M361K (Rusin) TEST 2 — April 13 2021

All questions count equally. Do your best. Remember the university Honor Code.
Submit a PDF of your answers through Canvas by 11:59pm on April 14 2021.

1. Suppose $f : \mathbf{R} \to \mathbf{R}$ is defined everywhere and is continuous at $x = 0$. (It may be continuous at other points, but that is not an assumption in this question.) Show that for every $\epsilon > 0$ there is an interval I around 0 such that for every x and y in I we have $|f(x) - f(y)| < \epsilon$.

2. A function $f : \mathbf{R} \to \mathbf{R}$ with the property that $f(x+1) = f(x)$ for all x is said to be *periodic*. Show that if such a function is continuous then there is a point x where $f(x) = f(x + \frac{1}{2})$. (Hint: What can you tell me about $g(x) = f(x) - f(x + \frac{1}{2})$?)

3. Show that the sum of two uniformly continuous functions on a set A is also uniformly continuous on A. For extra credit give an example to show that this result would be false if we change the word "sum" to "product".

4. Suppose $f : \mathbf{R} \to \mathbf{R}$ has the feature that $f(-x) = f(x)$ for all x. Show that if f is differentiable at the point $x = c$ then f is also differentiable at $x = -c$, and $f'(-c) = -f'(c)$.

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous everywhere and is known to be differentiable at all points except possibly at $x = 0$. Show that if $L = \lim_{x \to 0} f'(x)$ exists, then f is also differentiable at $x = 0$, and in fact $f'(0) = L$. (Hint: apply the Mean Value Theorem to the definition of $f'(0)$.)

(Remark: there exist functions f which satisfy the conditions of the first sentence but for which the limit L does not exist; for some such functions, $f'(0)$ exists, and for others it does not.)

6. We have not actually proved this in class, but there is a function called *exp* with a few properties:

(1) $\exp(x)$ is defined and continuous on all of $\mathbf{R}$

(2) $\exp(0) = 1$ and $\exp(1) < 3$.

(3) exp is differentiable everywhere, and $\exp'(x) = \exp(x)$ for all x.

Find a polynomial P for which $|P(x) - \exp(x)| < 0.01$ for all $x \in [0, 1]$, and prove that your polynomial really does have this property.