M346 Final Exam Solutions, December 11, 2004

1. On $\mathbb{R}_3[t]$, let L be the linear operator that shifts a function to the left by one. That is $(L\mathbf{p})(t) = \mathbf{p}(t+1)$. Find the matrix of L relative to the standard basis $\{1, t, t^2, t^3\}$

Solution: Note that $L(\mathbf{b}_1) = 1$, $L(\mathbf{b}_2) = 1+t$, $L(\mathbf{b}_3) = (1+t)^2 = 1+2t+1$, $L(\mathbf{b}_4) = (1+t)^3 = 1+3t+3t^2+t^3$. Take the coordinates of these to get the columns of

$$[L]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. a) Find the eigenvalues of the following matrix. You do NOT have to find the eigenvectors.

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 7 & 8 & 9 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 5 & 4 & 3 \end{pmatrix}$$

This matrix is block-triangular, with upper-left block $A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ and lower-right block $B = \begin{pmatrix} 4 & 0 & 0 \\ 3 & 3 & 4 \\ 6 & 4 & 3 \end{pmatrix}$. B is itself block-triangular, with upper left block $C = 4$ and lower right block $D = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$. The eigenvalue of C is obviously 4, and the eigenvalues of D are $3 \pm 4 = 7, -1$. All that's left is to find the eigenvalues of A.

Since the sum of each column is 5, $\lambda_1 = 5$. Since the determinant is zero, one eigenvalue must be zero. Since the trace is 6, the sum of the eigenvalues is 6, so the third eigenvalue is 1. Thus the eigenvalues of A are $(5, 0, 1)$ and the eigenvalues of our matrix are $(5, 0, 1, 4, -1, 7)$.

b) Find the eigenvalues AND eigenvectors of the matrix $\begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$. Eigenvalues $\lambda_\pm = \frac{1 \pm \sqrt{17}}{2}$, eigenvectors $\mathbf{b}_\pm = \begin{pmatrix} \lambda_\pm \\ 1 \end{pmatrix}$. You can also rescale this to $\begin{pmatrix} 1 \pm \sqrt{17} \\ 2 \end{pmatrix}$ or $\begin{pmatrix} -8 \\ 1 \mp \sqrt{17} \end{pmatrix}$.

For extra credit, find the eigenvalues AND eigenvectors of the matrix $\begin{pmatrix} 1 & -4 \\ 1 & 0 \end{pmatrix}$.

$\lambda_\pm = \frac{1 \pm i\sqrt{15}}{2}$, $\mathbf{b}_\pm = \begin{pmatrix} \lambda_\pm \\ 1 \end{pmatrix}$.

3. A 3×3 matrix A has eigenvalues 2, 1 and -1 and corresponding eigenvectors

$\mathbf{b}_1 = (1,2,3)^T$, $\mathbf{b}_2 = (1,1,-1)^T$ and $\mathbf{b}_3 = (-5,4,-1)^T$.
a) Decompose $(36,1,34)^T$ as a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$.

Let $\mathbf{x}_0 = (36,1,34)^T$, and write $\mathbf{x}_0 = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + a_3\mathbf{b}_3$. Since the $\mathbf{b}$'s are orthogonal, $a_1 = \langle \mathbf{b}_1|\mathbf{x}\rangle / \langle \mathbf{b}_1|\mathbf{b}_1\rangle = 140/14 = 10$. Likewise, $a_2 = 3/3 = 1$ and $a_3 = -210/42 = -5$, so $\mathbf{x}_0 = 10\mathbf{b}_1 + \mathbf{b}_2 - 5\mathbf{b}_3$. This could also have been obtained by row reduction.

b) If $d\mathbf{x}/dt = A\mathbf{x}$ and $\mathbf{x}(0) = (36,1,34)^T$, what is $\mathbf{x}(t)$? [You do NOT need to compute A to do this.]

Since each term goes as $e^{lambdat}$, $\mathbf{x}(t) = 10e^{2t}\mathbf{b}_1 + e^t\mathbf{b}_2 - 5e^{-t}\mathbf{b}_3$.

c) Is A Hermitian? Why or why not? Is A unitary?

Since A is diagonalizable, has real eigenvalues, and has orthogonal eigenvectors, A is Hermitian. Since $|\lambda_1| = 2 \neq 1$, A is not unitary.

4. Let A be a 3×3 matrix with eigenvalues -9, 0 and 4, and with eigenvectors $\mathbf{b}_1 = (1,1,1)^T$, $\mathbf{b}_2 = (1,2,3)^T$, and $\mathbf{b}_3 = (0,0,1)^T$.

a) Decompose $\mathbf{w} = (4,5,5)^T$ and $\mathbf{v} = (2,1,4)$ as linear combinations of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$.

By row-reduction, $\mathbf{w} = 3\mathbf{b}_1 + \mathbf{b}_2 - \mathbf{b}_3$ and $\mathbf{v} = 3\mathbf{b}_1 - \mathbf{b}_2 + 4\mathbf{b}_3$. Since the vectors are NOT orthogonal, you cannot get these coefficients by taking inner products (at least not easily).

b) Solve the system of differential equations $d^2\mathbf{x}/dt^2 = A\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \mathbf{w}$ and $\frac{d\mathbf{x}}{dt}|_{t=0} = \mathbf{v}$.

Since $\lambda_1 = -9 < 0$, the $\mathbf{b}_1$ terms go as $\cos(3t)$ and $\sin(3t)$. Since $\lambda_2 = 0$, the $\mathbf{b}_2$ terms go as 1 and t. Since $\lambda_3 = 4 > 0$, the $\mathbf{b}_3$ terms go as $\cosh(2t)$ and $\sinh(2t)$. All together,

$$\mathbf{x}(t) = (3\cos(3t) + \sin(3t))\mathbf{b}_1 + (1-t)\mathbf{b}_2 + (-\cosh(2t) + 2\sinh(2t))\mathbf{b}_3$$

c) Is A Hermitian? Why or why not? Is A unitary?

Neither Hermitian nor unitary, since the eigenvectors are not orthogonal.

5. Linearization. Consider the nonlinear difference equations

$$\begin{aligned} x_1(n+1) &= \frac{x_1(n)^2}{2} + \frac{x_2(n)^2}{2} - \frac{1}{8} \\ x_2(n+1) &= x_1(n)x_2(n) + \frac{1}{2} \end{aligned}$$

near the fixed point $\mathbf{a} = (1/2, 1)^T$.

a) Write down a LINEAR system of difference equations that (approximately) describes the evolution of $\mathbf{y} = \mathbf{x} - \mathbf{a}$.

$\mathbf{y}(n+1) \approx A\mathbf{y}(n)$, where $A = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}\Big|_{\mathbf{x}=\mathbf{a}} = \begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}$. (The first row of A is the gradient of $x_1^2/2 + x_2^2/2 - 1/8$, while the second is the gradiant of $x_1x_2 + 1/2$.)

b) How many stable modes are there? How many unstable? How many neutral?

Since the eigenvalues of A are 3/2 and -1/2, and $|3/2| > 1 > |-1/2|$, there is one unstable mode and one stable mode. The unstable mode has eigenvector $(1,1)^T$ while the stable mode has eigenvector $(1,-1)^T$.

c) Write down the general solution to the linear difference equations you found in (a).

$$\mathbf{y}(n) = c_1(3/2)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2(-1/2)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

6. Gram-Schmidt. Convert the following collections of vectors to orthogonal collections using the Gram-Schmidt process. In each case, we are using the usual inner product.

a) In $\mathbb{R}^4$, $\mathbf{x}_1 = (1,0,1,2)^T$, $\mathbf{x}_2 = (2,1,2,1)^T$, $\mathbf{x}_3 = (6,3,4,1)^T$.

$\mathbf{y}_1 = (1,0,1,2)^T$, $\mathbf{y}_2 = (1,1,1,-1)^T$, $\mathbf{y}_3 = (1,0,-1,0)^T$.

b) In $\mathbb{C}^3$, $\mathbf{x}_1 = (1+i, 1-i, 2i)^T$, $\mathbf{x}_2 = (3+3i, 3-i, 1+3i)^T$.

$$\mathbf{y}_1 = \begin{pmatrix} 1+i \\ 1-i \\ 2i \end{pmatrix}, \ \mathbf{y}_2 = \mathbf{x}_2 - 2\mathbf{y}_1 = \begin{pmatrix} 1+i \\ 1+i \\ 1-i \end{pmatrix}.$$

7. Least squares.

a) Find a least-squares solution to the linear equations

$$\begin{pmatrix} 1 & -2 & -1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 7 \\ 5 \end{pmatrix}$$

Calling the 5×3 matrix A and the right-hand-side $\mathbf{b}$, we have $A^TA = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$ and $A^T\mathbf{b} = \begin{pmatrix} 18 \\ 10 \\ -10 \end{pmatrix}$. Solving $A^TA\mathbf{x} = A^T\mathbf{b}$ gives $\mathbf{x} = (18/5, 1, -1)^T$.

b) Find the equation of the best line through the points (0,1), (1,3), (2,4), (3,5), and (4,4).

Setting $y = mx + b$, our data becomes

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 4 \end{pmatrix},$$

whose least-squares solution is $b = 9/5$, $m = 4/5$. So the best line is $y = (4/5)x + (9/5)$.

8. Working on the interval $x \in [0,1]$, let $g_0(x) = \begin{cases} x & \text{if } x < 1/2; \\ 1-x & \text{if } x \geq 1/2 \end{cases}$. In the book, we saw that g_0 can be expanded in a (sine) Fourier series: $g_0(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$, where $c_n = 4\sin(n\pi/2)/n^2\pi^2$.

Compute the solution to the wave equation: $\partial^2 f/\partial t^2 = \partial^2 f/\partial x^2$ on the interval $[0,1]$ with Dirichlet boundary conditions and with initial conditions

$f(x,0) = 0$, $\frac{\partial f}{\partial t}(x,0) = g_0(x)$. You may leave your answer as a Fourier series, but you should compute all the coefficients.

The eigenvalues of d^2/dx^2 are $\lambda_n = -n^2\pi^2/L^2 = -n^2\pi^2$, so $\omega_n = n\pi$, and

$$f(x,t) = \sum \frac{c_n}{\omega_n} \sin(n\pi x)\sin(\omega_n t) = \sum \frac{4\sin(n\pi/2)}{n^3\pi^3} \sin(n\pi x)\sin(n\pi t)$$

Extra credit: For fixed nonzero t, how smooth is $f(x,t)$ as a function of x? How many derivatives can you take? At what level do you get jump discontinuities. [I'm looking for an answer like "the first 15 derivatives of f are continuous, but the 16th derivative has jumps". (But no, that's not the correct answer)]

Since the coefficients go as n^{-3}, $f(x,t)$ has continuous values and first derivatives, but the 2nd derivative w.r.t. x has jumps.

9. True of False? Each question is worth 2 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) If a matrix A is Hermitian, then e^A is diagonalizable.

True. If A is Hermitian then A is diagonalizable, and all of the eigenvectors of A are also eigenvectors of e^A.

b) If A is Hermitian, then e^A is unitary.

No. It's e^{iA} that's unitary.

c) The eigenvalues of a real orthogonal matrix must be real.

False. The real orthogonal matrix $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ has complex eigenvalues $e^{\pm i\theta}$.

d) If $f(x)$ is a periodic function with Fourier coefficients $\hat{f}_n = e^{-n^2}$, then $f(x)$ is infinitely differentiable.

True, since e^{-n^2} decays faster than any power of n.

e) If the columns of a matrix A are orthogonal and nonzero, then the only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = 0$.

True. (Nonzero) orthogonal vectors are linearly independent.

f) If $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ are bases for a vector space, then $P_{\mathcal{B}\mathcal{D}}P_{\mathcal{D}\mathcal{E}}P_{\mathcal{E}\mathcal{B}} = I$.

True.

g) If $L : V \to W$ is a linear transformation, then the kernel of L is a subspace of W.

False. The kernel is a subspace of V, not of W.

h) If A is a block-triangular matrix, and if each block is diagonalizable, then A is diagonalizable.

False. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a counterexample.

i) The system $\mathbf{x}(n+1) = A\mathbf{x}(n)$ is stable if all the eigenvalues of A have negative real part.

False. The condition for stability is that the eigenvalues have magnitude less than 1.

j) If the determinant of a square matrix is zero, then zero is an eigenvalue.

True.