M365C (Rusin) HW9 – due TEST DAY, Thursday, Nov 7 2019

By class acclamation it was decided that this HW will be graded but the grade will only be used to replace your lowest HW grade so far (so it is in effect optional).

1. Suppose $f_n : \mathbf{R} \to \mathbf{R}$ is a sequence of uniformly continuous functions, and that the sequence converges uniformly to a function $f : \mathbf{R} \to \mathbf{R}$. Show that f is also uniformly continuous.

2. Find an open cover of the set $\mathbf{Q} \cap [0, 1]$ which has no finite subcover.

3. Show that the composite of two increasing functions is increasing. What can you say about the composite of two decreasing functions?

4. If f is a continuous function on $[a, b]$ then you may recall in Calculus that we refer to the quantity

$$M = \frac{1}{|b-a|} \int_z^b f(x)\, dx$$

as the *average value* (or mean value) of f on $[a, b]$. Show that this mean value M is actually the value of the function f at some point in the interval $[a, b]$. (In layman's terms, if you can drive more than 60 miles in less than an hour, then you must at some point have exceeded the 60mph speed limit!)

5. Use Taylor Series to show that $\sum_{n=1}^{\infty} (-1)^{n-1}/n \ = \ln(2)$

6. You may recall Newton's Method (for solving equations $f(x) = 0$) from your Calculus class: you were told that you should find an approximation x_0 to the solution, and then to iteratively improve your approximation by constructing a sequence $\{x_n\}$ of real numbers using the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

You were probably told this sequence would "usually" converge to a solution to the equation $f(x) = 0$. Let's discover some features of this algorithm.

(a) Show that this method fails to find a solution to the equation $f(x) = 0$ if $f(x) = x^{1/3}$ and $x_0 = 1$. (So: clearly we need to add some hypotheses if we hope to prove Newton's Method converges to a solution! What hypotheses will be enough?)

(b) Assume f is continuous on an interval (a, b) and has $f(a) < 0$ and $f(b) > 0$. Use the *Intermediate Value Theorem* (Theorem 4.23) to show there exists at least one point $c \in (a, b)$ where $f(c) = 0$.

(c) Assume also that f is differentiable on (a, b) and that $f'(x) > 0$ for all $x \in (a, b)$. Show that this c is the *only* solution to $f(x) = 0$ in this interval.

(d) Assume also that f is *twice* differentiable on (a, b). Show for every $x \in (a, b)$ there are points $y, z \in (a, b)$ where

$$f(x) = f'(c)(x - c) + f''(y)/2(x - c)^2$$
$$f'(x) = f'(c) + f''(z)(x - c)$$

(e) Assume also that there are positive numbers M and N such that $|f''(x)| \leq M$ and $|f'(x)| \geq N > 0$ for every $x \in (a, b)$. Conclude that for $x \in (a, b)$, $|x - c - f(x)/f'(x)| \leq (3M/2N)(x - c)^2$

(f) Conclude that if all these conditions are met then there is a constant K such that whenever $x_n \in (a, b)$ we have $|x_{n+1} - c| \leq K|x_n - c|^2$.

You saw a similar conclusion in HW5 (problem 5): this inequality may be stated $e_{n+1} \leq e_n^2$ where $e_n = K|x_n - c|$. By induction we see $e_n \leq (e_0)^{(2^n)}$ which will make the e_n decrease to zero very rapidly (we say we have "quadratic convergence" and numerically we see that the number of decimal digits that x_n shares with c will roughly *double* with every iteration) as long as our first approximation is good enough: $|x_0 - c| < 1/K$. Of course in order to carry out the induction we need some further constraints to ensure that x_{n+1} remains inside the interval (a, b) where all the assumptions apply, but again that will be true assuming x_0 is sufficiently close to c.