# Interpolation via weighted $\ell_1$ minimization

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December 12, 2014

Joint work with Holger Rauhut (Aachen University)

# Function interpolation

Given a function $f : \mathcal{D} \to \mathbb{C}$ on a domain $\mathcal{D}$, interpolate or approximate $f$ from sample values $y_1 = f(u_1), \ldots, y_m = f(u_m)$.

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Assume the form

$$f(u) = \sum_{j \in \Gamma} x_j \psi_j(u), \qquad |\Gamma| = N$$

Approaches:

- Standard interpolation: Choose $m = N$. Find appropriate $\Gamma$ and sampling points $u_1, u_2, \ldots, u_m$.
- Least squares regression: Choose $N < m$; minimize $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2$ where $A_{\ell,j} = \psi_j(u_\ell)$ is the *sampling matrix*.
- Compressive sensing methods: Choose $N > m$. Exploit approximate sparsity of coefficient vector $\mathbf{x}$ to solve the underdetermined system $\mathbf{y} = \mathbf{A}\mathbf{x}$.

# Orthonormal systems

$L_2$-normalized Legendre polynomials

$\mathcal{D}$: domain endowed with a probability measure $\nu$.

$\psi_j : \mathcal{D} \to \mathbb{C}$, $j \in \Gamma$ (finite or infinite)

$\{\psi_j\}_{j\in\Gamma}$ is an orthonormal system: $\int_{\mathcal{D}} \psi_j(t)\overline{\psi_k(t)}d\nu(t) = \delta_{j,k}$

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- Trigonometric system: $\psi_j(t) = e^{2\pi ijt}$,
  $\mathcal{D} = [0,1], d\nu(t) = dt, \|\psi_j\|_\infty \le 1$.

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- $L_2$-normalized Legendre polynomials $L_j$:
  $\mathcal{D} = [-1,1], d\nu(t) = dt, \quad \|L_j\|_\infty = \sqrt{2j+1}$.

# Orthonormal systems

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- Tensor products of Legendre polynomials:
  $\mathcal{D} = [-1,1]^d$, $\mathbf{j} = (j_1, j_2, \ldots, j_d)$, $L_{\mathbf{j}} = \prod_{k=1}^{d} L_{j_k}$,
  $\|L_{\mathbf{j}}\|_\infty = \prod_{k=1}^{d} \sqrt{2j_k + 1}$.

In high-dimensional problems, smoothness is not enough to avoid curse of dimension – too local! We will combine smoothness and sparsity.

# Smoothness and weights

In general, $\|f\|_{L^\infty} + \|f'\|_{L^\infty}$ promotes smoothness

Consider $\psi_j(t) = e^{2\pi i j t}$, $j \in \mathbb{Z}$, $t \in [0,1]$

Derivatives satisfy $\|\psi_j'\|_\infty = 2\pi|j|$, $j \in \mathbb{Z}$.

For $f(t) = \sum_j x_j \psi_j(t)$,

$$\begin{aligned}
\|f\|_{L^\infty} + \|f'\|_{L^\infty} &= \|\sum_j x_j \psi_j\|_{L^\infty} + \|\sum_j x_j \psi_j'\|_\infty \\
&\leq \sum_{j\in\mathbb{Z}} |x_j|(\|\psi_j\|_{L^\infty} + \|\psi_j'\|_{L^\infty}) \\
&= \sum_{j\in\mathbb{Z}} |x_j|(1 + 2\pi|j|) =: \|\mathbf{x}\|_{\omega,1}.
\end{aligned}$$

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Weighted $\ell_1$-coefficient norm promotes smoothness. It also promotes sparsity!

# Weighted norms and weighted sparsity

Weighted $\ell_1$ norm: pay a higher price to include certain indices

$$\|\mathbf{x}\|_{\omega,1} = \sum_{j\in\Gamma} \omega_j |x_j|$$

New: weighted sparsity:

$$\|\mathbf{x}\|_{\omega,0} := \sum_{j:x_j\neq 0} \omega_j^2$$

$\mathbf{x}$ is called *weighted $s$-sparse* if $\|\mathbf{x}\|_{\omega,0} \leq s$.


Weighted best $s$-term approximation error:
$\sigma_s(\mathbf{x})_{\omega,1} := \inf_{\mathbf{z}:\|\mathbf{z}\|_{\omega,0}\leq s} \|\mathbf{x}-\mathbf{z}\|_{\omega,1}$

# (Weighted) Compressive Sensing

Recover a weighted $s$-sparse (or weighted-compressible) vector $\mathbf{x}$ from measurements $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} \in \mathbb{C}^{m \times N}$ with $m < N$.

Weighted $\ell_1$-minimization

$$\min_{\mathbf{z} \in \mathbb{C}^N} \|\mathbf{z}\|_{\omega,1} \quad \text{subject to } \mathbf{A}\mathbf{z} = \mathbf{y}$$

"Noisy" version

$$\min_{\mathbf{z} \in \mathbb{C}^N} \|\mathbf{z}\|_{\omega,1} \quad \text{subject to } \|\mathbf{A}\mathbf{z} - \mathbf{y}\|_2 \leq \eta$$

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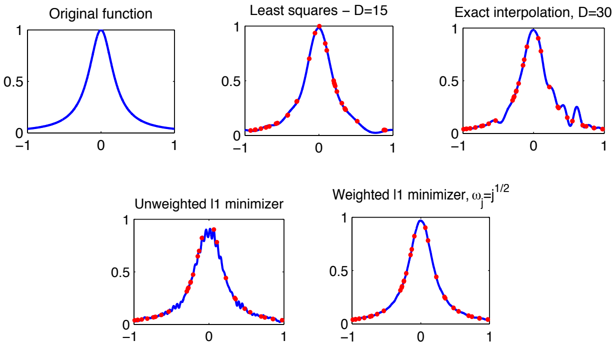
Keep in mind sampling matrix:

$$A = \begin{pmatrix} \psi_1(u_1) & \psi_2(u_1) & \dots & \dots & \psi_N(u_1) \\ \psi_1(u_2) & \psi_2(u_2) & \dots & \dots & \psi_N(u_2) \\ \vdots & \vdots & & & \vdots \\ \psi_1(u_m) & \psi_2(u_m) & \dots & \dots & \psi_N(u_m) \end{pmatrix}$$

# Numerical example - comparing different reconstructions




- $f(u) = \frac{1}{1+25u^2}$. Draw $m = 30$ sampling points $u_\ell$ i.i.d. from uniform measure on $[-1, 1]$.
- Interpolate (exactly or approximately) the samples $y_\ell = f(u_\ell)$ by various choices of $\{x_j\}$ in

$$f^\#(u) = \sum_{j=0}^{80} x_j e^{\pi i j u}$$

*Stability of unweighted $\ell_1$ minimization given *exact* sparsity: Rauhut, W. '09
*Stability of least squares regression: Cohen, Davenport, Leviatan 2011

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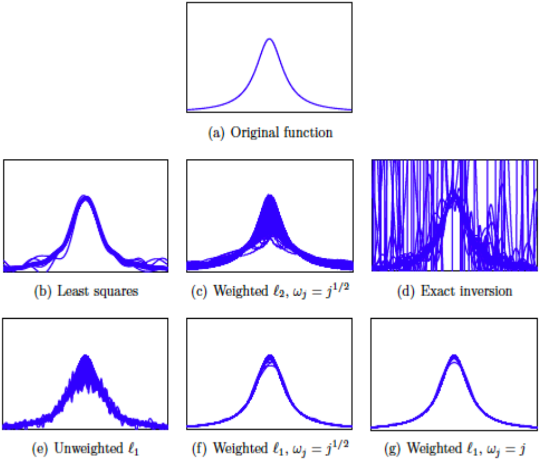


Different trials correspond to different random draws of $m = 30$ sampling points from uniform measure on $[-1, 1]$.

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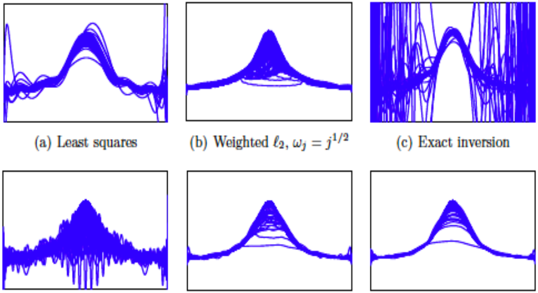


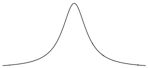
(a) Least squares (b) Weighted $\ell_2$, $\omega_j = j^{1/2}$ (c) Exact inversion

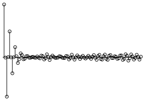
(d) Unweighted $\ell_1$ (e) Weighted $\ell_1$, $\omega_j = j^{1/2}$ (f) Weighted $\ell_1$, $\omega_j = j$

Same experiment, but now interpolating/approximating by Legendre polynomials, and sampling points are from Chebyshev measure on $[-1, 1]$, $d\nu(u) = \frac{du}{\pi\sqrt{1-u^2}}$.

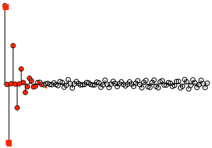
# What is going on?

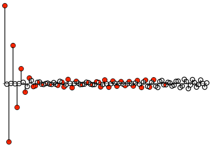
Runge's function and its Legendre polynomial coefficients

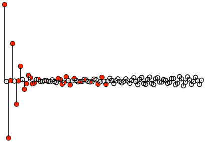




Compare coefficient indices picked up by various reconstruction methods:







Least squares | unweighted $\ell_1$ minimization | weighted $\ell_1$ minimization

# Back to Compressive Sensing setting

Recover a weighted $s$-sparse (or weighted-compressible) vector $\mathbf{x}$ from measurements $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} \in \mathbb{C}^{m \times N}$ with $m < N$.

Weighted $\ell_1$-minimization

$$\min_{\mathbf{z} \in \mathbb{C}^N} \sum_{j=1}^{N} \omega_j |z_j| \quad \text{subject to } \mathbf{A}\mathbf{z} = \mathbf{y}$$

Keep in mind sampling matrix:

$$\mathbf{A} = \begin{pmatrix} \psi_1(u_1) & \psi_2(u_1) & \dots & \dots & \psi_N(u_1) \\ \psi_1(u_2) & \psi_2(u_2) & \dots & \dots & \psi_N(u_2) \\ \vdots & \vdots & & & \vdots \\ \psi_1(u_m) & \psi_2(u_m) & \dots & \dots & \psi_N(u_m) \end{pmatrix}$$

# Weighted restricted isometry property (WRIP)

## Definition (with H. Rauhut '13)

The weighted restricted isometry constant $\delta_{\omega,s}$ of a matrix $\mathbf{A} \in \mathbb{C}^{m \times N}$ is defined to be the smallest constant such that

$$(1 - \delta_{\omega,s})\|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\|_2^2 \leq (1 + \delta_{\omega,s})\|\mathbf{x}\|_2^2$$

for all $\mathbf{x} \in \mathbb{C}^N$ with $\|\mathbf{x}\|_{\omega,0} = \sum_{\ell : x_\ell \neq 0} \omega_j^2 \leq s$.

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Unweighted "uniform" restricted isometry property (Candès, Tao '05): $\omega \equiv 1$.

Since we assume $\omega_j \geq 1$, WRIP is weaker than "uniform" RIP

Related: Model-based compressive sensing (Baraniuk, Cevher, Duarte, Hedge, Wakin '10)

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Weights allow us to analyze sampling matrices from function systems with unbounded $\|\cdot\|_\infty$ norm. Uniform RIP does not hold for such matrices!

$$\begin{pmatrix} \psi_1(u_1) & \psi_2(u_1) & \dots & \dots & \psi_N(u_1) \\ \psi_1(u_2) & \psi_2(u_2) & \dots & \dots & \psi_N(u_2) \\ \vdots & \vdots & & & \vdots \\ \psi_1(u_m) & \psi_2(u_m) & \dots & \dots & \psi_N(u_m) \end{pmatrix}$$

# Weighted RIP of random sampling matrix

$\psi_j : \mathcal{D} \to \mathbb{C}$, orthonormal system w.r.t. prob. measure $\nu$. and $\|\psi_j\|_\infty \leq \omega_j$.

Sampling points $u_1, \ldots, u_m$ taken i.i.d. at random according to $\nu$.
Random sampling matrix $\mathbf{A} \in \mathbb{C}^{m \times N}$ with entries $A_{\ell j} = \psi_j(u_\ell)$.

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Fix $s$, and choose $N$ so that $\omega_1, \omega_2, \ldots, \omega_N \leq s/2$.

Theorem (with H. Rauhut, '13)

*If*

$$m \geq C\delta^{-2} s \log^3(s) \log(N)$$

*then the weighted restricted isometry constant of* $\frac{1}{\sqrt{m}}\mathbf{A}$ *satisfies* $\delta_{\omega,s} \leq \delta$ *with high probability.*

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Generalizes previous results (Candès, Tao, Rudelson, Vershynin) for uniformly bounded systems, $\|\psi_j\|_\infty \leq K$ for all $j$.

# Recovery via weighted $\ell_1$-minimization

Theorem (with H. Rauhut)

*Let* $\mathbf{A} \in \mathbb{C}^{m \times N}$ *have the WRIP with weights* $(\omega)$ *and* $\delta_{\omega,3s} < 1/3$. *For* $\mathbf{x} \in \mathbb{C}^N$ *and* $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$ *with* $\|\mathbf{e}\|_2 \leq \eta$ *let* $\mathbf{x}^\sharp$ *be a minimizer of*

$$\min \|\mathbf{z}\|_{\omega,1} \quad \text{subject to } \|\mathbf{A}\mathbf{z} - \mathbf{y}\|_2 \leq \eta.$$

*Then*

$$\|\mathbf{x} - \mathbf{x}^\sharp\|_{\omega,1} \leq C_1 \sigma_s(\mathbf{x})_{\omega,1} + D_1 \sqrt{s}\eta$$

Generalizes unweighted $\ell_1$ minimization results (Candès, Romberg, Tao '06, Donoho '06).

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Should be of independent interest in structured sparse recovery problems / sparse recovery problems with nonuniform priors on support

# Prior work on weighted $\ell_1$ minimization

Weighted $\ell_1$ norm: $\|\mathbf{x}\|_{\omega,1} = \sum_{j\in\Gamma} \omega_j |x_j|$

Many previous works on the analysis of weighted $\ell_1$ minimization! Focused on the finite-dimensional setting, and with analysis based on unweighted sparsity

- von Borries, Miosso, and Potes 2007
- Vaswani and Lu 2009, Jacques 2010, Xu 2010
- Khajehnejad, Xu, Avestimehr, and Hassibi (2009 & 2010)
- Friedlander, Mansour, Saab, and Yilmaz 2012, Misra and Parrilo 2013
- Peng, Hampton, Doostan 2013 - sparse polynomial chaos approximation of high-dimensional stochastic functions

# Back to function interpolation

Suppose that $|\Gamma| = N < \infty$.

Given samples $y_1 = f(u_1), \ldots, y_m = f(u_m)$ of

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reconstruction amounts to solving $\mathbf{y} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} \in \mathbb{C}^{m \times N}$ is the sampling matrix $A_{\ell,j} = \psi_j(u_\ell)$

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- Previous analysis suggests: use weighted $\ell_1$-minimization with weights $\omega_j \geq \|\psi_j\|_{L^\infty}$ or $\omega_j \geq \|\psi_j'\|_{L^\infty}$ to recover weighted-sparse or weighted-compressible $\mathbf{x}$ when $m < N$.
- Choose $u_1, \ldots, u_m$ i.i.d. at random according to $\nu_\psi$ in order to analyze WRIP of sampling matrix.
- $\mathbf{x}$ will not be exactly sparse. Measure residual error $f - f^\#$ in which norm? Recall $\|g\|_{L^\infty} + \|g'\|_{L^\infty} \leq \|\!| g |\!\|_{\omega,1}$ if $\omega_j \geq \|\psi_j'\|_{L^\infty}$

# Interpolation via weighted $\ell_1$ minimization

Suppose $f(u) = \sum_{j \in \Gamma} x_j \psi_j(u), \quad |\Gamma| = N < \infty$

Set weights $\omega_j > \|\psi_j\|_\infty$

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**Theorem**

*From* $m \asymp s \log^3(s) \log(N)$ *samples* $y_1 = f(u_1), \ldots, y_m = f(u_m)$ *where* $u_j \sim \nu_\psi$, *if* $\mathbf{x}^\sharp$ *is the solution to*

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*and* $f^\sharp(u) = \sum_{j \in \Gamma} x_j^\sharp \psi_j(u)$,

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*and* $f^\sharp(u) = \sum_{j\in\Gamma} x_j^\sharp \psi_j(u)$, *then with high probability,*

$$\|f - f^\sharp\|_\infty \leq \|\!| f - f^\sharp \|\!|_{\omega,1} \leq C_1 \sigma(f)_{\omega,1}$$

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More realistic: $|\Gamma| = \infty$. How to pass to a finite dimensional approximation in a principled way?

# Weighted function spaces

$$\mathcal{A}_{\omega,p} = \{f : f(u) = \sum_{j \in \Gamma} x_j \psi_j(u), \; |||\, f \,|||_{\omega,p} := \|\mathbf{x}\|_{\omega,p} < \infty\}$$

Interesting range: $0 \leq p \leq 1$

$\mathcal{A}_{\omega,p} \subset \mathcal{A}_{\omega,q}$ if $p < q$

Solving $m \asymp s \log^3(s) \log(N)$ for $s$, a Stechkin-type error bound yields

$$\|f - f^\sharp\|_{L^\infty} \leq |||\, f - f^\sharp \,|||_{\omega,1} \leq C_1 \left( \frac{\log(N)^4}{m} \right)^{1/p-1} |||\, f \,|||_{\omega,p}$$

# Approximation in infinite-dimensional spaces

Suppose $|\Gamma| = \infty$, $\lim_{|j|\to\infty} \omega_j = \infty$ and $\omega_j > \|\psi_j\|_\infty$.

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*Suppose $f \in \mathcal{A}_{\omega,p}$ for some $0 < p < 1$. Fix $s$, and set $\Gamma_s = \{j \in \Gamma : \omega_j^2 \leq s/2\}$.*

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*Draw* $m \geq Cs\log^3(s)\log(|\Gamma_s|)$ *samples* $y_1 = f(u_1), \dots, y_m = f(u_m)$ *according to* $\nu_\psi$, *and let* $\mathbf{x}^\sharp$ *be the solution to*

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# Approximation in infinite-dimensional spaces

Suppose $|\Gamma| = \infty$, $\lim_{|j|\to\infty} \omega_j = \infty$ and $\omega_j > \|\psi_j\|_\infty$.

## Theorem

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$$\|f - f^\sharp\|_{L^\infty} \leq \||\, f - f^\sharp \||_{\omega,1} \leq C_1 \left(\frac{\log^4(N)}{m}\right)^{1/p-1} \||\, f \||_{\omega,p}$$

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**Using greedy alternative such as weighted iterative hard thresholding, do not need to know $\sqrt{m/s}\, \||\, f - f_{\Gamma_s} \||_{\omega,1}$, get same bound

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Important: For domains $\mathcal{D} = [-1, 1]^d$, don't want number of measurements $m$ in resulting bound to grow much with $d$.

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$L^2$-normalized Chebyshev polynomials

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Tensorized Chebyshev polynomials

$$T_{\mathbf{k}}(u) = \prod_{j=1}^{d} T_{k_j}(u_j) = \prod_{j \in \operatorname{supp} \mathbf{k}} T_{k_j}(u_j), \quad u = (u_j)_{j \geq 1}, \mathbf{k} \in \mathcal{F},$$

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Product probability measure

$$\eta = \bigotimes_{j\geq 1} \frac{du_j}{\pi\sqrt{1-u_j^2}}, \quad \in \mathcal{D}.$$

$\{T_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{F}}$ forms orthonormal basis for $L^2(\mathcal{D}, d\eta)$.

# Sparse recovery for tensorized Chebyshev polynomials

$L^\infty$-bound for the tensorized Chebyshev polynomials $T_{\mathbf{k}}$:

$$\|T_{\mathbf{k}}\|_\infty = 2^{\|\mathbf{k}\|_0/2}.$$

Choice of weights:

$$\omega_{\mathbf{k}} = \bigotimes_{j=1}^{d} (k_j + 1)^{1/2} \geq 2^{\|\mathbf{k}\|_0/2}$$

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Encourages both *sparse* and *low-order* tensor products

Subset of indices forms a hyperbolic cross:

$$\Lambda_0 = \{\mathbf{k} \in \mathbb{N}_0^d : \omega_{\mathbf{k}}^2 \leq s\}$$

[Kuhn, Sickel, Ulrich, 2014, Cohen, DeVore, Foucart, Rauhut 2011]:

$$|\Lambda_0| \leq e^d s^{2+\log(d)}$$

# Apply weighted $\ell_1$ theory:

Consider a function

$$f = \sum_{\mathbf{k}\in\Lambda} x_{\mathbf{k}} T_{\mathbf{k}} \quad \text{on } [-1,1]^d, \quad f \in \mathcal{A}_{\omega,p}$$

- Fix $s$, and fix $\Lambda_0 = \{\mathbf{k} \in \mathbb{N}^d : \omega_{\mathbf{k}}^2 \leq s\}$ Reduced basis
- Fix number of samples $m \geq Cs \log^4(s) \log(d)$
- Draw $m$ samples $y_\ell = f(u_\ell)$ according to Chebyshev measure
- Let $\mathbf{A} \in \mathbb{C}^{m\times N}$ be sampling matrix with entries $A_{\ell,\mathbf{k}} = T_{\mathbf{k}}(u_\ell)$.
- Let $\mathbf{x}^\#$ be solution of

$$\min \|\mathbf{x}\|_{\omega,1} \text{ subject to } \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2 \leq \sqrt{m/s}\, \||\, f - f_{\Lambda_0} \||_{\omega,1}$$

and set $f^\# = \sum_{\mathbf{k}\in\Lambda_0} x_{\mathbf{k}}^\# T_{\mathbf{k}}$.

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Then with probability exceeding $1 - e^{-d\log^3(s)}$,

$$\|f - f^{\sharp}\|_{L^\infty} \leq C_1 \left(\frac{\log^4(s)\log(d)}{m}\right)^{1/p-1} |\!|\!|\, f \,|\!|\!|_{\omega,p}$$

# Limitations of weighted $\ell_1$ approach

In the previous example,

$$N = |\Lambda_0| = e^d s^{2+\log_2(d)}$$

Weighted $\ell_1$ minimization as a reconstruction method on such large scale problems is impractical

Fix $\tilde{s} \ll s$. Least squares projection onto

$$\tilde{N} = e^d \tilde{s}^{2+\log_2(d)}$$

is faster, but too greedy

Can we meet in the middle?

# Summary

- We introduced weighted $\ell_1$ minimization for stable and robust function interpolation, as taking into account both sparsity and smoothness present in natural functions of interest
- Along the way, we extended the notion of *restricted isometry property* to *weighted restricted isometry property*, a more mild condition that allows us to treat function systems with increasing $\|\cdot\|_\infty$ norm
- Weighted $\ell_1$ minimization can overcome curse of dimension w.r.t. *number of samples* in high-dimensional approximation problems.

Extensions:

- We observe empirically that weighted $\ell_1$ minimization is "faster" than unweighted $\ell_1$ minimization. The steeper the weights, the faster. Justificaton?
- We observe that, with error on measurements $y = f(u) + \xi$, reconstruction results are similar, *provided* that the regularization parameter is chosen correctly in regularization methods. Equality constrained $\ell_1$ minimization, weighted or not, leads to overfitting. Why?

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