• Folland Section 7.3, problems 8, 9, and 10 (copied below). There is a typo in problem 10: $f' + cf = 0$ should be $f'(x) + cf(x) = 0$.

8. Suppose $f \in L^2(\mathbb{R})$, $\hat{f}(\omega) = 0$ for $|\omega| > \Omega$, and $\lambda > 1$.
   a. As in the proof of the sampling theorem, show that
   \[
   \hat{f}(\omega) = \frac{\pi}{2\Omega} \sum_{n=-\infty}^{\infty} f \left( \frac{\pi n}{\Omega} \right) e^{-i\pi nx/\Omega} \quad \text{for } |\omega| \leq \lambda\Omega.
   \]
   b. Let $\tilde{g}_l$ be the piecewise linear function sketched below. Show that the inverse Fourier transform of $\tilde{g}_l$ is
   \[
   g_l(t) = \frac{\cos \Omega t - \cos \frac{i\pi}{\Omega}}{\pi(n-1)\Omega^2}.
   \]
   c. Observe that $\tilde{f} = \tilde{g}_l \tilde{f}$. By substituting the expansion in part (a) into the Fourier inversion formula, show that
   \[
   f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega) \tilde{g}_l(\omega) e^{i\omega t} d\omega = \frac{\pi}{2\Omega} \sum_{n=-\infty}^{\infty} f \left( \frac{\pi n}{\Omega} \right) g_l \left( t - \frac{\pi n}{\Omega} \right).
   \]
   This gives a sampling formula for $f$ in which the basic functions $g_l(t)$ decay like $t^{-2}$ at infinity.

9. Suppose that $f$ satisfies the hypotheses of Heisenberg’s inequality, and let $F(x) = e^{-i\lambda x} f(x + a)$.
   a. Show that $\Delta_\alpha f = \Delta_\alpha F$.
   b. Show that $\tilde{F}(\xi) = e^{i\xi a} \tilde{f}(\xi + a)$ and thence that $\Delta_\alpha \tilde{f} = \Delta_\alpha \tilde{F}$.

10. Show that Heisenberg’s inequality ($\Delta_\alpha f(\Delta_\alpha \tilde{f}) \geq \frac{1}{4}$) is an equality if and only if $f + cf = 0$ where $c$ is a real constant, and hence show that the functions that minimize the uncertainty product ($\Delta_\alpha f(\Delta_\alpha \tilde{f})$) are precisely those of the form $f(x) = Ce^{-i\xi x}$ for some $c > 0$. (Hint: Examine the proof of Heisenberg’s inequality and recall that the Cauchy-Schwarz inequality $|\langle f, g \rangle| \leq \|f\| \|g\|$ is an equality if and only if $f$ and $g$ are scalar multiples of one another.) What are the minimizing functions for the uncertainty product ($\Delta_\alpha f(\Delta_\alpha \tilde{f})$) for general $\alpha, \alpha^*$? (Cf. Exercise 9.)

• Folland Section 3.3, problem 4: Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a, b)$. Suppose $c > 0$ and $d \in \mathbb{R}$, and let $\psi_n(x) = e^{i/2} \phi_n(cx + d)$. Show that $\{\psi_n\}$ is an orthonormal basis for $L^2(\frac{a-d}{c}, \frac{b-d}{c})$.

• Folland Section 3.3, problem 11: Suppose $f$ is of class $C^{(1)}$, 2$\pi$-periodic, and real-valued. Show that $f'$ is orthogonal to $f$ in $L^2(-\pi, \pi)$ in two ways: (a) by expanding $f$ in a Fourier series and using Parseval’s Theorem: $\langle f, g \rangle = \sum (f, \phi_n)(g, \phi_n)$, (b) directly from the fact that $2ff' = (f'^2)'$. 

Folland 2.6, problem 1 (Gibbs’ phenomenon) (copied below). The referenced equations (2.10) and (2.12) are that the $N$th Dirichlet kernel is given by

$$D_N(\phi) = \frac{1}{2\pi} \sum_{n=-N}^{n=N} e^{in\phi} = \frac{1}{2\pi} \frac{\sin((N + \frac{1}{2})\phi)}{\sin(\frac{1}{2}\phi)}$$

Figure 2.8. Graph of $2\sum_{n=1}^{\infty} n^{-1} \sin n\theta$, $-2\pi < \theta < 2\pi$ (an illustration of the Gibbs phenomenon).

**Exercise**

1. Recall from Table 1, §2.1, that $f(\theta) = 2\sum_{n=1}^{\infty} n^{-1} \sin n\theta$ is the $2\pi$-periodic function that equals $\pi - \theta$ for $0 < \theta < 2\pi$. Let

$$g_N(\theta) = \frac{1}{2} \sum_{n=1}^{N} \frac{\sin n\theta}{n} - (\pi - \theta),$$

so that $g(\theta)$ is the difference between $f(\theta)$ and its $N$th partial sum for $0 < \theta < 2\pi$.

   a. Show that $g_N'\theta) = 2\pi D_N(\theta)$ where $D_N$ is the Dirichlet kernel (2.10).

   b. Using (2.12), show that the first critical point of $g_N(\theta)$ to the right of zero occurs at $\theta_N = \pi/(N + \frac{1}{2})$, and that

$$g_N(\theta_N) = \int_{0}^{\theta_N} \sin(n + \frac{1}{2})\theta \sin \frac{1}{2}\theta d\theta - \pi.$$

   c. Show that

$$\lim_{N \to \infty} g_N(\theta_N) = 2 \int_{0}^{\pi} \frac{\sin \phi}{\phi} d\phi - \pi.$$

(Hint: Let $\phi = (N + \frac{1}{2})\theta$.) This limit is approximately equal to .562. Thus the difference between $f(\theta)$ and the $N$th partial sum of its Fourier series develops a spike of height .562 (but of increasingly narrow width) just to the right of $\theta = 0$ as $N \to \infty$. (There is another such spike on the left.)