M346 Practice Second Exam
Originally given November, 2, 2000

Problem 1: Find all the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & -3 & 5 \end{pmatrix}.$$

Note that the matrix is block-triangular, with an upper left 1×1 block and a lower right 2×2 block. The eigenvalue of the 1×1 block is 1, and the eigenvalues of the 2×2 block (which is itself triangular) are 2 and 5. Thus our eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 5$.

Another way to get the eigenvalues is to note that the sum of each row is 2, that the trace is 8, and that the determinant is 10. Or you could compute the characteristic polynomial and find the roots.

You then find the eigenvectors by solving $(A - \lambda I)x = 0$ by row reduction. The answers are $\mathbf{b}_1 = (1,0,0)^T$, $\mathbf{b}_2 = (1,1,1)^T$, $\mathbf{b}_2 = (1,0,4)^T$.

Problem 2: Find a matrix with eigenvalues 1, 2 and 3 and corresponding eigenvectors $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$,

$A = PDP^{-1}$, where $P = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. We compute $P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ -3 & 1 & -2 \end{pmatrix}$ and $A = PDP^{-1} = \begin{pmatrix} 5 & -1 & 2 \\ -2 & 2 & -2 \\ -4 & 1 & -1 \end{pmatrix}$.

Problem 3: The eigenvalues of the matrix $A = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$ are 5, 0, and -5.

a) Find the eigenvectors.

By row-reduction, $\mathbf{b}_1 = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$.

b) Decompose the vector $(50,0,0)^T$ as a linear combination of eigenvectors.

By row-reduction, or by computing P^{-1}, we get $\begin{pmatrix} 50 \\ 0 \\ 0 \end{pmatrix} = 3\mathbf{b}_1 + 8\mathbf{b}_2 + 3\mathbf{b}_3$.

c) Solve the differential equation $d\mathbf{x}/dt = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = (50,0,0)^T$.

$$\mathbf{x}(t) = 3e^{5t}\mathbf{b}_1 + 8\mathbf{b}_2 + 3e^{-5t}\mathbf{b}_3 = \begin{pmatrix} 9e^{5t} + 32 + 9e^{-5t} \\ 15e^{5t} - 15e^{-5t} \\ 12e^{5t} - 24 + 12e^{-5t} \end{pmatrix}.$$

Problem 4: Consider discrete-time evolution equations

$$\begin{aligned} x_1(n) =& x_1(n-1) + 2x_2(n-1) \\ x_2(n) =& x_1(n-1) + 3x_2(n-1). \end{aligned}$$

a) How many stable modes does this system have? How many neutrally stable modes? How many unstable modes?

The matrix is $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, whose eigenvalues are $2 \pm \sqrt{3}$ and eigenvectors are $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 1+\sqrt{3} \end{pmatrix}$ (or $\begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix}$) and $\mathbf{b}_2 = \begin{pmatrix} 2 \\ 1-\sqrt{3} \end{pmatrix}$ (or $\begin{pmatrix} \sqrt{3}+1 \\ -1 \end{pmatrix}$). One eigenvalue is bigger than one (unstable), and one is less than one (stable).

b) Write down the general solution to this system of equations.

$\mathbf{x}(n) = c_1(2+\sqrt{3})^n \mathbf{b}_1 + c_2(2-\sqrt{3})^n \mathbf{b}_2$, with $\mathbf{b}_1$ and $\mathbf{b}_2$ as above.

c) Describe qualitatively the behavior of $\mathbf{x}(n)$ for large n (both size and direction), given typical initial conditions.

Typically, both c_1 and c_2 will be nonzero, so $c_1(2+\sqrt{3})^n$ will grow, while $c_2(2-\sqrt{3})^n$ will go to zero. For n large, the system will point in the $\mathbf{b}_1$ direction (that is, the ratio x_1/x_2 will be close to $2/(1+\sqrt{3}) = \sqrt{3}-1$), and the size will multiply by approximately $2+\sqrt{3}$ in each step.

Problem 5: Consider the nonlinear system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} =& x_1(3 - x_1 - 2x_2) \\ \frac{dx_2}{dt} =& x_2(2 - x_1 - x_2) \end{aligned}$$

a) Find the fixed points. [There are four of them]

The points where $d\mathbf{x}/dt = 0$ are $(0,0)^T$, $(0,2)^T$, $(3,0)^T$, and $(1,1)^T$.

b) For each fixed point, find a linear system of equations that approximates the dynamics near the fixed point.

Taking derivatives of $x_1(3-x_1-2x_2)$ and $x_2(2-x_1-x_2)$ with respect to x_1 and x_2 gives the matrix of the linearization, namely $A = \begin{pmatrix} 3-2x_1-2x_2 & -2x_1 \\ -x_2 & 2-x_1-2x_2 \end{pmatrix}$. At $(0,0)^T$ this is $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, at $(3,0)^T$ it is $\begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$, at $(0,2)^T$ it is $\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$, and at $(1,1)^T$ it is $\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$.

c) Which (if any) of the fixed points are stable?

At $(0,0)$, both eigenvalues are positive, so the system is unstable. At $(3,0)^T$, both eigenvalues are negative, so the system is stable. At $(0,2)^T$, both eigenvalues are negative, so the system is stable. At $(1,1)^T$, the eigenvalues are $-1 \pm \sqrt{2}$. One eigenvalue is positive, so the system is unstable.