M346 First Midterm Exam, September 21, 2004

1. Let V be the subspace of $\mathbb{R}^4$ defined by the equation $x_1+x_2+x_3+x_4=0$.

a) Find the dimension of V.

b) Find a basis for V. [Any basis will do, but the simpler your answer, the easier part (c) will be. Be sure that each of your vectors really is in V, and that they are linearly independent]

c) Let $L(\mathbf{x}) = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{pmatrix}$. Note that L takes V to V, and can be viewed as an operator *on* V. Find the matrix $[L]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis you found in part (b).

2. In $\mathbb{R}^2$, consider the basis $\mathbf{b}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

a) Find the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$, where $\mathcal{E}$ is the standard basis.

b) If $\mathbf{v} = \begin{pmatrix} 13 \\ -2 \end{pmatrix}$, find $[\mathbf{v}]_{\mathcal{B}}$.

c) Let $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_1+x_2 \end{pmatrix}$. Find $[L]_{\mathcal{E}}$ and $[L]_{\mathcal{B}}$.

3. Consider the coupled first-order differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 + 2x_2 \\ \frac{dx_2}{dt} &= 2x_1 + x_2 \end{aligned}$$

Define the new variables $y_1(t) = x_1(t) + x_2(t)$, $y_2(t) = x_1(t) - x_2(t)$.

a) Rewrite the system of equations completely in terms of y_1 and y_2. (That is, express dy_1/dt and dy_2/dt as functions of y_1 and y_2.)

b) Given the initial conditions $x_1(0) = 1$, $x_2(0) = 0$, find $x_1(t)$ and $x_2(t)$.

4. Let $V = \mathbb{R}_3[t]$, and let $L : V \to V$ be defined by $L(\mathbf{p})(t) = \mathbf{p}'(t) + 2\mathbf{p}''(t)$.

a) Find $[L]_{\mathcal{E}}$, where $\mathcal{E} = \{1, t, t^2, t^3\}$ is the standard basis.

b) What is the dimension of the kernel of L? What is the dimension of the range of L?

c) Find a basis for the kernel of L.

d) Find a basis for the range of L.

5. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

For (a) and (b), suppose that a 4×4 matrix A row-reduces to $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

a) The null space of A is the span of $(-2, -1, 1, 0)^T$.

b) The column space of A is the span of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

For (c) and (d), suppose that $L : \mathbb{R}_2[t] \to M_{2,2}$ is a linear transformation, and that $B = [L]_{\mathcal{E}\mathcal{E}}$ is the matrix of L relative to the standard bases for $\mathbb{R}_2[t]$ and $M_{2,2}$.

c) If B row-reduces to something with 3 pivots, then L is 1–1.

d) If $\begin{pmatrix} 1 & 3 \\ 4 & 7 \end{pmatrix}$ is in the range of L, then $\begin{pmatrix} 1 \\ 3 \\ 4 \\ 7 \end{pmatrix}$ is in the column space of B.

e) $\mathbb{R}^3$ is the internal direct sum of the x_1-x_2 and x_1-x_3 planes.