I was asked to show how to evaluate the four integrals we saw in today’s class.

1. Compute

\[ \int_{x=1}^{x=4} \frac{1}{x^2} \cos \left( \frac{2}{x} \right) \, dx \]

That first denominator suggests \( u = x^2 \) but that won’t work well. (Try it!)

The input to the cosine term might be a better choice, so let’s let \( u = 2/x \). Then \( du/dx = -2x^{-2} \), i.e. \( dx = (-2x^2/2)du \). So we can remove the \( x \)'s from the \( dx \), then from the integrand, then from the limits of integration, and get simply

\[ \int_{u=1/2}^{u=2} (-1/2) \cos(u) \, du = (1/2) \sin(2) - (1/2) \sin(1/2) \]

(This is a slight correction from an answer I posted in an earlier version!)

Another good choice for the substitution comes from a factor of the integrand. You would be hard-pressed to find a substitution \( u \) for which \( \cos(2/x) \, dx \) would be exactly \( du \), but the other factor works well here: we would be looking for a choice of a \( u \) for which \( (1/x^2) \, dx \) will be exactly \( du \). That requires \( du/dx = 1/x^2 \), i.e. the derivative of this mysterious \( u \) is supposed to be \( 1/x^2 \), which is to say that \( u = \int (1/x^2) \, dx \). Well, that’s not hard to find: letting \( u = -1/x \) would make this true, and as you would see, it’s not a bad choice for a substitution in this particular integral.

By the way, I chose this example in order to make a subtle point you might enjoy. The reason the endpoints look “backwards” when expressed as values of \( u \) instead of as values of \( x \) is because the function \( u = 2/x \) that relates them is a decreasing function (which literally means the bigger of the two inputs \( x \) yields the smaller of the two outputs \( u \)). You know from calculus that decreasing functions are recognizable by their negative derivative, and sure enough \( du/dx = -2/x^2 \) is certainly negative because squares are always positive. You could reverse the limits of integration by tossing in a minus sign, but that same minus sign would then replace this negative derivative with its absolute value, so that we would be deciding that

\[ \int_{x=0}^{x=1} \left| \frac{du}{dx} \right| f(u(x)) \, dx = \int_{u=1/2}^{u=2} f(u) \, du \]

where we match the intervals rather than the endpoints: the first integral is over \([1, 4]\) and the second is over \([1/2, 2]\). I give you this form of the substitution formula now so that when you get to multivariable calculus you’ll understand why the corresponding formula requires absolute value signs that were not used in our course.

2. Compute

\[ \int_{x=0}^{x=1} \sqrt{1 - x^2} \, dx \]
We looked at this very integral on the first day of class: it represents the area of a quarter-circle of radius 1, so it evaluates to $\pi/4$. But we’d like to compute it using substitution. What shall we take for $u$?

Obvious choice: let $u = 1 - x^2$, or equivalently $x = \sqrt{1 - u}$. Then $du/dx = -2x$, i.e. $dx = \left(\frac{-1}{2x}\right) du$. So we can get rid of the $dx$ in favor of $du$ (and while we’re at it, we can change the endpoints) to get something equivalent to the original integral:

$$\frac{-1}{2} \int_{u=0}^{u=1} \sqrt{1 - x^2} \frac{du}{x}$$

We have only to get rid of the remaining $x$s, which leaves us with

$$\frac{1}{2} \int_{u=0}^{u=1} \frac{du}{\sqrt{1 - u}}$$

(I interchanged the endpoints.) This is not horrible but doesn’t look much easier and certainly doesn’t suggest any way to get a final answer.

Instead, we choose to use a “trigonometric” substitution. This is actually a whole separate section of the text that we get to later, so I won’t dwell on it now, but it follows pretty much the same as any other substitution we’ve done, except that we choose our formula for $u$ in terms of $x$ by first choosing our $x$ in terms of $u$. That is, in this problem we observe it would be really helpful if $1 - x^2$ were a perfect square; that would be true if for example $x$ were the sine of something, thanks to that trig identity: if we let $x = \sin(u)$ then $\sqrt{1 - x^2} = \sqrt{1 - \sin^2(u)} = \sqrt{\cos^2(u)} = |\cos(u)|$. Not bad (especially since we’ll get $x$s between 0 and 1 as we need, if we take $u$s between 0 and $\pi/2$, and in that case $\cos(u)$ will be positive and we won’t need the absolute value bars.)

So let’s state exactly that relationship backwards: let $u = \arcsin(x)$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$ and thus $dx = \sqrt{1 - x^2} du$ which (as in the previous paragraph) simplifies to $dx = \cos(u) du$. So our integral has been transformed into

$$\int_{u=0}^{u=\pi/2} \cos(u) \cdot \cos(u) du$$

Now this is a definite integral which is not hard to evaluate: we need the area lying under the graph of $y = \cos^2(u)$ and over $[0, \pi/2]$; by symmetry that’s the same as the area lying under the graph of $y = \sin^2(u)$, and yet the sum of those two (equal) areas is the area under the line $y = 1$, which itself will be $\pi/2$. So

$$\int_{u=0}^{u=\pi/2} \cos^2(u) du = \frac{\pi}{4} \quad \text{(and likewise)} \quad \int_{u=0}^{u=\pi/2} \sin^2(u) du = \frac{\pi}{4}$$

You can also evaluate this last integral using the trig identity $\cos^2(x) = (1+\cos(2x))/2$. 

Remark: in class the second integral I assigned you was actually over the interval $[-1, 1]$ instead of over $[0, 1]$. I changed it here for clarity. You will run into trouble if you try to use $u = 1 - x^2$ as your transformation on the larger interval because you would then not be able to invert this substitution properly: $x = \sqrt{1 - u}$ is only valid on half the interval; you need $x = -\sqrt{1 - u}$ on the other half. (The condition is that there must be a one-to-one correspondence between the $u$s and the $x$s, or equivalently $du/dx$ must be either always positive or always negative.) In general you don’t need to worry about that subtlety but I didn’t want to same something incorrect in a typed document!

3. Compute

$$\int 2^x \, dx$$

It’s generally true that if you see an exponent that involves a variable in any way, you’ll be better off if you can write the whole exponential expression as one in which the base is just “$e$”; remember that $2 = e^{\ln(2)}$ so $2^x = e^{\ln(2)x}$.

Then you need to find an antiderivative but in this case you can simply let $u = \ln(2)x$, so that $du/dx = \ln 2$ and $dx = du/\ln(2)$. Then

$$\int 2^x \, dx = \int e^u \frac{du}{\ln 2} = \frac{e^u}{\ln 2} = \frac{2^x}{\ln(2)} + C$$

4. Compute

$$\int_{x=0}^{x=9} \sqrt{4 - \sqrt{x}} \, dx$$

Let $u = 4 - \sqrt{x}$; then $du/dx = \frac{-1}{2\sqrt{x}}$. Also $x = (4-u)^2$, so $dx = -2\sqrt{x}du = -2(4-u)\, du$ and the limits of integration are $u = 4$ and $u = 1$. Thus the integral becomes

$$\int_{u=4}^{u=1} \sqrt{u}(-2(4-u)) \, du = 2 \int_{u=1}^{u=4} (4u^{1/2} - u^{3/2}) \, du = (16/3)u^{3/2} - (4/5)u^{5/2} \bigg|_{u=1}^{u=4} = 188/15$$

where I used a minus sign to interchange the limits of integration.