Theorem: If X is any set, and P is its power set, and $f : X \to P$ is any function, then f is not a surjection.

Corollary: For every set X, its power set is "bigger than" X.

Corollary: There are infinitely many different "sizes" of infinite sets!

Proof of Theorem: For every $x \in X$, $f(x)$ is in P, which means $f(x)$ is a subset of X. This subset might or might not have x itself in it. So the elements of X can be split into two groups:

$A = \{x \in X \mid x \in f(x)\}$

$B = \{x \in X \mid x \notin f(x)\}$

But now A and B are a couple of elements of P, which is the codomain of f. Are they in the *image* of f? In particular, is there some element $x_0 \in X$ for which $f(x_0) = B$?

The answer is NO, and thus f is not a surjection.

Indeed, if there were such an element $x_0 \in X$, it would have to lie in either A or B, right?

Well, if it's in A, then by definition of A, that means $x_0 \in f(x_0)$. But if $f(x_0)$ is supposed to be B, then we could write the end of that last sentence to say $x_0 \in B$; but the beginning of the same sentence assumed $x_0 \in A$. Since A and B are disjoint, that's a contradiction.

So the last paragraph just shows $x_0 \in A$ leads to a contradiction. In exactly the same way we get a contradiction from assuming $x_0 \in B$.

So there is no where to look for such an element x_0 and thus we have proved that B is not an element of the image of f, so f is not a surjection.