M365C (Rusin) HW7 – due Thursday, Oct 24 2019

1. Prove that addition of real numbers is continuous. That is, let $F : \mathbf{R}^2 \to \mathbf{R}$ be the function $F(x, y) = x + y$. Here $\mathbf{R}$ is the set of real numbers with the usual metric, and $\mathbf{R}^2$ is the Cartesian plane where distance between two points is computed by the high-school distance formula. Is this F continuous?

2. As I noted in class, it is more common to use the *supremum metric*

$$d_\infty(f, g) = \max_{x \in [0,1]} (|f(x) - g(x)|)$$

on the set $C^0[0, 1]$, instead of the L^1 metric I have been using: it's just one example of the whole family of metrics known as the L^p metrics, defined by

$$d_p(f, g) = \left(\int_{x \in [0,1]} |f(x) - g(x)|^p \, dx \right)^{1/p}$$

For the functions

$$f_n(x) = \begin{cases} n - n^2 x & \text{if } x < 1/n \\ 0 & \text{if } x \geq 1 \end{cases}$$

compute $d_\infty(f_n, 0)$ and $d_p(f_n, 0)$ for each $p > 0$. In which metrics does the sequence $\{f_n\}$ converge to the zero function? Does the sequence converge to 0 pointwise?

I encourage you to think about why the notation d_∞ becomes appropriate for the supremum metric now.

3. For every $a \in [0, 1]$ we may define the function $e_a : C^0[0, 1] \to \mathbf{R}$ by

$$e_a(f) = f(a)$$

(That is, e_a is the operation of simply evaluating a function at a.) Use the $\delta - \epsilon$ definition of continuity to decide whether e_a is continuous when we use the metric d_1 on $C^0[0, 1]$ (as we have until now) and then similarly decide whether e_a is continuous when we use the supremum metric d_∞.

4. I asserted in class that every rational function (a ratio of two real polynomials) continuous (i.e. continuous at each point p in its domain). Prove this, using the theorems found in our book.

5, Recall that we say a function $f : X \to Y$ is *uniformly continuous* if: for every $\epsilon > 0$ we can find a $\delta > 0$ such that for all elements x_1 and x_2 in X with $d_X(x_1, x_2) < \delta$ we have $d_Y(f(x_1), f(x_2)) < \epsilon$.

(a) Find an interval $X \subseteq \mathbf{R}$ on which the function $f(x) = x^3$ is uniformly continuous.

(b) Find an interval $X \subseteq \mathbf{R}$ on which the function $f(x) = x^3$ is not uniformly continuous.