M328K – Rusin – HW5 – Due Thursday, Mar 9 2017

1. Give a formula for the number of positive divisors of a number n based on its factorization into primes. That is, if

$$n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

then determine how many divisors n has.

2. Show that if a and b are coprime integers and $a \cdot b$ is a perfect cube, then a and b are perfect cubes too. The corresponding statement for squares is almost true, but there's a little subtlety; can you find it?

3. Show that for every integer $n > 1$, $n^3 + 1$ is composite. (Hint: you may find a few examples to be instructive. Try $n = 1, 2, 4, 6$, and 16.)

4. *Twin primes* are primes p and q which differ by 2. For example 11 and 13 are twin primes. Prove that there are infinitely many primes which are NOT part of a twin-prime pair. How many primes p are there for which $p, p+2$, and $p+4$ are all prime?

No one knows whether or not there are infinitely many pairs of twin primes, although it *is* known that they are fairly sparse, in the sense that the sum of their reciprocals is finite!

5. For each integer n let C_n denote the central binomial coefficient $C_n = \binom{2^{n+1}}{2^n}$. Compute C_0, C_1, C_2. Show that for every integer M, gcd(M,C_n) is divisible by all the prime divisors of M that lie between 2^n and 2^{n+1}.

Note that in particular, we can factor every integer M with only about $\log(M)$ gcd computations. This is very fast! The only difficult part is computing the numbers C_n in the first place!