M346 First Midterm Exam Solutions, September 21, 2000

The exam is closed book, but you may have a single hand-written 8.5×11 crib sheet. There are 4 problems, each worth 25 points. I hope to have the exam returned to you next Thursday.

Good luck!

---

1. Let V be the subspace of $\mathbb{R}_3$ consisting of polynomials $\mathbf{p}$ with $\mathbf{p}(0) = 0$. Let $\mathbf{b}_1 = -t + t^2$, $\mathbf{b}_2 = t + t^2 + t^3$, $\mathbf{b}_3 = -7t - 5t^2 + 2t^3$. Is the set $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ linearly independent? Does it span V? Is it a basis for V?

Solution: Note that V is a 3-dimensional space with basis $\{t, t^2, t^3\}$. Relative to this basis, our three vectors have coordinates $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} -7 \\ -5 \\ 2 \end{pmatrix}$. Since the 3×3 matrix $\begin{pmatrix} -1 & 1 & -7 \\ 1 & 1 & -5 \\ 0 & 1 & 2 \end{pmatrix}$ is invertible (just row reduce it, or take its determinant), the columns form a basis for $\mathbb{R}^3$, hence the original vectors form a basis for V (and are linearly independent and span, or course).

2. In $\mathbb{R}_2[t]$, let $\mathbf{b}_1(t) = 1 + t + t^2$, $\mathbf{b}_2(t) = 2 + 3t + t^2$, $\mathbf{b}_3(t) = 1 + 2t + t^2$, and $\mathbf{v}(t) = 5 - 2t + 3t^2$. Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis. Find $P_{\mathcal{E}\mathcal{B}}$, $P_{\mathcal{B}\mathcal{E}}$, and $[\mathbf{v}]_{\mathcal{B}}$.

Solution:

$$P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} [b_1]_{\mathcal{E}} & [b_2]_{\mathcal{E}} & [b_3]_{\mathcal{E}} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}$$

$$[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}; \qquad [\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} 10 \\ 2 \\ -9 \end{pmatrix}.$$

You should double-check that $\mathbf{v}$ really does equal $10\mathbf{b}_1 + 2\mathbf{b}_2 - 9\mathbf{b}_3$.

3. On $\mathbb{R}_3[t]$, let L be the linear operator that shifts a function over to the left by one. That is, $(L\mathbf{p})(t) = \mathbf{p}(t+1)$. Find the matrix of L relative to the standard basis $\{1, t, t^2, t^3\}$.

Solution: Note that $L(\mathbf{b}_1) = 1$, $L(\mathbf{b}_2) = 1+t$, $L(\mathbf{b}_3) = (1+t)^2 = 1+2t+1$, $L(\mathbf{b}_4) = (1+t)^3 = 1+3t+3t^2+t^3$. Take the coordinates of these to get the columns of

$$[L]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. In $\mathbb{R}_2[t]$, let $\mathbf{b}_1(t) = 1+t+t^2$, $\mathbf{b}_2(t) = 2+3t+t^2$, $\mathbf{b}_3(t) = 1+2t+t^2$, as in problem 2. Let $L = d/dt$ be the derivative operator. Find the matrix of L relative to the basis $\mathcal{B}$. [You may find your answers to problem 2 to be useful.]

Solution: Let $\mathcal{E}$ be the standard basis, and compute $[L]_{\mathcal{E}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Then $[L]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[L]_{\mathcal{E}}P_{\mathcal{E}\mathcal{B}} =$

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & -4 & -2 \end{pmatrix}.$$