1. Let $V$ be the subspace of $\mathbb{R}_3$ consisting of polynomials $p$ with $p(0) = 0$. Let $b_1 = -t + t^2$, $b_2 = t + t^2 + t^3$, $b_3 = -7t - 5t^2 + 2t^3$. Is the set \{b_1, b_2, b_3\} linearly independent? Does it span $V$? Is it a basis for $V$?

Solution: Note that $V$ is a 3-dimensional space with basis \{t, t^2, t^3\}. Relative to this basis, our three vectors have coordinates $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} -7 \\ -5 \\ 2 \end{pmatrix}$. Since the $3 \times 3$ matrix $\begin{pmatrix} -1 & 1 & -7 \\ 1 & 1 & -5 \\ 0 & 1 & 2 \end{pmatrix}$ is invertible (just row reduce it, or take its determinant), the columns form a basis for $\mathbb{R}^3$, hence the original vectors form a basis for $V$ (and are linearly independent and span, of course).

2. In $\mathbb{R}_2[t]$, let $b_1(t) = 1 + t + t^2$, $b_2(t) = 2 + 3t + t^2$, $b_3(t) = 1 + 2t + t^2$, and $v(t) = 5 - 2t + 3t^2$. Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis. Find $P_{\mathcal{E}B}$, $P_{\mathcal{B}E}$, and $[v]_B$.

Solution:

\[
P_{\mathcal{E}B} = \begin{pmatrix} [b_1]_\mathcal{E} & [b_2]_\mathcal{E} & [b_3]_\mathcal{E} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}
\]

\[
P_{\mathcal{B}E} = P_{\mathcal{E}B}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}
\]

\[
[v]_\mathcal{E} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}; \quad [v]_\mathcal{B} = P_{\mathcal{B}E}[v]_\mathcal{E} = \begin{pmatrix} 10 \\ 2 \\ -9 \end{pmatrix}.
\]

You should double-check that $v$ really does equal $10b_1 + 2b_2 - 9b_3$. 

M346 First Midterm Exam Solutions, September 21, 2000

The exam is closed book, but you may have a single hand-written 8.5 x 11 crib sheet. There are 4 problems, each worth 25 points. I hope to have the exam returned to you next Thursday.

Good luck!
3. On \( \mathbb{R}_3[t] \), let \( L \) be the linear operator that shifts a function over to the left by one. That is, \((Lp)(t) = p(t + 1)\). Find the matrix of \( L \) relative to the standard basis \( \{1, t, t^2, t^3\} \).

Solution: Note that 
\[
L(b_1) = 1, \quad L(b_2) = 1 + t, \quad L(b_3) = (1 + t)^2 = 1 + 2t + 1, \\
L(b_4) = (1 + t)^3 = 1 + 3t + 3t^2 + t^3.
\]

Take the coordinates of these to get the columns of 
\[
[L]_B = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

4. In \( \mathbb{R}_2[t] \), let \( b_1(t) = 1 + t + t^2, \quad b_2(t) = 2 + 3t + t^2, \quad b_3(t) = 1 + 2t + t^2, \) as in problem 2. Let \( L = \frac{d}{dt} \) be the derivative operator. Find the matrix of \( L \) relative to the basis \( B \). [You may find your answers to problem 2 to be useful.]

Solution: Let \( E \) be the standard basis, and compute 
\[
[L]_E = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{pmatrix}.
\]

Then 
\[
[L]_B = P_{BE}[L]_E P_{EB} = \\
\begin{pmatrix}
1 & -1 & 1 \\
1 & 0 & -1 \\
-2 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 1 \\
1 & 3 & 2 \\
1 & 1 & 1
\end{pmatrix} = \\
\begin{pmatrix}
-1 & 1 & 0 \\
1 & 3 & 2 \\
0 & -4 & -2
\end{pmatrix}.
\]