M346 Final Exam, December 11, 2004

1. On $\mathbb{R}_3[t]$, let L be the linear operator that shifts a function to the left by one. That is $(L\mathbf{p})(t) = \mathbf{p}(t+1)$. Find the matrix of L relative to the standard basis $\{1, t, t^2, t^3\}$

2. a) Find the eigenvalues of the following matrix. You do NOT have to find the eigenvectors.

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 7 & 8 & 9 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 5 & 4 & 3 \end{pmatrix}$$

b) Find the eigenvalues AND eigenvectors of the matrix $\begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$. Also find the eigenvalues AND eigenvectors of the matrix $\begin{pmatrix} 1 & -4 \\ 1 & 0 \end{pmatrix}$.

3. A 3×3 matrix A has eigenvalues 2, 1 and -1 and corresponding eigenvectors $\mathbf{b}_1 = (1,2,3)^T$, $\mathbf{b}_2 = (1,1,-1)^T$ and $\mathbf{b}_3 = (-5,4,-1)^T$.

a) Decompose $(36,1,34)^T$ as a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$.

b) If $d\mathbf{x}/dt = A\mathbf{x}$ and $\mathbf{x}(0) = (36,1,34)^T$, what is $\mathbf{x}(t)$? [You do NOT need to compute A to do this.]

c) Is A Hermitian? Why or why not? Is A unitary?

4. Let A be a 3×3 matrix with eigenvalues -9, 0 and 4, and with eigenvectors $\mathbf{b}_1 = (1,1,1)^T$, $\mathbf{b}_2 = (1,2,3)^T$, and $\mathbf{b}_3 = (0,0,1)^T$.

a) Decompose $\mathbf{w} = (4,5,5)^T$ and $\mathbf{v} = (2,1,4)$ as linear combinations of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$.

b) Solve the system of differential equations $d^2\mathbf{x}/dt^2 = A\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \mathbf{w}$ and $\frac{d\mathbf{x}}{dt}|_{t=0} = \mathbf{v}$.

c) Is A Hermitian? Why or why not? Is A unitary?

5. Linearization. Consider the nonlinear difference equations

$$\begin{aligned} x_1(n+1) &= \frac{x_1(n)^2}{2} + \frac{x_2(n)^2}{2} - \frac{1}{8} \\ x_2(n+1) &= x_1(n)x_2(n) + \frac{1}{2} \end{aligned}$$

near the fixed point $\mathbf{a} = (1/2, 1)^T$.

a) Write down a LINEAR system of difference equations that (approximately) describes the evolution of $\mathbf{y} = \mathbf{x} - \mathbf{a}$.

b) How many stable modes are there? How many unstable? How many neutral?

c) Write down the general solution to the linear difference equations you found in (a).

6. Gram-Schmidt. Convert the following collections of vectors to orthogonal collections using the Gram-Schmidt process. In each case, we are using the usual inner product.

a) In $\mathbb{R}^4$, $\mathbf{x}_1 = (1, 0, 1, 2)^T$, $\mathbf{x}_2 = (2, 1, 2, 1)^T$, $\mathbf{x}_3 = (6, 3, 4, 1)^T$.

b) In $\mathbb{C}^3$, $\mathbf{x}_1 = (1 + i, 1 - i, 2i)^T$, $\mathbf{x}_2 = (3 + 3i, 3 - i, 1 + 3i)^T$.

7. Least squares.

a) Find a least-squares solution to the linear equations

$$\begin{pmatrix} 1 & -2 & -1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 7 \\ 5 \end{pmatrix}$$

b) Find the equation of the best line through the points (0,1), (1,3), (2,4), (3,5), and (4,4).

8. Working on the interval $x \in [0, 1]$, let $g_0(x) = \begin{cases} x & \text{if } x < 1/2; \\ 1 - x & \text{if } x \geq 1/2 \end{cases}$. In the book, we saw that g_0 can be expanded in a (sine) Fourier series: $g_0(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$, where $c_n = 4\sin(n\pi/2)/n^2\pi^2$.

Compute the solution to the wave equation: $\partial^2 f/\partial t^2 = \partial^2 f/\partial x^2$ on the interval $[0, 1]$ with Dirichlet boundary conditions and with initial conditions $f(x, 0) = 0$, $\frac{\partial f}{\partial t}(x, 0) = g_0(x)$. You may leave your answer as a Fourier series, but you should compute all the coefficients.

Extra credit: For fixed nonzero t, how smooth is $f(x, t)$ as a function of x? How many derivatives can you take? At what level do you get jump discontinuities. [I'm looking for an answer like "the first 15 derivatives of f are continuous, but the 16th derivative has jumps". (But no, that's not the correct answer)]

9. True of False? Each question is worth 2 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) If a matrix A is Hermitian, then e^A is diagonalizable.

b) If A is Hermitian, then e^A is unitary.

c) The eigenvalues of a real orthogonal matrix must be real.

d) If $f(x)$ is a periodic function with Fourier coefficients $\hat{f}_n = e^{-n^2}$, then $f(x)$ is infinitely differentiable.

e) If the columns of a matrix A are orthogonal and nonzero, then the only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = 0$.

f) If $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ are bases for a vector space, then $P_{\mathcal{B}\mathcal{D}} P_{\mathcal{D}\mathcal{E}} P_{\mathcal{E}\mathcal{B}} = I$.

g) If $L : V \to W$ is a linear transformation, then the kernel of L is a subspace of W.

h) If A is a block-triangular matrix, and if each block is diagonalizable, then A is diagonalizable.

i) The system $\mathbf{x}(n+1) = A\mathbf{x}(n)$ is stable if all the eigenvalues of A have negative real part.

j) If the determinant of a square matrix is zero, then zero is an eigenvalue.