\text{there is a } \delta > 0 \text{ so that } \|Px_n\| \geq \delta \text{ for all } n. \tag{109}$$

But then by Theorem 10 part 5, we may choose a subsequence (x'_n) of (x_n) so that

$$(Px'_n) \ \text{ is equivalent to the } \ell^p\text{-basis.} \tag{110}$$

Now if some subsequence (x''_n) of (x'_n) satisfies that $\|(I-P)x''_n\| \to 0$, then just as before, we get a subsequence of (x'_n) equivalent to the ℓ^p basis, so suppose this also is not the case. But then (after dropping a few terms if necessary), there is a $\delta' > 0$ such that

$$\|(I-P)x'_n\| \geq \delta' \ \text{ for all } \ n \ , \tag{111}$$

and now by the standard property of the ℓ^p basis, there is a subsequence (x''_n) such that also

$$((I-P)x''_n) \ \text{ is equivalent to the } \ell^1\text{-basis.} \tag{112}$$

But then also by (110),

$$(Px''_n) \ \text{ is equivalent to the } \ell^p\text{-basis.} \tag{113}$$

The continuity of P and $I-P$ and (126),(127) now easily yield that (x''_n) is equivalent to the ℓ^p basis. $\square$

Remarks. 1. Going through the proofs of Theorems 6 and 11 is a difficult way to see that condition 4 in the Remark (following the statement of Theorem 11) holds if 2 holds, for this may more easily be seen directly as follows. Since $(JT_p)^*$ has the metric approximation property,

$$JT_p^* \overset{\wedge}{\otimes} JT_q^* \ \text{ is isometric to a closed linear subspace of } I(JT_p, JT_q^*) = (JT_p \overset{\vee}{\otimes} JT_q)^* \ , \tag{114}$$

and thus

$$\mathcal{L}_{p,q} \ \text{ is isometric to a quotient space of } (JT_p \overset{\vee}{\otimes} JT_q)^{**}, \tag{115}$$

(where $\mathcal{L}_{p,q}$ is as in the above remark). Using Theorem 10 part 4 and (77), let $Q : (JT_p)^* \to \ell^{p^*}_{\mathfrak{c}}$ be a quotient map and $j : \ell^q_{\mathfrak{c}} \to (JT_q)^{**}$ be an isometric injection. Since $p^* \leq q$, the ℓ^{**}-basis dominates the ℓ^q-basis. Then given $\eta \in \{0,1\}^{\mathfrak{c}}$, there exists a unique linear contraction $T_\eta : \ell^{p^*}_{\mathfrak{c}} \to \ell^?_{\mathfrak{c}}$ such that

$$\text{for all } \alpha < \mathfrak{c}, \ (T_\eta)(\alpha) = 1 \text{ if } \eta(\alpha) = 1, \ (T_\eta)(\alpha) = 0 \text{ if } \eta(\alpha) = 0. \tag{116}$$

Now defining $\tilde{T}_\eta \in \mathcal{L}_{p,q}$ by

$$\tilde{T}_\eta = jT_\eta Q \ , \tag{117}$$

it follows that $\eta \neq \eta \Rightarrow \tilde{T}_\eta \neq \tilde{T}_{\eta'}$, proving that card $\mathcal{L}_{p,q} = 2^{\mathfrak{c}}$ (and also showing directly that $(JT_p \overset{\vee}{\otimes} JT_q)^{**}$ has cardinality $2^{\mathfrak{c}}$ by (115), thus giving that ℓ^1 embeds in $JT_p \overset{\vee}{\otimes} JT_q$ by [OR].

2. An inspection of the operator V constructed in the above proof shows that its range is actually contained in $(JT_q)_*$, and thus condition 3. of Theorem 10 may be strengthened to

$3'$. *There exists an integral operator from JT_p to (JT_q), which is not nuclear*

4. It's conceivable that $(JT)^* \overset{\wedge}{\otimes} (JT)^*$ (or more generally, $(JT_p)^* \otimes (JT_p^*)^*$ for some $1 < p, q < \infty$, $(p^* \leq q)$, is a counterexample to Problem 4. I reduced this to a separable issue, which, however, I cannot decide. That is, I proved that if p, q are as above and L^1 *embeds in* $(JT_p)^* \overset{\wedge}{\otimes} (J_q)^*$, *then* L^1 *embeds in* $(JT_p)_* \overset{\wedge}{\otimes} (JT_q)_*$. An alternate approach to Problem 3, using embeddings of unconditional families in place of embeddings of L^1, does not work. In fact, it follows that $Y = ((JT)^* \overset{\wedge}{\otimes} JT^*)/(JT)_* \overset{\wedge}{\otimes} (JT)_*)$ is isometric to $\ell^2_{\mathfrak{c}} \overset{\wedge}{\otimes} \ell^2_{\mathfrak{c}}$; the space of "diagonal" operators in this space is isometric to $\ell^1_{\mathfrak{c}}$. Thus, Y contains a subspace isometric to $\ell^1_{\mathfrak{c}}$, whence by the lifting property of this space, $(JT)^* \overset{\wedge}{\otimes} (JT)^*$ contains a subspace isometric to $\ell^1_{\mathfrak{c}}$. However $(JT)^*$ has no uncountable unconditional family by Theorem 1.

We conclude this section with an application of Theorem 8 to the relationship of Problem 1 with the RNP. We first need

Lemma 12. *$(JT_q)_*$ fails the RNP for all q, $1 < q < \infty$.*

Proof. We shall show that $(JT_q)_*$ has a "δ-tree." We first motivate the construction. Let μ be the Cantor measure on the Borel subsets of $\mathcal{B}$ and $V : L^1(\mu) \to (JT_q)^*$ be the operator constructed at the beginning of the proof of Theorem 11. Then in fact V is valued in $(JT_q)_*$, and is not representable by a Bochner integrable function. Rather than proving this, however, let us just examine: $V(\mathbf{1})$. Since (e_j^*) is a w^*-basis for $(JT_q)^*$, $V(\mathbf{1})$ must have an expansion; $V(\mathbf{1}) = \sum_{j=1}^\infty c_j e_j^*$, the series converging in the w^*-topology. The coefficients (c_j) are determined by: $c_j = \langle V(\mathbf{1}), e_j \rangle$ for all j. Let us instead, label the coefficient corresponding to e_j as c_α, where $j = \tau(\alpha)$. Then for any $\alpha \in \mathcal{D}$,

$$\langle V\mathbf{1}, e_{\tau(\alpha)} \rangle = \int_{\mathcal{B}} w^*(e_{\tau(\alpha)})\, d\mu(w) = \frac{1}{2^{|\alpha|}} \ , \tag{118}$$

because $w^*(e_{\tau(\alpha)}) = 1$ if $w \in \mathcal{B}_\alpha$ and 0 otherwise.

But we have that

$$\sum_{\alpha \in \mathcal{D}} \frac{1}{2^{|\alpha|}} e^*_{\tau(\alpha)} \quad \text{converges in norm.} \tag{119}$$

Indeed, let y_n be defined by

$$y_n = \sum_{j=1}^{2^n} \frac{1}{2^n} (e_j^n)^* = \frac{1}{2^n} \sum_{|\alpha|=n} e^*_{\tau(\alpha)} \ . \tag{120}$$

Then

$$\|y_n\| = \frac{1}{2^{n/q^*}} \qquad \text{by (75).} \tag{121}$$

Thus

$$\sum_{n=0}^{\infty} \|y_n\| = \frac{1}{1-\frac{1}{2^{n/q^*}}} = \frac{2^{n/q^*}}{2^{n/q^*}-1} \stackrel{\text{def}}{=} c_q < \infty \ , \tag{122}$$

and of course $\sum_{n=0}^{\infty} y_n = \sum_{\alpha\in\infty} \frac{1}{2^{|\alpha|}} e^*_{q(\alpha)}$.

It is then easily verified that for any α,

$$T(\chi_{\mathcal{B}_\alpha}) = \sum_{\gamma\in\mathcal{B}_\alpha} \frac{1}{|\alpha|} e^*_{\tau(\alpha)} \tag{123}$$

and this series converges in norm, by exactly the same reason we gave for (119).

Now define $(t_\alpha)_{\alpha\in\mathcal{D}}$ by

$$t_\alpha = 2^{|\alpha|} \sum_{\gamma\in\mathcal{B}_\alpha} \frac{1}{2^{|\alpha|}} e^*_{\tau(\gamma)} = T(2^{|\alpha|}\chi_{\mathcal{B}_\alpha} \ . \tag{124}$$

Since $[e^*_{\tau(\gamma)} : \gamma \in \mathcal{B}_\alpha]$ is canonically isometric to $(JT_q)_*$, we thus have that

$$\|t_\alpha\| \le c_q \ \text{ for all } \ \alpha \ . \tag{125}$$

It follows by (124) and the linearity of T (or by direct computation) that for all $\alpha \in \mathcal{D}$,

$$t_\alpha = \frac{1}{2}(t_{\alpha_0} + t_{\alpha_1}) \tag{126}$$

and

$$\|t_{\alpha_0} - t_{\alpha_1}\| \ge \|e^*_{\alpha_0} - e^*_{\alpha_1}\| = 2^{1/q^*} \ . \tag{127}$$

Thus $(t_\alpha)_{\alpha\in\mathcal{D}}$ is a bounded "δ-tree" with $\delta = 2^{1/q^*}$, and so $(JT_q)_*$ fails the RNP (cf. [DU]). $\square$

Corollary 13. *Let $1 < p < \infty$ and let $Y_p = (JT_p)_* \oplus JT_p$.*

1. *Y_p and Y_p^* fail the RNP.*
2. *ℓ^1 does not embed in $K(Y_p)$ if $1 < p < 2$.*
3. *ℓ^1 embeds in $K(Y_p)$ and moreover there exists an integral non-nuclear operator on Y_p^* if $2 \le p < \infty$.*

Proof. 1. Y_p fails the RNP by the preceding result and

$$Y_p^* = JT_p \oplus (JT_p)^* \tag{128}$$

fails the RNP because $(JT_p)^*$ does.

Next, we have for any p that

$$K(Y_p) = \left[(JT_p)_* \overset{\vee}{\otimes} JT_p\right] \oplus [(JT_p)^* \otimes JT_p] \oplus \left[(JT_p)_* \overset{\vee}{\otimes} (JT_p)^*\right] \oplus \left[JT_p \overset{\vee}{\otimes} JT_p\right] \ . \tag{129}$$

ℓ^1 does not embed in the first three summands in (129) by Corollary 4, because in each case, the dual of one of the factors in the injective tensor product has the RNP (and of course the factors all have the map, and their duals do not contain ℓ^1 isomorphically).

If $1 < p < 2$, then $p^* > p$, and so ℓ^1 does not embed in the fourth summand by Theorem 11, so ℓ^1 does not embed in $K(Y_p)$.

If $2 \le p$, then $p^* \le p$, and thus ℓ^1 embeds in the fourth summand by Theorem 11, part 3 of which also shows part 3 of the Corollary, using (129). □

Remark. Notice by Lemma 12 that we also obtain an integral operator from JT_p into $(JT_p)_*$ which is not nuclear if $2 \le p < \infty$, while if $1 < p < 2$, every integral operator from JT_p into $(JT_p)^*$ is nuclear by Theorem 11.

Appendix

We give a somewhat new proof of (8) via

Proposition 14. *Suppose that Z is a $\mathcal{L}_\infty$ space which is isomorphic to a quotient of a subspace of a Banach space X. Then Z^* is isomorphic to a subspace of X^*.*

Thus to obtain (8), if ℓ^1 is isomorphic to a subspace of X, $C([0,1])$ is isomorphic to a quotient of that subspace, and hence the Proposition applies. (For properties of $\mathcal{L}_\infty$ spaces, see [LP] and [LR]; the definition will appear in our proof.)

Proof of Proposition 14. Let $\tilde{X}$ be a (closed linear) subspace of X and $T : \tilde{X} \to Z$ a surjective bounded linear map, with $\|T\| = 1$. We may choose a "$\mathcal{P}_1$" space W with $Z \subset W$ (e.g., $W = \ell^\infty(\mathrm{Ba}\, X^*)$), and so we can then choose $\tilde{T} : X \to W$ with $\|\tilde{T}\| = 1$ such that $\tilde{T}$ extends T. Now choose $\lambda > 1$ such that for all finite dimensional subspaces E of Z, there exists a subspace $F_E \supset E$ with

$$d(F_E, \ell_n^\infty) \le \lambda \ , \quad \text{where} \quad \dim F_E < \infty \tag{130}$$

(and the first term in (130) denotes the Banach-Mazur distance from F_E to ℓ_n^∞). Therefore we may choose $P_E : W \to Z$ with

$$\|P_E\| \le \lambda \ , \quad P_E^2 = P_E \ , \quad \text{and} \quad P|_{F_E} = I|_{F_E} \ . \tag{131}$$

Now let $\mathcal{D}$ be the family of all finite dimensional subspaces of Z, directed by reverse inclusion. Then a compactness argument, using the fact that $\lambda\, \mathrm{Ba}\, Z^*$ is w^*-compact, shows there is a linear operator $S : Z^* \to W^*$ so that

$$\|S\| \le \lambda \ \text{ and for all } \ z \in Z \ \text{ and } \ z^* \in Z^* \ , \ \langle Sz^*, z\rangle = \langle z^*, z\rangle \ . \tag{132}$$

Thus if $R : W^* \to Z^*$ denotes the canonical restriction map, we have by (132) that

$$RS = I_{Z^*} \ . \tag{133}$$

Then setting $Y = S(Z^*)$, Y is λ-isomorphic to Z^*, since (133) gives that S is an isomorphism with the inverse of $S : Z^* \to Y$ given by $R|Y$. Moreover,

$$\begin{array}{l}\text{Given } \varepsilon > 0 \text{ and } y \in Y\text{, there exists a } z \in Z \text{ with } \|z\| = 1 \\ \text{such that } |y(z)| > \frac{1}{\lambda}\|y\| - \varepsilon.\end{array} \tag{134}$$

(It also follows from (133) that Y is complemented in Z^*, by $Z^\perp$, but we don't use this.)

Now (134) implies that

$$\tilde T^* \mid Y \text{ is an isomorphism, with } \tilde T(Y) \text{ being } \lambda\|(T^*)^{-1}\|\text{-isomorphic to } Z^*. \tag{135}$$

Indeed, given $y \in Y$ and $\varepsilon > 0$, choose $z \in Z$ satisfying (134), then choose $x \in \tilde X$ with $\|x\| \le \|(T^*)^{-1}\| + \varepsilon$ so that $Tx = z$. Thus letting $\tau = \frac{1}{\|(T^*)^{-1}\|} + \varepsilon$,

$$\begin{aligned} \|(\tilde T^*)(y)\| &= \tau|\langle \tilde T^* y, x\rangle| \\ &= \tau|\langle y, \tilde T x\rangle| = \tau|\langle y, Tx\rangle| \\ &= \tau|\langle y, z\rangle| \\ &\ge \frac{\tau}{\lambda}\|y\| - \varepsilon \ . \end{aligned} \tag{136}$$

Since $\varepsilon > 0$ was arbitrary, (135) is proved. □

Remark. We have kept track of the constants in the above proof because it then yields the following result: *Suppose X and Z are Banach spaces so that Z is an "L^1-predual," meaning that Z^* is isometric to $L^1(\mu)$ for some (not necessarily σ-finite) measure μ, and suppose for all $\varepsilon > 0$, there exists a subspace $\tilde X$ of X so that Z is $1+\varepsilon$-isomorphic to a quotient space of $\tilde X$. Then Z^* is $1+\varepsilon$-isomorphic to a subspace of X^* for all $\varepsilon > 0$.*

Indeed, we need only apply our proof, using the standard fact (as follows from local reflexivity) that Z is an $\mathcal{L}_{\infty,\lambda}$ space for all $\lambda > 1$. Now assuming ℓ^1 embeds in X, then by a result of James [J1], given $\varepsilon > 0$, there is a subspace $\tilde X$ of X which $(1+\varepsilon)$-isomorphic to ℓ^1, and consequently $C([0,1])$ is $(1+\varepsilon)$-isomorphic to a quotient space of $\tilde X$. Hence we deduce that *for all $\varepsilon > 0$, $(C(\Lambda))^*$ is $(1+\varepsilon)$-isomorphic to a subspace of X^*.*

As we show in Proposition 17, James' result that ℓ^1 is not distortable, holds for the spaces ℓ^1_κ as well, κ an infinite cardinal. We thus obtain the following generalization of the above quantitative version of (8).

Theorem 15. *Let κ be an infinite cardinal number, and suppose X contains a subspace isomorphic to ℓ^1_κ. Then given Z an L^1 predual of density character at most κ, X^* contains a subspace $(1+\varepsilon)$-isomorphic to Z^* for all $\varepsilon > 0$.*

The following result, which reduces to Pełczyński's theorem for $\kappa = \aleph_0$, is an immediate consequence.

Corollary 16. *Let κ be an infinite cardinal number and suppose X contains a subspace isomorphic to ℓ^1_κ. Then for all $\varepsilon > 0$, X^* contains a subspace $(1+\varepsilon)$-isomorphic to $[C(\{0,1\}^\kappa)]^*$.*

□

Finally, we give the extension of James' theorem to the spaces ℓ^1_κ (which is apparently a new result).

Proposition 17. *Let κ be an infinite cardinal number, and assume X contains a subspace isomorphic to ℓ^1_κ. Then X contains a subspace $(1+\varepsilon)$-isomorphic to ℓ^1_κ for all $\varepsilon > 0$.*

Proof. We identify cardinals with initial ordinals. Let then $[e_\alpha)_{\alpha<\kappa}$ be a normalized basis of cardinality κ in X, equivalent to the ℓ^1_κ basis. For each cardinal $\alpha < \kappa$, define δ_α by

$$\begin{cases} \delta_\alpha = \sup\{\delta > 0 : \|\sum_{\gamma\ge\alpha} c_\gamma e_\gamma\| \ge \delta \text{ for} \\ \text{all families of scalars } (c_\gamma)_{\gamma\ge\alpha} \text{ such that } \sum_{\gamma\ge\alpha} |c_\gamma| = 1\} \end{cases} \tag{137}$$

It is obvious that $\alpha \to \delta_\alpha$ is an increasing function. Hence

$$\lim_{\alpha\to\kappa} \delta_\alpha \stackrel{\text{def}}{=} \delta \text{ exists.} \tag{138}$$

(Of course if κ is of uncountable cofinality, then $\delta_\alpha = \delta$ for all α sufficiently large; however this fact is irrelevant for the proof.)

Now let $0 < \eta < \delta$ be given. It follows easily by induction and the fact that the family of all finite subsets of κ also has cardinality κ, and also because $\operatorname{card}\gamma < \kappa$ for $\gamma < \kappa$ and hence $\operatorname{card}\{\alpha : \gamma \le \alpha < \kappa\} = \kappa$ for all $\gamma < \kappa\}$, that we may choose a family $(f_\alpha)_{\alpha<\kappa}$ of finite linear combinations of the e_α's such that for each α, there exist ordinals $\alpha \le a_\alpha \le b_\alpha$ and c_γ's, only finitely many non-zero, with

$$f_\alpha = \sum_{a_\alpha \le \gamma \le b_\alpha} c_\gamma e_\gamma \tag{153i}$$

$$\|f_\alpha\| = 1 \quad \text{and} \quad \sum_{a_\alpha \le \gamma \le b_\alpha} |c_\gamma| > \frac{1}{\delta_0 + \eta} \tag{153ii}$$

$$\delta_{\alpha_0} > \delta - \eta \tag{153iii}$$

$$\text{for all } \ \alpha < \alpha' < \kappa \ , \quad b_\alpha < a'_\alpha \ . \tag{153iv}$$

Now let $(x_\alpha)_{\alpha<\kappa}$ be a family of scalars, only finitely many non-zero, such that $\sum_{\alpha<\kappa} |x_\alpha| = 1$. Thus we have

$$\begin{aligned} \|\sum_\alpha x_\alpha f_\alpha\| &= \|\sum_\alpha x_\alpha \sum_{a_\alpha \le \gamma \le b_\alpha} c_\gamma e_\gamma\| \\ &\ge \delta_{\alpha_0} \sum_\alpha |x_\alpha| \sum_{a_\alpha \le \gamma \le b_\alpha} |c_\gamma| \ \text{ by definition of } \ \delta_{\alpha_0} \\ &> \frac{\delta_{\alpha_0}}{\delta_0 + \eta} \sum |x_\alpha| \ \text{ by (153ii)} \\ &> \frac{\delta - \eta}{\delta + \eta} \ \text{ by (138).} \end{aligned} \tag{140}$$

Thus given $\varepsilon > 0$, we may choose η so small that $\frac{\delta-\eta}{\delta+\eta} > \frac{1}{1+\varepsilon}$, proving Proposition 17. □

Remark. Similar reasoning shows that the "predual" formulation of Proposition 17 also holds. That is,

Proposition 18. *Let κ be a infinite cardinal number, and assume X contains a subspace isomorphic to $c_0(\kappa)$. Then X contains a subspace $(1+\varepsilon)$-isomorphic to $c_0(\kappa)$ for all $\varepsilon > 0$.*

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