M328K – Rusin – HW6 – Due Tuesday, Mar 28 2017

My apologies for failing to make this HW accessible earlier. I have reset the due date to be next Tuesday.

I was asked to provide some comparable problems from the book for you to consider. OK, try sect. 3.1#17, sect. 3.3#9, sect. 3.4#9, sect. 3.5#63, sect. 3.5#83

1. Suppose a and b are positive integers. Show that if $a^3|b^2$ then $a|b$. Can we also conclude that $a|b$ if instead we are instead told that $a^2|b^3$?

2. For each positive integer n, let us write M_n for the nth Mersenne number, that is, $M_n = 2^n - 1$.

(a) Show that whenever $k|n$ then $M_k|M_n$.

(b) Show that if d divides two Mersenne numbers M_k and M_n with $k < n$, then it divides M_{n-k}.

I won't assign it but you might accept the following challenge: show that $\gcd(M_r, M_s) = M_{\gcd(r,s)}$.

3. Suppose a and b are coprime integers, and that one of them is even and the other is odd. Show that $a - b$ and $a^3 + b^3$ are also coprime.

4. *Twin primes* are primes p and q which differ by 2. For example 11 and 13 are twin primes. Prove that there are infinitely many primes which are NOT part of a twin-prime pair.

(I asked this question last time too but then I realized you didn't yet have the tool I intended you to use to solve this: Dirichlet's Theorem about primes in arithmetic progressions. Well, now you have the theorem, so .... Here is a hint: find a few primes which are NOT part of a twin-prime pair. Try reducing them modulo 30 and see if you see anything suggestive.)

5. A vague but important question is: how far apart are the primes? That is, if we number the primes in order,

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \quad p_4 = 7, \quad p_5 = 11, \quad \ldots$$

then can we estimate how big the gap $p_{n+1} - p_n$ is, compared to p_n itself? Obviously the size of that gap will vary: for example, if it turns out that the Twin Prime Conjecture is true, then there will be infinitely many values of n for which $p_{n+1} - p_n$ is just 2. On the other hand, there can be arbitrarily long gaps between the primes (see Theorem 3.5). But the size of the gap from p_n to p_{n+1} can be bounded by the size of p_n:

(a) Find Bertrand's Conjecture in the book. (This conjecture is known to be true.) Use it to show that $p_{n+1} - p_n < p_n$,

(b) Find Legendre's Conjecture in the book. (This conjecture is NOT yet known to be true.) Show that if it's true, then $p_{n+1} - p_n < 4\sqrt{p_n} + 2$.

(Researchers think that the gaps are *never* even close to the sizes shown in this problem; it's probably true that the gaps are never more than roughly $\log(p_n)^2$.)