M408M First Midterm Exam, October 5, 2006

1. Parametrized curves. Consider the curve $x = t^2$, $y = t - t^3/3$, with t running from 0 to 2.

a) Find the velocity (vector!) at time $t = 1$, and the slope of the curve at $t = 1$.

Since $dx/dt = 2t$ and $dy/dt = 1 - t^2$, the velocity at $t = 1$ is (2,0) and the slope is 0/2=0.

b) Find the length of the curve. (Yes, you CAN do this integral! This is very similar to a homework problem.)

$\int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^2 \sqrt{(1-t^2)^2 + (2t)^2} dt = \int_0^2 \sqrt{1 + 2t^2 + t^4} dt = \int_0^2 1 + t^2 dt = 14/3$.

2. Polar coordinates. Consider the curve C given by the formula $r = 3 + 2\cos(\theta)$.

a) Find the points of intersection, in polar coordinates, of this curve with the circle $r = 4$.

If $3 + 2\cos(\theta) = 4$, we must have $\cos(\theta) = 1/2$, or $\theta = \pm\pi/3$. Our two points of intersection (in polar coordinates) are then $(4, \pi/3)$ and $(4, -\pi/3)$.

b) Find the Cartesian coordinates of the points you found in part (a).

$x = 4\cos(\pi/3) = 2$, $y = 4\sin(\pm\pi/3) = \pm 2\sqrt{3}$, so our two points are $(2, \pm 2\sqrt{3})$.

c) Find the area of the region inside the curve C but outside the circle $r = 4$. [You get partial credit for setting up this integral, and full credit for solving it. The final answer does involve $\sqrt{3}$ and π.]

$\int_{-\pi/3}^{\pi/3} \frac{(3+2\cos(\theta))^2 - 4^2}{2} d\theta = \int_{-\pi/3}^{\pi/3} \frac{12\cos(\theta) + 4\cos^2(\theta) - 7}{2} d\theta = \int_{-\pi/3}^{\pi/3} \frac{12\cos(\theta) + 2\cos(2\theta) - 5}{2} d\theta = \frac{12\sin(\theta) + \sin(2\theta) - 5\theta}{2}\Big|_{-\pi/3}^{\pi/3} = 13\sqrt{3}/2 - 5\pi/3$.

3. Equations of lines and planes.

a) Find the equation of the line through the point $(3, 2, 5)$ and parallel to the vector $(1, 2, -1)$. Express your answer FIRST in parametric form (or in vector form) and THEN in symmetric form.

Vector form: $(x, y, z) = (3, 2, 5) + (1, 2, -1)t$. Parametric form: $x = 3 + t$, $y = 2 + 2t$, $z = 5 - t$. Symmetric form: $x - 3 = (y - 2)/2 = -(z - 5)$.

b) Find the equation of the plane through the point $(3, 2, 5)$ perpendicular to the vector $(1, 2, -1)$.

If $P_0 = (3, 2, 5)$ and $v = (1, 2, -1)$, then our equation is $\mathbf{v} \cdot (x, y, z) = \mathbf{v} \cdot P_0$, or explicitly $x + 2y - z = 2$.

4. Cosines.

a) Find the cosine of the angle between the line $x = y = z$ and the line $z = -y = z/2$.

The first line is along the vector $\mathbf{v} = (1, 1, 1)$, while the second is along $\mathbf{w} = (1, -1, 2)$, so the cosine of the angle between them is $\mathbf{v} \cdot \mathbf{w}/|\mathbf{v}||\mathbf{w}| = 2/(\sqrt{3}\sqrt{6}) = \sqrt{2}/3$.

b) Find the cosine of the angle between the plane $x + y + z = 17$ and the plane $x - y + 2z = 24$.

The first plane is perpendicular to $\mathbf{v}$ and the second is perpendicular to $\mathbf{w}$, so this is precisely the same calculation as in part (a), and the answer is $\sqrt{2}/3$.

5. Triangles and planes. Consider the three points $P = (1, 2, 3)$, $Q = (2, 3, 4)$, $R = (3, 3, 3)$.

a) Find the area of the triangle PQR.

Let $\mathbf{v} = PQ = (1, 1, 1)$ and let $\mathbf{w} = PR = (2, 1, 0)$. Then $\mathbf{v} \times \mathbf{w} = (-1, 2, -1)$, and the area of the triangle is $|\mathbf{v} \times \mathbf{w}|/2 = \sqrt{6}/2$.

b) Find the equations of the plane through the three points P, Q and R.

Since the normal vector is $(-1, 2, -1)$, our plane is of the form $-x+2y-z =$ some number determined by plugging in P, Q or R into the equation, which turns out to be zero. Our equation is then $-x + 2y - z = 0$, or equivalently $x - 2y + z = 0$.