M346 Final Exam Solutions, December 10, 2003

1. In the vector space $\mathbb{R}_2[t]$, consider the basis $\mathcal{B} = \{1, 1+t, 1+t+t^2\}$, the basis $\mathcal{D} = \{1+t+t^2, t+t^2, t^2\}$ and the vector $\mathbf{v} = 2t^2 - 1$.

a) Find $[\mathbf{v}]_\mathcal{B}$ and $[\mathbf{v}]_\mathcal{D}$. (That is, find the coordinates of $\mathbf{v}$ in the $\mathcal{B}$ basis, and the coordinates of $\mathbf{v}$ in the $\mathcal{D}$ basis.)

Since $\mathbf{v} = 2t^2 - 1 = 2(\mathbf{b}_3 - \mathbf{b}_2) - \mathbf{b}_1 = 2\mathbf{b}_3 - 2\mathbf{b}_2 - \mathbf{b}_1$, $[\mathbf{v}]_\mathcal{B} = (-1, -2, 2)^T$. Since $\mathbf{v} = 2t^2 - 1 = 2\mathbf{d}_3 - 1 = 2\mathbf{d}_3 + \mathbf{d}_2 - \mathbf{d}_1$, $[\mathbf{v}]_\mathcal{D} = (-1, 1, 2)^T$.

b) Find the change-of-basis matrices $P_{\mathcal{BD}}$ and $P_{\mathcal{DB}}$.

$\mathbf{b}_1 = \mathbf{d}_1 - \mathbf{d}_2$, $\mathbf{b}_2 = \mathbf{d}_1 - \mathbf{d}_3$ and $\mathbf{b}_3 = \mathbf{d}_1$, so $P_{\mathcal{DB}} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$.

Similarly, $\mathbf{d}_1 = \mathbf{b}_3$, $\mathbf{d}_2 = \mathbf{b}_3 - \mathbf{b}_1$, $\mathbf{d}_3 = \mathbf{b}_3 - \mathbf{b}_2$, so $P_{\mathcal{BD}} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

The results of parts (a) and (b) could also have been obtained by writing down $P_{\mathcal{EB}}$ and $P_{\mathcal{ED}}$, taking their inverses to get $P_{\mathcal{BE}}$ and $P_{\mathcal{DE}}$, using these to compute $[\mathbf{v}]_\mathcal{B}$ and $[\mathbf{v}]_\mathcal{D}$, and taking products of matrices to get $P_{\mathcal{BD}}$ and $P_{\mathcal{DB}}$.

2. Consider the operator $L : \mathbb{R}_2[t] \to \mathbb{R}_2[t]$ defined by $(L\mathbf{p})(t) = \mathbf{p}(t+1) - \mathbf{p}(t)$.

a) Find the matrix of L in the standard basis $\{1, t, t^2\}$.

Let $\mathbf{e}_1 = 1$, $\mathbf{e}_2 = t$, $\mathbf{e}_3 = t^2$. We compute $L\mathbf{e}_1 = 1 - 1 = 0$, $L\mathbf{e}_2 = (t+1) - t = 1 = \mathbf{e}_1$ and $L\mathbf{e}_3 = (t+1)^2 - t^2 = 2t + 1 = \mathbf{e}_1 + 2\mathbf{e}_2$, so

$$[L]_\mathcal{E} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

b) Find the matrix of L in the basis $\mathcal{B} = \{1, 1+t, 1+t+t^2\}$ (this is the same basis $\mathcal{B}$ you saw in problem 1).

Either compute $[L]_\mathcal{B} = P_{\mathcal{BE}}[L]_\mathcal{E} P_{\mathcal{EB}}$ or work directly in the $\mathcal{B}$ basis: $L\mathbf{b}_1 = 0$, $L\mathbf{b}_2 = 1 = \mathbf{b}_1$, $L\mathbf{b}_3 = 2t + 2 = 2\mathbf{b}_2$. Either way, we have

$$[L]_\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. Matrices and eigenvalues:

a) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 2 & 1 & 4 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

This is block triangular. The eigenvalue of the upper left block is $\lambda_1 = 2$, while those of the lower right block are $\lambda_2 = 3$ and $\lambda_3 = 1$. The corresponding eigenvectors (obtained by solving $(\lambda I - A)\mathbf{x} = 0$) are $\mathbf{b}_1 = (1, 0, 0)^T$, $\mathbf{b}_2 = (5, 1, 1)^T$, $\mathbf{b}_3 = (3, 1, -1)^T$.

b) Find a matrix whose eigenvalues are $1, 3, 4$ and eigenvectors are $\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$. [Hint: there is an easy way to compute P^{-1}]

Note that the three eigenvectors are orthonormal, so the matrix $P = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$ is orthogonal, so $P^{-1} = P^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$. We then compute $A = PDP^{-1} =$

$$\frac{1}{9}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 29 & -8 & -2 \\ -8 & 23 & -10 \\ -2 & -10 & 20 \end{pmatrix}.$$

Note that A is Hermitian. This is to be expected, since A has real eigenvalues and orthogonal eigenvectors.

4. Let $A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$. (You may find useful the fact that A is Hermitian, but you don't need this fact to solve the problem).

a) Find the eigenvalues and eigenvectors of the matrix A.

Eigenvalues $\lambda_1 = 0$, $\lambda_2 = -5$, eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. (Note that the eigenvectors are orthogonal).

b) Find the most general solution to the system of equations $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}$.

$\mathbf{x}(t) = (c_1 + c_2 t)\mathbf{b}_1 + (c_3 \cos(\sqrt{5}t) + c_4 \sin(\sqrt{5}t))\mathbf{b}_2$.

c) Find the solution to the system of equations $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\dot{\mathbf{x}}(0) = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

Since $\mathbf{x}(0) = c_1\mathbf{b}_1 + c_3\mathbf{b}_2$, we have $c_1 = \langle \mathbf{b}_1|\mathbf{x}(0)\rangle/\langle \mathbf{b}_1|\mathbf{b}_1\rangle = 8/5$, and $c_3 = \langle \mathbf{b}_2|\mathbf{x}(0)\rangle/\langle \mathbf{b}_2|\mathbf{b}_2\rangle = -1/5$. (You could also get c_1 and c_3 by row-reduction). Likewise, $\dot{\mathbf{x}}(0) = c_2\mathbf{b}_1 + \sqrt{5}c_4\mathbf{b}_2$, so $c_2 = \langle \mathbf{b}_1|\dot{\mathbf{x}}(0)\rangle/\langle \mathbf{b}_1|\mathbf{b}_1\rangle = 18/5$, $\sqrt{5}c_4 = \langle \mathbf{b}_2|\dot{\mathbf{x}}(0)\rangle/\langle \mathbf{b}_2|\mathbf{b}_2\rangle = -1/5$. Combining, we get

$$\begin{aligned} \mathbf{x}(t) &= \left(\frac{8}{5} + \frac{18}{5}t\right)\mathbf{b}_1 + \left(\frac{-1}{5}\cos(\sqrt{5}t) - \frac{1}{5\sqrt{5}}\sin(\sqrt{5}t)\right)\mathbf{b}_2 \\ &= \begin{pmatrix} \frac{16+36t}{5} - \frac{1}{5}\cos(\sqrt{5}t) - \frac{1}{5\sqrt{5}}\sin(\sqrt{5}t) \\ \frac{8+18t}{5} + \frac{2}{5}\cos(\sqrt{5}t) + \frac{2}{5\sqrt{5}}\sin(\sqrt{5}t) \end{pmatrix} \end{aligned}$$

5. Linearization in one dimension.

a) Consider the first order differential equation $dx/dt = f(x)$, where $f(x) = \frac{1}{9}\sin(\pi x^2)$. This has fixed points at $x = 1$, $x = 2$, and $x = 3$ (and lots of other points, which we'll ignore). Which of these three points is stable, unstable, or neutrally stable?

$f'(x) = \frac{2\pi x}{9}\cos(\pi x^2)$. At $x = 1$ this equals $-2\pi/9$, which is negative, so 1 is stable. $f'(2) = 4\pi/9 > 0$, so 2 is unstable. $f'(3) = -2\pi/3 < 0$, so 3 is stable.

b) Next, consider the second-order differential equation $d^2x/dt^2 = f(x)$, where $f(x) = \frac{1}{9}\sin(\pi x^2)$, as before. Again, this has fixed points at $x = 1, 2, 3$. Which of these three points is stable, unstable, or neutrally stable?

$f'(1) < 0$, so 1 is neutrally stable. $f'(2) > 0$, so 2 is unstable. $f'(3) < 0$, so 3 is neutrally stable.

c) Finally, consider the difference equation $x(n+1) = g(x(n))$, where now $g(x) = x + \frac{1}{9}\sin(\pi x^2)$. Yet again, this has fixed points at $x = 1, 2, 3$. Which of these three points is stable, unstable, or neutrally stable?

$g'(1) = 1 - 2\pi/9$, which is less than one, so 1 is stable. $g'(2) = 1 + 4\pi/9 > 1$, so 2 is unstable. $f'(3) = 1 - 2\pi/3 < -1$, so 3 is unstable. Remember that for this type of problem what matters is the norm $|\lambda|$, not the sign of λ.

6. Consider the vectors $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 6 \\ 4 \\ -9 \end{pmatrix}$ in $\mathbb{R}^3$. Note that these vectors are orthogonal.

a) Decompose the vector $\mathbf{v} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ as a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$. [Warning: this problem involves somewhat messy fractions.]

We wish to write $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$. Since the $\mathbf{b}$'s are orthogonal, we have $c_1 = \langle \mathbf{b}_1|\mathbf{v}\rangle/\langle \mathbf{b}_1|\mathbf{b}_1\rangle = 32/14 = 16/7$, $c_2 = \langle \mathbf{b}_2|\mathbf{v}\rangle/\langle \mathbf{b}_2|\mathbf{b}_2\rangle = 14/38 = 7/19$, $c_3 = \langle \mathbf{b}_3|\mathbf{v}\rangle/\langle \mathbf{b}_3|\mathbf{b}_3\rangle = -25/133$. Doing this problem by row-reduction, while certainly possible, would have been very, very messy.

b) Find the matrices for the projections $P_{\mathbf{b}_1}$ and $P_{\mathbf{b}_2}$.

$$P_{\mathbf{b}_1} = \frac{|\mathbf{b}_1\rangle\langle\mathbf{b}_1|}{\langle\mathbf{b}_1|\mathbf{b}_1\rangle} = \frac{1}{14}\begin{pmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & 4 \end{pmatrix}, \text{ while } P_{\mathbf{b}_2} = \frac{|\mathbf{b}_2\rangle\langle\mathbf{b}_2|}{\langle\mathbf{b}_2|\mathbf{b}_2\rangle} = \frac{1}{38}\begin{pmatrix} 25 & -15 & 10 \\ -15 & 9 & -6 \\ 10 & -6 & 4 \end{pmatrix}.$$

c) Find an orthogonal matrix whose columns are proportional to the three vectors $\mathbf{b}_i$.

Just rescale the $\mathbf{b}$'s to be unit vectors. Our matrix is

$$\begin{pmatrix} 1/\sqrt{14} & 5/\sqrt{38} & 6/\sqrt{133} \\ 3/\sqrt{14} & -3/\sqrt{38} & 4/\sqrt{133} \\ 2/\sqrt{14} & 2/\sqrt{38} & -9/\sqrt{133} \end{pmatrix}$$

7. Working on the interval $x \in [0, 1]$, let

$$f(x) = \begin{cases} 1 & \text{if } 1/4 < x < 3/4; \\ 0 & \text{otherwise} \end{cases}; \qquad g(x) = 2x - 1.$$

The functions $f(x)$ and $g(x)$ can each be written as (sine) Fouries series:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x); \qquad g(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

a) Compute a_1, a_2, a_3 and a_4.

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx = 2\int_{1/4}^{3/4} \sin(n\pi x) = \frac{2}{n\pi}\left[\cos(n\pi/4) - \cos(3n\pi/4)\right],$$

from which we get $a_1 = 2\sqrt{2}/\pi$, $a_2 = 0$, $a_3 = -2\sqrt{2}/3\pi$, $a_4 = 0$.

b) Compute $\sum_{n=1}^{\infty} |a_n|^2$.

Note that $1/2 = \langle f|f\rangle = \sum |a_n|^2 \langle \mathbf{b}_n|\mathbf{b}_n\rangle = \frac{1}{2}\sum |a_n|^2$, so $\sum |a_n|^2 = 1$.

c) Compute $\sum_{n=1}^{\infty} a_n b_n$. [Note: you do NOT need the results of (a) to do (b) and (c)]

Note that $0 = \int_0^1 \bar{f}(x)g(x)dx = \langle f|g\rangle = \frac{1}{2}\sum \bar{a}_n b_n$. However, the coefficients are all real, so $\sum a_n b_n = \sum \bar{a}_n b_n = 0$.

8. We wish to solve the differential equation

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2}$$

on the interval $x \in (0, \pi)$ with Dirichlet boundary conditions $f(0,t) = f(\pi, t) = 0$. [This is called the heat equation, and can be attacked by generalizing the methods of section 5.1 to spaces of functions, just as the vibrating string problem was solved by generalizing the methods of section 5.3 to functions.]

(a) Find the most general solution to this equation.

This equation is of the form $\dot{f} = Lf$, where $L = d^2/dx^2$. The eigenvectors of L are $\sin(nx)$, with eigenvalue $-n^2$, so the most general solution is

$$f(x,t) = \sum_n c_n e^{\lambda_n t} \mathbf{b}_n = \sum_n c_n e^{-n^2 t} \sin(nx).$$

b) Given the initial conditions $f(x,0) = 3\sin(x) - 5\sin(2x) + 37\sin(3x)$, find $f(x,t)$ for all $x \in (0,\pi)$ and all t. [Note that the equation involves the first derivative with respect to time, so our initial conditions are just the value $f(x,0)$ of the function at time $t = 0$, and doesn't involve $\dot{f}(x,0)$.]

$$f(x,t) = 3e^{-t}\sin(x) - 5e^{-4t}\sin(2x) + 37e^{-9t}\sin(3x).$$

9. True of False? Each question is worth 2 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) If a matrix is Hermitian, then the geometric multiplicity of each eigenvalue equals the algebraic multiplicity.

TRUE. The matrix is diagonalizable.

b) If A is a real anti-symmetric matrix ($A^T = -A$), then e^A is an orthogonal matrix.

TRUE. Since A is anti-Hermitian, e^A is unitary, and since A is real, e^A is real, hence e^A is orthogonal.

c) Every solution to the wave equation on the real line is either a forward traveling wave or a backwards traveling wave.

FALSE. Every solution is the SUM of a forward traveling wave and a backwards traveling wave.

d) There exists a Hermitian matrix with eigenvalue $2 + i$.

FALSE. Hermitian matrices have real eigenvalues.

e) The equation $A\mathbf{x} = \mathbf{b}$ has a least-squares solution only if $\mathbf{b}$ is in the column space of A.

FALSE. Least-squares solutions always exist.

f) Every change-of-basis matrix is invertible.

TRUE. (You can always change back!)

g) If A is a 4×7 matrix, then the null space of A is 3-dimensional.

FALSE. The dimension is AT LEAST 3. (If A is a matrix of zeroes, then the null space is 7-dimensional).

h) If the columns of a square matrix are linearly dependent, then zero is an eigenvalue.

TRUE. The matrix has a null vector, hence a vector for which $A\mathbf{v} = 0\mathbf{v}$.

i) The system $d\mathbf{x}/dt = A\mathbf{x}$ is stable if all the eigenvalues of A lie inside the unit circle.

FALSE. The system is stable if all the eigenvalues have negative real part.

j) If $\mathcal{B}, \mathcal{D}$ and $\mathcal{E}$ are bases for the same vector space, then the change-of-basis matrices satisfy $P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{B}\mathcal{E}} P_{\mathcal{E}\mathcal{D}}$.

TRUE. Converting from $\mathcal{D}$ to $\mathcal{E}$, and then from $\mathcal{E}$ to $\mathcal{B}$, is the same as converting from $\mathcal{D}$ to $\mathcal{B}$.