1. Find the 10th derivative of \( \frac{6}{x^3 + x^2 - 2x} \)

**ANSWER** Try using the technique of Partial Fractions to write this function as a sum of simple parts. The function turns out to equal
\[
\frac{2}{x - 1} + \frac{1}{x + 2} - \frac{3}{x}
\]
from which we see that the \( n \)th derivative is
\[
(-1)^n \cdot n! \cdot \left( 2 \cdot (x - 1)^{-(n-1)} + (x + 2)^{-(n-1)} - 3 \cdot (x + 2)^{-(n-1)} \right)
\]

2. Sasha Student has prepared poorly for the Calculus test and thinks that for all differentiable functions \( f \) and \( g \) it is true that
\[
\frac{d}{dx} (f(x)g(x)) = f'(x)g'(x)
\]

Amazingly, Sasha used this false result on a particular such product and nonetheless obtained the correct derivative of \( f(x)g(x) \) ! Find a pair \( \{f(x), g(x)\} \) of non-constant functions for which this is possible. (A few extra points will be awarded for finding additional, substantially different, such pairs.)

**ANSWER** We are looking for two functions \( f(x) \) and \( g(x) \) for which
\[
f(x)g'(x) + f'(x)g(x) = f'(x)g'(x)
\]

Divide both sides by \( f'(x)g'(x) \) to see that we require
\[
\frac{f(x)}{f'(x)} + \frac{g(x)}{g'(x)} = 1.
\]
The easiest way to accomplish this is to have each of the summands equal \( 1/2 \), which requires \( f'(x) = 2f(x) \) and \( g'(x) = 2g(x) \). This occurs if (and only if) each of the two functions is a multiple of \( e^{2x} \).

Many other solutions are possible: we can have
\[
\frac{f(x)}{f'(x)} = \left( \frac{1}{2} \right) (1 + h(x)), \quad \frac{g(x)}{g'(x)} = \left( \frac{1}{2} \right) (1 - h(x))
\]
for any function $h$. You may recognize the expression $f'(x)/f(x)$ from “logarithmic differentiation”: it’s the derivative of $F(x) = \ln(|f(x)|)$. So we need to compute functions $F$ and $G$ which satisfy $F'(x) = 2/(1 + h(x))$ and $G'(x) = 2/(1 - h(x))$, from which we will obtain our answers:

$$f(x) = e^2 \int dx/(1+h(x)), \quad g(x) = e^2 \int dx/(1-h(x))$$

There are many kinds of solutions you could try. For example, if $h(x) = kx$ for some constant $k$ then the solutions are

$$f(x) = (1 + kx)^{2/k}, \quad g(x) = (1 - kx)^{-2/k}$$

e.g. $\{(1 + 2x), 1/(1 - 2x)\}$. Using Partial Fractions you can compute solutions whenever $h(x)$ is any rational function, e.g. for $h(x) = x^2$ we obtain $\{(1 + x)/(1 - x), e^{2\arctan(x)}\}$.

And you have the skills to handle certain transcendental functions as well, e.g. when $h(x) = \tan(x)$ we obtain the solution $\{e^x(\cos(x) + \sin(x)), e^x/(\cos(x) - \sin(x))\}$. Other solutions noted by students include $\{x, 1/(1 - x)\}$ and $\{e^x(x - 1), e^{x^2/2}\}$.

3. The equation $x = 2y + 3y^2 + 4y^3$ defines $y$ implicitly as a function of $x$. (That is, the graph of this equation is the graph of some function $y = f(x)$.) Compute the 0th through 3rd terms of the Taylor series of this function at the origin.

**ANSWER** We can compute the derivatives of $y$ by implicit differentiation: since $\frac{dx}{dy} = 2 + 6y + 12y^2$, we conclude $dy/dx = 1/(2 + 6y + 12y^2)$. Then differentiate using the Chain Rule: $d^2y/dx^2 = -(6 + 24y)/(2 + 6y + 12y^2)^2 \cdot (dy/dx) = -(6 + 24y)/(2 + 6y + 12y^2)^3$ and similarly (albeit with more effort) we can compute

$$\frac{d^3y}{dx^3} = \frac{15(24y^2 + 12y + 1)}{8(6y^2 + 3y + 1)^5}$$

Evaluating all these for $x = 0$ (i.e. for $y = 0$) we find

$$f(0) = 0, \quad f'(0) = \frac{1}{2}, \quad f''(0) = \frac{-3}{4}, \quad f'''(0) = \frac{15}{8}$$

so $y = \frac{1}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$.

It is probably easier to use the method of undetermined coefficients, though: write $y = ax + bx^2 + cx^3 + dx^4 + \ldots$, substitute into the defining equation for $y$, expand, and collect powers of $x$. This leads to the equations

$$2a - 1 = 0, \quad 3a^2 + 2b = 0, \quad 4a^3 + 6ab + 2c = 0, \quad 12a^2b + 6ac + 3b^2 + 2d = 0, \quad \ldots$$
which can be solved successively for the unknown coefficients to obtain

\[ a = 1/2, \quad b = -3/8, \quad c = 5/16, \quad d = -15/128, \ldots \]

4. For what values of \( x \) does this series converge?

\[
\sum_{n=1}^{\infty} \frac{n^n x^{(n^2)}}{n!} = x + 2x^4 + \frac{9}{2}x^9 + \frac{32}{3}x^{16} + \ldots
\]

**ANSWER** The Ratio Test as usually presented for power series does not apply because so many coefficients are zero (that is, \( \lim |a_{n+1}/a_n| \) does not exist). But we may still use the Ratio Test for each \( x \), just thinking of this as a series whose \( n \) term is as given. That is, the series converges if \( L < 1 \) where \( L \) is

\[
\lim_{n \to \infty} \left| \frac{(n+1)^{n+1} x^{(n+1)^2}}{(n+1)!} \cdot \frac{n^n x^{(n^2)}}{n!} \right| = \lim_{n \to \infty} \left| \left( 1 + \frac{1}{n} \right)^n x^{2n+1} \right| = e \lim_{n \to \infty} \left| x^{2n+1} \right|
\]

If \( |x| < 1 \), the last limit is zero, so \( L = 0 \) and the series converges. If \( |x| > 1 \), then \( L = \infty \) so the series diverges. If \( |x| = 1 \) then the limit is \( L = e > 1 \) so the series diverges again. Thus the series converges for \( x \) in \((-1, 1)\).

5. For what values of \( k \) does \( f(x, y) = \frac{x^k y}{x^6 + y^2} \) have a (finite) limit as \( (x, y) \to (0, 0) \)?

**ANSWER** Since \( ab/(a^2 + b^2) < 1/2 \) for all real \( a, b \), we see that for any \( k > 3 \) we have \( f(x, y) < |x|^{k-3}/2 \to 0 \). For \( k \leq 3 \) we consider the limit along the curves \( y = rx^3 \): \( f(x, y) = rx^{k-3}/(1 + r^2) \) there, which converges to \( \infty \) if \( k < 3 \), and which has different limits for different \( r \) if \( k = 3 \). So the limit exists if and only if \( k > 3 \).

Alternatively, replace \( x \) with a different coordinate \( u = x^3 \); in terms of \( u \) and \( y \) the question asks for the limit of \( (u^{k/3} y)/(u^2 + y^2) \) as \( (u, y) \to (0, 0) \). This in turn can be addressed by using polar coordinates in the \( u, y \) plane: we need the limit of \( r^{(k/3)-1} \cos(\theta) \sin(\theta) \) as \( r \to 0 \). This limit is zero if \( k > 3 \), and undefined for \( k \leq 3 \).