Math 408D (Rusin) — FINAL EXAM — May 17 2011. Here are some answers.

1. Calculate \( \lim_{x \to \infty} 4x(e^{1/x} - 1) \)

**Answer:** This is an indeterminate form of the \( \infty \cdot 0 \) type. Write the function instead in the form \( \frac{4(e^{1/x} - 1)}{1/x} \) and apply L’Hopital’s Rule: the limit is the same as that of

\[
\frac{4(e^{1/x})(-1/x^2)}{-1/x^2} = 4e^{1/x} \to 4
\]

2. For what positive numbers \( a \) does this improper integral converge? Explain.

\[
\int_a^\infty \frac{4}{(x-3)^2} \, dx
\]

**Answer:** The integral is (potentially) improper in both senses: on (say) the interval \([10, \infty)\) the function is continuous but the region of integration is unbounded so the integral over this interval is defined to be

\[
\lim_{T \to \infty} \int_{10}^T \frac{4}{(x-3)^2} \, dx
\]

which can be easily evaluated by a \( u \)-substitution: if \( u = x - 3 \), the antiderivative is \(-4/u\) and so the integral is \(-4/(T-3)) + (4/7)\), which approaches \(4/7\) as \(T \to \infty\).

Separately, on the interval \([a, 10]\) we have a bounded interval but an integrand \( f(x) \) which is not continuous on \([a, 10] \) if \( a \leq 3 \). So the integral over this interval is, for such \( a \), defined as

\[
\lim_{L \to 3} \int_a^L f(x) \, dx + \lim_{R \to 3^+} \int_R^{10} f(x) \, dx
\]

assuming both those limits exist. As above, the antiderivative is \(-4/(x-3)\) so we need to assess the limits

\[
\lim_{L \to 3^-} \frac{-4}{L-3} - \frac{-4}{a-3} \quad \text{and} \quad \lim_{R \to 3^+} \frac{-4}{10-3} - \frac{-4}{R-3}
\]

But neither of these limits exists (they both diverge to \(+\infty\)) so the integral does not converge for any \( a \leq 3 \).

3. Let \( \{a_n\} \) be the sequence whose \( n \)th term is \( a_n = \sqrt{(n^2 + 6n)} - n \). Determine whether this sequence converges, and if so, to what.

**Answer:** You can turn this into a L’Hopital’s Rule problem of the \( \infty - \infty \) type and then into the \( \infty \cdot 0 \) type, but I think it’s faster to rewrite \( a_n \) as

\[
\frac{(n^2 + 6n) - (n)^2}{\sqrt{(n^2 + 6n)} + n}
\]
and then view this as a “∞/∞” kind of problem. Applying L’Hopital’s Rule we get the same limit as that of $6/(n^2 + 3\sqrt{n^2 + 6n}) + 1)$. That fraction in the denominator is the square root of $(n^2 + 6n + 9)/(n^2 + 6n)$ and hence approaches 1, which means the denominator approaches 2 while the numerator is just 6, making the fraction tend towards 3. By L’Hopital’s Rule, our original limit is then also equal to 3.

4. Evaluate $\sum_{n=0}^{\infty} \frac{1 + 3^n}{7^n}$

**Answer:** This is the sum of two geometric series,

$$\sum_{n=0}^{\infty} \frac{1}{7^n} + \sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n$$

and so the sum is $\frac{1}{1 - (1/7)} + \frac{1}{1 - (3/7)} = \frac{35}{12}$.

5. Is this series convergent or divergent? $\sum_{m \geq 0} \frac{1 + (-1)^n}{n!}$

**Answer:** It’s convergent. You could do a comparison with $\sum_{m \geq 0} \frac{1 + (+1)^n}{n!} = \sum_{m \geq 0} \frac{2}{n!}$; you can even recognize the series as the sum of two parts, $\sum_{m \geq 0} \frac{1}{n!} + \sum_{m \geq 0} \left(-\frac{1}{n!}\right) = e + e^{-1}$.

6. Find the interval of convergence for the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}(x+1)^k$

**Answer:** If you try, say, the Ratio Test, you find that the series converges absolutely if

$$\lim_{k \to \infty} \frac{1/(k+1)^2)|x+1|^{k+1}}{1/k^2)|x+1|^k} < 1$$

and diverges if this limit is greater than 1. Since $\lim_{k \to \infty} \frac{k}{k+1} = 1$, the limit is less than 1 iff $|x+1| < 1$, i.e. iff $x \in (-2,0)$. If $x < -2$ or $x > 0$ then the limit is more than 1, and the series diverges.

If $x = -2$, the series is simply $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which is convergent by the integral test. (It’s a “p-series”, with $p > 1$.) If $x = 0$, the series is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$, which is strictly alternating and decreasing, hence again convergent by the Alternating Series Test.

So the interval of convergence is exactly $[-2,0]$.

7. Give a power series representation of $f(x) = \frac{2x}{1-x^2}$. 
**Answer:** This \( f(x) \) matches the presentation of the sum of a geometric series starting with \( 2x \) and having common ratio \( x^2 \), so \( f(x) = 2x + 2x^3 + 2x^5 + \ldots \)

You could use Partial Fractions and write this function as \( \frac{1}{1-x} - \frac{1}{1+x} \), which we similarly recognize as geometric series: \[
\sum_{n \geq 0} x^n - \sum_{n \geq 0} (-x)^n = \sum_{n \geq 0} (1 - (-1)^n)x^n = \sum_{n \geq 0} 2x^{2k+1} = 2x + 2x^3 + 2x^5 + \ldots .
\]

Or you might have recognized \( f \) as the derivative of \(-\ln(1 - x^2)\), so you could write the series for the latter, and then differentiate.

Just don’t attempt to use Taylor’s Formula .

8. Find the degree-3 Taylor polynomial \( T_3 \) centered at \( x = 25 \) for the square-root function \( f(x) = \sqrt{x} \). Use your polynomial to estimate \( \sqrt{26} \).

**Answer:** The derivatives of \( f(x) = x^{1/2} \) are \( f'(x) = (1/2)x^{-1/2}, f''(x) = (-1/4)x^{-3/2}, f'''(x) = (3/8)x^{-5/2} \), etc. The values of these functions at 25 are, respectively, 5, 1/10, −1/500, and 3/25000. So the Taylor polynomial we want is

\[
\]

When \( x = 26 \) this gives us the estimate

\[
f(26) \approx T_3(26) = 5 + (1/10) + (-1/1000) + (1/50000) = 5 + .1 - .001 + .00002 = 5.09902.
\]

The correct value is \( \sqrt{26} = 5.09901951359278483002822410902 \ldots \) Approximately :-).

9. The equation (in polar coordinates) \( r = \cos^2(\theta) \) defines a curve in the \( xy \)-plane. Find the equation of the line tangent to the curve at the point on that curve having \( \theta = \pi/4 \).

**Answer:** The point has \( r = \cos^2(\pi/4) = 1/2 \), and so its Cartesian coordinates are \( x = r \cos(\theta) = \sqrt{2}/4 \) and \( y = \sqrt{2}/4 \). Now we just need the slope of the line, which is (using the Chain Rule to compute \( dx \) and \( dy \), and then noting that \( dr = -2 \cos(\theta) \sin(\theta) d\theta \))

\[
\frac{dy}{dx} = \frac{\sin(\theta) dr + r \cos(\theta) d\theta}{\cos(\theta) dr - r \sin(\theta) d\theta} = \frac{-\sin(\theta)2 \cos(\theta) \sin(\theta) + r \cos(\theta)}{-\cos(\theta)2 \cos(\theta) \sin(\theta) - r \sin(\theta)}
\]

When \( \theta = \pi/4 \) and \( r = 1/2 \) this comes out to 1/3, so the line is \( (y - \sqrt{2}/4) = \frac{1}{3}(x - \sqrt{2}/4) \).

10. Consider the triangle whose vertices are the points \( P = (2,1,0), Q = (3, -1, 2), \) and \( R = (4, -1, -1) \). Find (a) the area of this triangle, and (b) the angle in this triangle at the vertex \( P \).

**Answer:** Two sides of the triangle form the vectors \( PQ = (1, -2, 2) \) and \( PR = (2, -2, -1) \) respectively. The area of the triangle is half that of the corresponding parallelogram, i.e.
(1/2)\|PQ \times PR\| = (1/2)\|6i + 5k + 2k\| = \sqrt{65}/2. The angle at \(P\) has a cosine equal to 
\((PQ \cdot PR)/(\|PQ\| \cdot |PR|)) = 4/(3 \cdot 3),\) that is, the angle is \(\arccos(4/9)\).

11. Find the point on the plane \(x + 2y - z = 2\) that is closest to the origin.

**Answer:** The distance from a point \((x, y, z)\) to the origin is \(\sqrt{x^2 + y^2 + z^2}\); since on this plane we have \(z = x + 2y - 2\), our job is to find a pair \((x, y)\) which minimizes the expression \(\sqrt{x^2 + y^2 + (x + 2y - 2)^2}\). It’s actually sufficient to minimize the square of this expression, \(x^2 + y^2 + (x + 2y - 2)^2 = 2x^2 + 4xy + 5y^2 - 4x - 8y + 4\). The minimum occurs when the gradient is zero, i.e. when \(4x + 4y - 4 = 4x + 10y - 8 = 0\). That gives two linear equations in two unknowns, whose unique solution is \(x = 1/3, y = 2/3\) (where \(z = -1/3\)). So there is only this one critical point, and it is indeed the location of a (local, hence global) minimum: the Hessian matrix is

\[
\begin{pmatrix}
4 & 4 \\
4 & 10
\end{pmatrix}
\]

with determinant \(24 > 0\) and trace \(14 > 0\). So the point \((x, y, z) = (1/3, 2/3, -1/3)\) is on the plane and closer to the origin than any other.

12. A certain curve is parameterized by \(x = 2t^2, y = \cos(\pi t), z = e^{t-1}\). There is a plane which passes through the origin and also is tangent to this curve at the point \((2, -1, 1)\). Find the equation of this plane.

**Answer:** We pass through that point \(P\) only when \(t = 1\). At that moment, the tangent line points in the direction of the velocity vector \((x', y', z') = (4t, -\pi \sin(\pi t), e^{t-1}) = (4, 0, 1)\).

So the normal vector is perpendicular to both this vector and to the vector \(OP = (2, -1, 1)\), which means the normal vector is parallel to their cross product, \((1, -2, -4)\). Therefore the plane is of the form \(x - 2y - 4z = d\) for some number \(d\) but obviously \(d = 0\) since the plane passes through the origin.

13. Suppose \(w = f(x, y, z)\) where \(f\) is a function with partial derivatives given by \(f_x = ye^x + yz^3, f_y = e^x + xz^3 + z,\) and \(f_z = 3xyz^2 + y\). Also suppose that \(x, y,\) and \(z\) are themselves functions of time \(t\), so that \(w\) also varies with time. If \(x(0) = 0, y(0) = 1, z(0) = 2, x'(0) = 3, y'(0) = 4,\) and \(z'(0) = 5,\) what is \(w'(0)\)?

**Answer:** \(w'(0) = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = 9 \cdot 3 + 3 \cdot 4 + 1 \cdot 5 = 44\

Incidentally, the function \(f\) has to differ by a constant from \(ye^x + xyz^3 + yz\).

14. The surfaces \(xy^3z^4 = 1\) and \(xyz = 1\) intersect at the point \((-1, -1, 1)\). Find the angle between the surfaces at this point. (Hint: that means the angle between the two tangent planes there, which equals the angle between the two planes’ normal lines.)

**Answer:** The gradient to the surface \(xyz = 1\) is \((yz, xz, xy)\), which means in particular that at the point \((-1, -1, 1)\) the vector \((-1, -1, 1)\) points perpendicular to the surface. Likewise the gradient to the other surface at \((x, y, z)\) is \((y^3z^4, 3xyz^3, 4xyz^3)\) and at our
point that vector is \((-1, -3, 4)\). The angle between the surfaces is the angle between these vectors, an angle with cosine \(8/\sqrt{78}\), so that our angle is \(\arccos(8/\sqrt{78})\).

15. Minimize \(x^2 + 2y^2 + 2z^2\) subject to the constraint \(x + y + z = 1\).

**Answer:** You can do this by eliminating \(z\) (say) or use Lagrange Multipliers: the minimum value occurs when \(\nabla f = (2x, 4y, 4z)\) is parallel to \(\nabla g = (1, 1, 1)\). That requires \(2x = 4y = 4z\) (as well as \(x + y + z = 1\)) so \((x, y, z) = (1/2, 1/4, 1/4)\).

In the remaining three problems, evaluate the integral \(\int_A f(x, y) \, dx\,dy\):

16. \(f(x, y) = x^2\) and \(A = \{(x, y) : 1 \leq x \leq 2, 3 \leq y \leq 5\}\).

**Answer:** It will be \((5 - 3) \int_1^2 x^2 \, dx = 14/3\).

17. \(f(x, y) = x^2 + y\) and \(A = \{(x, y) : 0 \leq x \leq 2, -x \leq y \leq x\}\).

**Answer:** This is

\[
\int_0^2 \int_{-x}^x (x^2 + y) \, dy \, dx = \int_0^2 (x^2y + \frac{y^2}{2}) \bigg|_{y=-x}^{y=x} \, dx = \int_0^2 2x^3 \, dx = 8
\]

18. \(f(x, y) = x^2 + y^2\) and \(A = \{(x, y) : 0 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\}\).

**Answer:** Use polar coordinates: the integral equals

\[
\int_0^2 \int_{-\pi/2}^{\pi/2} r^2 \, r \, d\theta \, dr = \pi \int_0^2 r^3 \, dr = 4\pi
\]