M365C (Rusin) HW4 – due Thursday, Sept 26 2019

1. Find two subsets A, B of $\mathbf{R}$ with $A \cap B = \emptyset$ but $\bar{A} \cap \bar{B} \neq \emptyset$.

2. Notice that the definition of what makes a set open depends on the metric used. Thus if we have two different metrics on a set, some of its subsets can be open in one metric but not the other. For example, we could use the *discrete metric* on $\mathbf{R}$, i.e.

$$d(x, y) = 1 \quad \text{whenever} \quad x \neq y,$$

instead of the usual metric. What subsets of $\mathbf{R}$ are open when using the discrete metric?

3. The *taxicab metric* on $\mathbf{R}^2$ is defined by

$$d(\ (a, b),\ (c, d)\ ) = d(a, c) + d(b, d)$$

Show that a subset of $\mathbf{R}^2$ is open under the taxicab metric iff it is open under the usual (Euclidean) metric. (We say two metrics on a set X are "equivalent" when they declare the same subsets of X as being open.)

4. Suppose X is a metric space with metric d. If $x \in X$ and $A \subseteq X$ then we define

$$d(x, A) = \inf\{d(x, a) \,|\, a \in A\}$$

(a) Show that if y is another point of X that

$$d(y, A) \leq d(x, A) + d(x, y)$$

(b) Under what conditions will $d(x, A) = 0$?

5. Show that if A and B are disjoint closed subsets of any metric space, then there are open sets U and V with $A \subseteq U$ and $B \subseteq V$ and $U \cap V = \emptyset$. (Hint: use the preceding problem.)

6. Suppose A is any subset of $\mathbf{R}$, and let $\{O_\gamma | \gamma \in \Gamma\}$ be a collection of open sets with $A \subseteq \bigcup_{\gamma \in \Gamma} O_\gamma$. (That is these open sets form an *open cover* of A.) Show that there is a countable subset $\Gamma' \subseteq \Gamma$ of this collection of open sets which still covers A. (This is called the *Lindelöf Property*.)

(Hint: For each $a \in A$, pick an O_γ that contains it. If A is countable, we're done. Otherwise, for each such a, pick an interval (p, q) around a that is contained in O_γ *and* has rational endpoints p and q. (Why is this possible?) Some of the intervals (p, q) will be picked for multiple points a. (Why?) For each such (p, q), pick one of the O_γ that contain it. Let Γ' be the collection of these γs. Why is this a countable collection and why does it still cover A?)