Math 343K (Rusin) Exam 2, Apr 21, 2011. Please put your name on each sheet of paper you want me to grade. Please leave whitespace on which I can provide feedback.

1. Suppose G is a group containing two *normal* subgroups H and K. On the previous exam you showed that their intersection $N = H \cap K$ is again a subgroup of G. Prove now that N is *normal* in G.

2. Let Q denote the quaternion group of order 8, with elements denoted

$$\{1, -1, i, -i, j, -j, k, -k\}$$

The subgroup $N = \{1, -1\}$ is normal in Q. (You don't have to prove this.)

Construct the multiplication table for the quotient group Q/N.

3. Suppose G_0, G_1, G_2 are groups, and that $\phi_1 : G_0 \longrightarrow G_1$ and $\phi_2 : G_0 \longrightarrow G_2$ are homomorphisms. Show that the function $\psi : G_0 \longrightarrow G_1 \times G_2$ defined by $\psi(g) = (\phi_1(g), \phi_2(g))$ is a homomorphism too. What is its kernel?

4. Suppose R is a ring, and suppose $a \in R$. Let $I = aR$, the set of multiples of a in R; thus for every element $b \in I$ there exist elements $r \in R$ with $b = ar$.

Show that if R is an integral domain, then for nonzero a, this r is unique (i.e. if $b = ar$ and also $b = ar'$, then $r = r'$.)

Extra Credit: Prove that a finite integral domain must be a field. (Hint: you have just shown that multiplication-by-a is a one-to-one function from R to R; thus it is also onto.)

5. Suppose I is an ideal in a ring R. Show that if $1 \in I$ then $I = R$.

Conclude that if R is a field then either $I = 0$ or $I = R$. (Hint: if the first case does not apply, then I contains an element a of the field R which is different from 0. So what else is in I?)

6. Let R be a ring with the property that every element is its own square (that is, for each $r \in R$ we have $r^2 = r$). Show that R is commutative. (Hint: apply the condition to ring elements like $r = r_1, r_2$ and $r_1 + r_2$, including the case where $r_1 = r_2$.)

Much extra credit: Prove the same for a ring in which $r^3 = r$ for each r. (Spectacular amounts of extra credit are available for suitably worthy generalizations!)

Possible answers to these questions may be found at

`http://www.math.utexas.edu/users/rusin/343K/ans2.pdf`