Allow me to comment on a couple other questinos from HW1.

5. Prove that between any two irrational numbers there exists another irrational number.

As a lemma, you might prove that between any rational number r and any irrational number x there exist irrational numbers; for example the midpoint $m = (x+r)/2$ is surely not rational (because then $x = 2m - r$ would be rational).

So now given irrational numbers a and b, consider the midpoint $c = (a + b)/2$. If it is irrational we are done. If instead it is rational, we may then use the lemma to find an irrational between a and c and again we are done.

Alternatively, you might consider the two numbers $c = (a + b)/2$ and $d = (2a + b)/3$ between a and b. Since $a = 2d - 2c$ is irrational, c and d cannot both be rational, so at least one of them is the irrational we seek.

8. Let $S = \{x \,|\, x^3 > 7\}$. Show that if y is a rational number in S then there is a smaller rational number y' also in S. (Hint: Find y' using Newton's method. In order to show $y' \in S$ you may wish to write $y = a + h$ where a is the (real) cube root of 7.)

Newton's method proposes finding solutions to equations $f(X) = 0$ by iteration: if y is an approximate solution, then $y' = y - f(y)/f'(y)$ is typically a closer approximation. In the case $f(X) = X^3 - 7$ this method proposes replacing a number y which is close to $a = 7^{1/3}$ with the number $y' = (2y^3 + 7)/(3y^2)$. I suggested this as a hint because it has the features we want. The very definition makes y' smaller than y since $f(y) = y^3 - 7 > 0$ and $f'(y) = 3y^2 > 0$, Our last formula shows y' is clearly rational if y is. All we have left is to show $y' \in S$. We can demonstrate this in a couple of ways.

First of all, if y is close to a, let's write it as $y = a + h$. Then $y' = (2(a + h)^3 + 7)/(3(a + h)^2)$ is also close to a and so we will write it as $y' = a + h'$, where $h' = y' - a$ may be computed algebraically as

$$h' = \frac{2(a+h)^3 - 3a(a+h)^2 + 7}{3(a+h)^2} = \frac{3ah^2 + 2h^3 + (7 - a^3)}{3(a+h)^2} = \frac{h^2(3a + 2h)}{3(a+h)^2}$$

Since $y > a > 0$, $3a + 2h > 0$ which means $h' > 0$, i.e. $y' > a$ so that $y' \in S$.

Alternatively, we can actually calculate $(y')^3 - 7$ and see that it is positive, placing y' into S. (This is more similar to Rudin's approach.) Indeed, if we write Y for y^3, we compute

$$(y')^3 - 7 = \frac{(2Y + 7)^3}{27Y^2} - 7 = \frac{8Y^3 - 105Y^2 + 294Y + 343}{27Y^2} = \frac{(8Y + 7)(Y - 7)^2}{27Y^2}$$

which is clearly positive.

These two calculations show that whether we measure how far y is from a or how far y^3 is from 7, we will see that in each iteration the distance get quadratically smaller: there will be roughly twice as many zeros after the decimal point each time. Newton's method computes $7^{1/3}$ fast!